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THE HOPF ALGEBRA STRUCTURE OF A CROSSED PRODUCT IN A BRAIDED MONOIDAL CATEGORY

By

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Abstract. In this paper we define conditions under which a tensor product $A \otimes H$, in a braided monoidal category, together with a crossed product structure $A \sharp_\sigma H$ and a smash coproduct structure $A \propto H$ is a Hopf algebra. When $\sigma = \varepsilon_H \otimes \varepsilon_H \otimes \eta_A$, Radford's theorems characterizing the biproduct are obtained and when the antipode of H is a σ-antipode we find an analogous result with the one due to Wang, Jiao and Zhao.

1. Introduction

For a bialgebra H, an H-comodule coalgebra A with weak action and a normal twisted cocycle $\sigma : H \otimes H \to A$ Wang, Jiao and Zhao, using the crossed product and the smash coproduct, constructed in [11] the bialgebra $A \overset{\times}{\sharp_\sigma} H$ in a symmetric context. When σ is trivial this construction recovers the classical theorem, due to Radford [9], where we can find necessary and sufficient conditions for the smash product algebra and the smash coproduct coalgebra to form a bialgebra. Moreover, in [11], if H is a bialgebra with σ-antipode (which is necessarily an ordinary antipode), the authors have found an antipode for the bialgebra $A \overset{\times}{\sharp_\sigma} H$ but in the construction of this antipode the cocycle σ disappears.

In this paper, we will explain how when in a braided monoidal category $(\mathscr{C}, c)$, an algebra coalgebra A with a left weak action $\varphi_A : H \otimes A \to A$ satisfying $\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A$ and a structure of left H-comodule coalgebra r_A verifying the condition $r_A \circ \eta_A = \eta_H \otimes \eta_A$, if σ is a normal twisted cocycle such that $\varepsilon_A \circ \sigma = \varepsilon_H \otimes \varepsilon_H$ it is possible to obtain a bialgebra denoted here by $A\,{}_\sigma\!\bowtie H$. Moreover, if H is a Hopf algebra with invertible antipode λ_H and λ_A is the inverse of id_A in $Reg(A, A)$ then $A\,{}_\sigma\!\bowtie H$ is a Hopf algebra with antipode

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$$\lambda_{A_\sigma \bowtie H} = (\mu_A \otimes H) \circ (\gamma \otimes \varphi_A \otimes H) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A)$$
$$\circ (\lambda_H \otimes \lambda_A) \circ (\mu_H \otimes A) \circ (H \otimes c_{A,H}) \circ (r_A \otimes H)$$

being $\gamma = \sigma^{-1} \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H$.

In our construction we don't use the σ-antipode condition and we prove that if we assume that the antipode of H is a σ-antipode then the morphism $\gamma = \varepsilon_H \otimes \eta_A$ and therefore the Hopf algebras $A_\sigma \bowtie H$ and $A \overset{\times}{\sharp}_\sigma H$ are equal. Then, as a consequence, to obtain a Hopf algebra structure in $A \overset{\times}{\sharp}_\sigma H$ the condition of σ antipode is very strong.

Also, when r_A is trivial, the Hopf algebra $A_\sigma \bowtie H$ is the Hopf algebra $A_\sigma \rtimes H$ studied in [1]. This Hopf algebra is a generalization of the ones defined by S. Majid in [6], [7] and [8]. In [8] the Hopf algebra $A_\sigma \rtimes H$ is defined in a symmetric context (modules over a commutative ring K) and in [7] the Hopf algebra $A_\sigma \rtimes H$ is defined in a braided category but for trivial cocycle. Moreover, in [1] the authors have proved that recent constructions of crossed products of Hopf algebras in a category of modules over a field, for example, the Hopf algebra $A \star H$ considered by Wang and Li in [10], are particular cases of the Majid Hopf algebra $A \rtimes H$ and, of course, of the braided Hopf algebra $A_\sigma \rtimes H$. Therefore, as a consequence, when (A, H, τ) is an invertible skew pair of Hopf algebras and the antipode of H is bijective, the bialgebra $A \bowtie_\tau H$, defined by Doi and Takeuchi in [2], is an example of $A_\sigma \rtimes H$, because in this conditions the bialgebra $A \bowtie_\tau H$ is an special instance of $A \star H$ (see [10]).

2. Preliminaries

In this paper as the base category we consider a braided monoidal category $\mathscr{C}$. These categories were introduced by Joyal and Street in [3], motivated by the theory of braids and links in topology (see [4] for more details). Using Maclane's coherence theorem we may assume without loss of generality that $\mathscr{C}$ is a strict monoidal category. A braiding of a monoidal category $(\mathscr{C}, \otimes, K)$ consists of a natural isomorphism of functors $c_{A,B} : A \otimes B \to B \otimes A$ such that

b1) $c_{A \otimes B, D} = (c_{A,D} \otimes B) \circ (A \otimes c_{B,D})$
b2) $c_{D, A \otimes B} = (A \otimes c_{D,B}) \circ (c_{D,A} \otimes B)$

As a consequence, it is easy to obtain that $c_{A,K} = c_{K,A} = id_A$. Moreover, by the naturality of $c_{A \otimes B, D}$ ⊗—we obtain the identity (called the *Yang-Baxter* identity):

$$(c_{B,D} \otimes A) \circ (B \otimes c_{A,D}) \circ (c_{A,B} \otimes D) = (D \otimes c_{A,B}) \circ (c_{A,D} \otimes B) \circ (A \otimes c_{B,D})$$

In what follows, $(\mathscr{C}, c)$ denotes the braided monoidal category with braiding c and $(\mathscr{C}, c^{-1})$ the braided monoidal category with the inverse braiding.

An algebra in $(\mathscr{C}, c)$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathscr{C}$ and $\eta_A : K \to A$, $\mu_A : A \otimes A \to A$ are morphisms in $\mathscr{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are algebras in $\mathscr{C}$, the algebra product is

$$AB = (A \otimes B, \eta_{A\otimes B} = \eta_A \otimes \eta_B, \mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)).$$

A coalgebra in $(\mathscr{C}, c)$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathscr{C}$ and $\varepsilon_D : D \to K$, $\delta_D : D \to D \otimes D$ are morphisms in $\mathscr{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are coalgebras in $\mathscr{C}$, the coalgebra product is

$$DE = (D \otimes E, \varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E, \delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)).$$

Let A be an algebra. (M, ψ_M) is a right A-module if M is an object in $(\mathscr{C}, c)$ and $\psi_M : M \otimes A \to M$ is a morphism in $\mathscr{C}$ satisfying $\psi_M \circ (M \otimes \eta_A) = id_M$, $\psi_M \circ (\psi_M \otimes A) = \psi_M \circ (M \otimes \mu_A)$. Given two right A-modules (M, ψ_M) and (N, ψ_N), $f : M \to N$ is a morphism of right A-modules if $\psi_N \circ (f \otimes A) = f \circ \psi_M$. We denote the category of right A-modules by $\mathscr{C}_A$. In an analogous way we define the left A-modules and we denote this category by ${}_A\mathscr{C}$.

Let D be a coalgebra. (M, ρ_M) is a right D-comodule if M is an object in $(\mathscr{C}, c)$ and $\rho_M : M \to M \otimes D$ is a morphism in $\mathscr{C}$ satisfying $(M \otimes \varepsilon_D) \circ \rho_M = id_M$, $(\rho_M \otimes D) \circ \rho_M = (M \otimes \delta_D) \circ \rho_M$. Given two right D-comodules (M, ρ_M) and (N, ρ_N), $f : M \to N$ is a morphism of right D-comodules if $\rho_N \circ f = (f \otimes D) \circ \rho_M$. We denote the category of right D-comodules by $\mathscr{C}^D$. Analogously, ${}^D\mathscr{C}$ denotes the category of left D-comodules.

Definition 2.1. A bialgebra in $(\mathscr{C}, c)$ means an algebra and a coalgebra $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H)$ such that ε_H and δ_H are algebra morphisms (equivalently η_H and μ_H are coalgebra morphisms). If there exists a morphism $\lambda_H : H \to H$ verifying the condition

$$\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \varepsilon_H \otimes \eta_H = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H$$

we say that $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ is a Hopf algebra in $(\mathscr{C}, c)$.

If $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ is a Hopf algebra in $(\mathscr{C}, c)$ we have that

(2.1.1) $$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}$$

and

$$\delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H \tag{2.1.2}$$

Let $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ be a Hopf algebra in $(\mathscr{C}, c)$ with bijective antipode. Then $H^{op} = (H, \eta_H, \mu_H \circ c_{H,H}^{-1}, \varepsilon_H, \delta_H, \lambda_H^{-1})$ and $H^{coop} = (H, \eta_H, \mu_H, \varepsilon_H, c_{H,H}^{-1} \circ \delta_H, \lambda_H^{-1})$ are Hopf algebras in $(\mathscr{C}, c^{-1})$ ([5]).

DEFINITION 2.2. Let H be a bialgebra in $(\mathscr{C}, c)$ and let M be an algebra. We say that $\varphi_M : H \otimes M \to M$ is a left weak action if

a) $\varphi_M \circ (\eta_H \otimes M) = id_M$
b) $\varphi_M \circ (H \otimes \eta_M) = \varepsilon_H \otimes \eta_M$
c) $\varphi_M \circ (H \otimes \mu_M) = \mu_M \circ (\varphi_M \otimes \varphi_M) \circ (H \otimes c_{H,M} \otimes M) \circ (\delta_H \otimes M \otimes M)$

If moreover $\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M)$, we say that (M, φ_M) is a left H-module algebra.

Analogously, if M is a coalgebra $r_M : M \to H \otimes M$ is a left weak coaction if:

a) $(\varepsilon_H \otimes M) \circ r_M = id_M$
b) $(H \otimes \varepsilon_M) \circ r_M = \eta_H \otimes \varepsilon_M$
c) $(H \otimes \delta_M) \circ r_M = (\mu_H \otimes M \otimes M) \circ (H \otimes c_{M,H} \otimes M) \circ (r_M \otimes r_M) \circ \delta_M$

If moreover $(H \otimes r_M) \circ r_M = (\delta_H \otimes M) \circ r_M$, we say that (M, r_M) is a left H-comodule coalgebra.

3. The Hopf Algebra $A_\sigma \bowtie H$

DEFINITION 3.1. Let H be a bialgebra in $(\mathscr{C}, c)$. If A is an algebra, φ_A a left weak action and $\sigma : H \otimes H \to A$ a morphism, we will say that σ is a normal twisted cocycle if:

a) $\mu_A \circ (\varphi_A \otimes \sigma) \circ (H \otimes \sigma \otimes H \otimes \mu_H) \circ \delta_{H \otimes H \otimes H}$
$= \mu_A \circ (\sigma \otimes \sigma) \circ (H \otimes H \otimes \mu_H \otimes H) \circ (\delta_{H \otimes H} \otimes H)$
b) $\mu_A \circ (\varphi_A \otimes A) \circ (H \otimes \varphi_A \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes H \otimes \sigma \otimes A) \circ$
$(\delta_{H \otimes H} \otimes A)$
$= \mu_A \circ (A \otimes \varphi_A) \circ (\sigma \otimes \mu_H \otimes A) \circ (\delta_{H \otimes H} \otimes A)$
c) $\sigma \circ (\eta_H \otimes H) = \sigma \circ (H \otimes \eta_H) = \varepsilon_H \otimes \eta_A$

In this definition a) is the condition of cocycle, b) is the twisted condition and c) is the normal condition.

REMARK 3.2. Let H be a bialgebra in $(\mathscr{C}, c)$. If A is an algebra, φ_A a left weak action and $\sigma : H \otimes H \to A$ a morphism, we define

$$\eta_{A \sharp_\sigma H} = \eta_A \otimes \eta_H$$

$$\mu_{A \sharp_\sigma H} = (\mu_A \otimes H) \circ (\mu_A \otimes \sigma \otimes \mu_H) \circ (A \otimes \varphi_A \otimes \delta_{H \otimes H}) \circ (A \otimes H \otimes c_{H,A} \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H)$$

It is well know that $(A \otimes H, \eta_{A \sharp_\sigma H}, \mu_{A \sharp_\sigma H})$ is an algebra in $(\mathscr{C}, c)$, called the crossed product, iff σ is a normal twisted cocycle. When $\sigma = \varepsilon_H \otimes \varepsilon_H \otimes \eta_A$ the crossed product is the smash product. In this case we denote $A \sharp_\sigma H$ by $A \sharp H$ and then

$$\mu_{A \sharp H} = (\mu_A \otimes H) \circ (A \otimes \varphi_A \otimes \mu_H) \circ (A \otimes H \otimes c_{H,A} \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H)$$

On the other hand, if (A, r_A) is a left H-comodule coalgebra, we have that $(A \otimes H, \varepsilon_{A \propto H}, \delta_{A \propto H})$, where $\varepsilon_{A \propto H} = \varepsilon_A \otimes \varepsilon_H$ and

$$\delta_{A \propto H} = (A \otimes \mu_H \otimes A \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes H) \circ (A \otimes r_A \otimes H \otimes H) \circ (\delta_A \otimes \delta_H),$$

is a coalgebra, called the smash coproduct of A and H in $(\mathscr{C}, c)$.

PROPOSITION 3.3. *Let H be a bialgebra in $(\mathscr{C}, c)$ and let A be an algebra coalgebra such that $\eta_A \otimes \eta_A = \delta_A \circ \eta_A$ and $\varepsilon_A \circ \mu_A = \varepsilon_A \otimes \varepsilon_A$. If φ_A is a left weak action verifying that $\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A$, (A, r_A) is a left H-comodule coalgebra satisfying the condition $r_A \circ \eta_A = \eta_H \otimes \eta_A$ and $\sigma : H \otimes H \to A$ is a normal twisted cocycle such that $\varepsilon_A \circ \sigma = \varepsilon_H \otimes \varepsilon_H$ then*

$$A \,{}_\sigma\!\bowtie H = (A \otimes H, \eta_{A \sharp_\sigma H}, \mu_{A \sharp_\sigma H}, \varepsilon_{A \propto H}, \delta_{A \propto H})$$

is a bialgebra iff we have the next conditions:

a) $(\mu_H \otimes \sigma) \circ \delta_{H \otimes H} = (\mu_H \otimes A) \circ (H \otimes c_{A,H}) \circ (r_A \otimes H) \circ (\sigma \otimes \mu_H) \circ \delta_{H \otimes H}$
b) $\delta_A \circ \sigma = (\sigma \otimes \sigma) \circ \delta_{H \otimes H}$
c) $\delta_A \circ \varphi_A = (\mu_A \otimes A) \circ (\varphi_A \otimes \sigma \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes c_{H,H} \otimes A) \circ (\delta_H \otimes A \otimes H \otimes r_A) \circ \delta_{H \otimes A}$

d) $r_A \circ \mu_A = \mu_{H\otimes A} \circ (r_A \otimes r_A)$
e) $\delta_A \circ \mu_A = (\mu_A \otimes A) \circ (A \otimes \mu_A \otimes A) \circ (A \otimes \varphi_A \otimes \sigma \otimes \mu_A)$
$\circ (A \otimes H \otimes c_{H,A} \otimes c_{A,H} \otimes A) \circ (A \otimes \delta_H \otimes c_{A,A} \otimes r_A)$
$\circ (A \otimes r_A \otimes A \otimes A) \circ (\delta_A \otimes \delta_A)$
f) $(A \otimes \mu_H \otimes A) \circ (A \otimes H \otimes c_{A,H}) \circ (A \otimes r_A \otimes H) \circ (\delta_A \otimes H)$
$= (\mu_A \otimes H \otimes A) \circ ((A \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes A)) \circ (A \otimes H \otimes c_{A,H})$
$\circ (A \otimes r_A \otimes H) \circ (\delta_A \otimes H)$
g) $(\mu_H \otimes \varphi_A) \circ (H \otimes c_{H,H} \otimes A) \circ (\delta_H \otimes r_A)$
$= (\mu_H \otimes A) \circ (H \otimes c_{A,H}) \circ ((r_A \circ \varphi_A) \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A).$

PROOF. The proof is similar with the one developed in Theorem 2.5 of [11]. □

REMARK 3.4. Note that the condition f) of 3.3 implies that

$$\delta_{A\propto H} = (\mu_A \otimes H \otimes A \otimes H) \circ (A \otimes ((\sigma \otimes \mu_H) \circ \delta_{H\otimes H}) \otimes A \otimes H)$$
$$\circ (A \otimes H \otimes c_{A,H} \otimes H) \circ (A \otimes r_A \otimes H \otimes H) \circ (\delta_A \otimes \delta_H).$$

DEFINITION 3.5. Let D be a coalgebra and let A be an algebra. By $Reg(D,A)$ we denote the set of invertible morphisms $f : D \to A$ in $\mathscr{C}$ respect to the convolution operation $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$. $Reg(D,A)$ is a monoid where the unit element is $\varepsilon_D \circ \eta_A = \varepsilon_D \otimes \eta_A$.

Next we state the main result of this paper.

PROPOSITION 3.6. *Let H be a Hopf algebra in $(\mathscr{C}, c)$ and let A be an algebra coalgebra in the conditions of proposition 3.3. If the antipode λ_H of H is invertible and id_A has inverse λ_A in $Reg(A,A)$ we have that $A\,{}_\sigma\!\bowtie H$ is a Hopf algebra in $(\mathscr{C}, c)$ with antipode*

$$\lambda_{A_\sigma \bowtie H} = (\mu_A \otimes H) \circ (\gamma \otimes \varphi_A \otimes H) \circ (\delta_H \otimes c_{H,A}) \circ (\delta_H \otimes A)$$
$$\circ (\lambda_H \otimes \lambda_A) \circ (\mu_H \otimes A) \circ (H \otimes c_{A,H}) \circ (r_A \otimes H)$$

being $\gamma = \sigma^{-1} \circ (H \otimes \lambda_H^{-1}) \circ c_{H,H}^{-1} \circ \delta_H$ where $\sigma^{-1} = \lambda_A \circ \sigma$ denotes the inverse of σ in $Reg(H \otimes H, A)$.

PROOF. By 3.3 $A_{\sigma}\bowtie H$ is a bialgebra in $(\mathscr{C}, c)$. Now we show that $\lambda_{A_{\sigma}\bowtie H}$ is the antipode.

$$\mu_{A\sharp_{\sigma}H} \circ (\lambda_{A_{\sigma}\bowtie H} \otimes A \otimes H) \circ \delta_{A\propto H}$$

$$= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma \otimes \mu_H) \circ (A \otimes A \otimes H \otimes c_{H,H} \otimes H)$$

$$\circ (\gamma \otimes \varphi_A \otimes \delta_H \otimes \delta_H) \circ (\delta_H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes \mu_A \otimes H)$$

$$\circ (\lambda_H \otimes \lambda_A \otimes A \otimes H) \circ (\mu_H \otimes A \otimes A \otimes H) \circ (H \otimes c_{A,H} \otimes A \otimes H)$$

$$\circ (r_A \otimes \mu_H \otimes A \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes H) \circ (A \otimes r_A \otimes H \otimes H)$$

$$\circ (\delta_A \otimes \delta_H)$$

$$= (\mu_A \otimes H) \circ (\mu_A \otimes \sigma \otimes \mu_H) \circ (A \otimes A \otimes H \otimes c_{H,H} \otimes H)$$

$$\circ (\gamma \otimes \varphi_A \otimes \delta_H \otimes \delta_H) \circ (\delta_H \otimes c_{H,A} \otimes H)$$

$$\circ (\delta_H \otimes (\mu_A \circ (\lambda_A \otimes A) \circ \delta_A) \otimes H) \circ (\lambda_H \otimes A \otimes H) \circ (\mu_H \otimes A \otimes H)$$

$$\circ (A \otimes c_{A,H} \otimes H) \circ (r_A \otimes \delta_H)$$

$$= (\mu_A \otimes H) \circ (\gamma \otimes \sigma \otimes \mu_H) \circ (H \otimes \delta_{H\otimes H}) \circ ((\delta_H \circ \lambda_H) \otimes H) \circ (\varepsilon_A \otimes \delta_H)$$

$$= (\mu_A \otimes H) \circ (A \otimes \sigma \otimes \mu_H) \circ (A \otimes H \otimes c_{H,H} \otimes H)$$

$$\circ (\sigma^{-1} \otimes \lambda_H \otimes \lambda_H \otimes H \otimes H) \circ (\lambda_H \otimes H \otimes c_{H,H} \otimes H \otimes H)$$

$$\circ (\delta_H \otimes \delta_H \otimes \delta_H) \circ (c_{H,H} \otimes H) \circ (\delta_H \otimes H) \circ (\varepsilon_A \otimes \delta_H)$$

$$= ((\sigma^{-1} * \sigma) \otimes H) \circ (\lambda_H \otimes H \otimes \mu_H) \circ (\delta_H \otimes \lambda_H \otimes H)$$

$$\circ ((c_{H,H} \circ \delta_H) \otimes H) \circ (\varepsilon_A \otimes \delta_H)$$

$$= \varepsilon_A \otimes \varepsilon_H \otimes \eta_A \otimes \eta_H.$$

The first and the second equalities follow from the condition of weak action for φ_A, the structure of H-comodule coalgebra on A and the naturality of the braiding. In the third one we use that λ_A is the convolution inverse of id_A and the naturality of the braiding. The fourth one follows from the naturality of the braiding and by (2.1.2). In the fifth one we apply the naturality of the braiding and finally, in the sixth one, we use the naturality of the braiding and the Hopf algebra structure of H.

On the other hand

$$
\begin{aligned}
&\mu_{A\sharp_\sigma H}\circ(A\otimes H\otimes\lambda_{A_\sigma\bowtie H})\circ\delta_{A\propto H}\\
&\quad=(\mu_A\otimes H)\circ(\mu_A\otimes\sigma\otimes\mu_H)\circ(A\otimes\varphi_A\otimes\delta_{H\otimes H})\circ(A\otimes H\otimes c_{H,A}\otimes H)\\
&\qquad\circ(A\otimes\delta_H\otimes\mu_A\otimes H)\circ(A\otimes H\otimes\gamma\otimes\varphi_A\otimes H)\circ(A\otimes H\otimes\delta_H\otimes c_{H,A})\\
&\qquad\circ(A\otimes H\otimes(\delta_H\circ\lambda_H)\otimes\lambda_A)\circ(A\otimes(\delta_H\circ\mu_H)\otimes A)\\
&\qquad\circ(A\otimes H\otimes c_{A,H})\circ(A\otimes r_A\otimes H)\circ(\delta_A\otimes H)\\
&\quad=(\mu_A\otimes\eta_H)\circ(\mu_A\otimes A)\circ(A\otimes\varphi_A\otimes\sigma)\circ(A\otimes H\otimes\mu_A\otimes H\otimes H)\\
&\qquad\circ(A\otimes H\otimes\gamma\otimes\varphi_A\otimes H\otimes H)\circ(A\otimes H\otimes\delta_H\otimes c_{H,A}\otimes H)\\
&\qquad\circ(A\otimes H\otimes c_{H,H}\otimes c_{H,A})\circ(A\otimes\delta_H\otimes(\delta_H\circ\lambda_H)\otimes\lambda_A)\\
&\qquad\circ(A\otimes(\delta_H\circ\mu_H)\otimes A)\circ(A\otimes H\otimes c_{A,H})\circ(A\otimes r_A\otimes H)\circ(\delta_A\otimes H)\\
&\quad=(\mu_A\otimes\eta_H)\circ(\mu_A\otimes A)\circ(A\otimes(\mu_A\circ(\varphi_A\otimes\varphi_A))\otimes A)\\
&\qquad\circ(A\otimes H\otimes c_{H,A}\otimes A\otimes A)\circ(A\otimes\delta_H\otimes\gamma\otimes\varphi_A\otimes A)\\
&\qquad\circ(A\otimes H\otimes H\otimes H\otimes c_{A,A})\circ(A\otimes H\otimes\delta_H\otimes\sigma\otimes A)\\
&\qquad\circ(A\otimes H\otimes c_{H,H}\otimes H\otimes A)\circ(A\otimes\delta_H\otimes(\delta_H\circ\lambda_H)\otimes\lambda_A)\\
&\qquad\circ(A\otimes(\delta_H\circ\mu_H)\otimes A)\circ(A\otimes H\otimes c_{A,H})\circ(A\otimes r_A\otimes H)\circ(\delta_A\otimes H)\\
&\quad=(\mu_A\otimes\eta_H)\circ(\mu_A\otimes\mu_A)\circ(A\otimes\varphi_A\otimes\sigma\otimes\varphi_A)\\
&\qquad\circ(A\otimes H\otimes\gamma\otimes H\otimes H\otimes\mu_H\otimes A)\circ(A\otimes H\otimes H\otimes\delta_{H\otimes H}\otimes A)\\
&\qquad\circ(A\otimes\delta_{H\otimes H}\otimes A)\circ(A\otimes H\otimes\lambda_H\otimes\lambda_A)\circ(A\otimes(\delta_H\circ\mu_H)\otimes A)\\
&\qquad\circ(A\otimes H\otimes c_{A,H})\circ(A\otimes r_A\otimes H)\circ(\delta_A\otimes H)\\
&\quad=(\mu_A\otimes\eta_H)\circ(\mu_A\otimes A)\circ(A\otimes[\mu_A\circ(\varphi_A\otimes\sigma)\circ(H\otimes A\otimes((H\otimes\lambda_H)\circ\delta_H))\\
&\qquad\circ(H\otimes c_{H,A})\circ(\delta_H\otimes(\gamma\circ\lambda_H))\circ\delta_H]\otimes\lambda_A)\circ(A\otimes\mu_H\otimes A)\\
&\qquad\circ(A\otimes H\otimes c_{A,H})\circ(A\otimes r_A\otimes H)\circ(\delta_A\otimes H)\\
&\quad=(\mu_A\otimes\eta_H)\circ(\mu_A\otimes A)\circ(A\otimes[\mu_A\circ(\varphi_A\otimes A)\circ(H\otimes c_{A,A})\circ(H\otimes\sigma\otimes\sigma^{-1})
\end{aligned}
$$

$$\circ (H \otimes H \otimes \lambda_H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H \otimes \delta_H) \circ (H \otimes \delta_H) \circ \delta_H] \otimes \lambda_A)$$

$$\circ (A \otimes \mu_H \otimes A) \circ (A \otimes H \otimes c_{A,H}) \circ (A \otimes r_A \otimes H) \circ (\delta_A \otimes H)$$

$$= (\mu_A \otimes \eta_H) \circ (\mu_A \otimes A) \circ (A \otimes \varepsilon_H \otimes \eta_A \otimes \lambda_A) \circ (A \otimes \mu_H \otimes A)$$

$$\circ (A \otimes H \otimes c_{A,H}) \circ (A \otimes r_A \otimes H) \circ (\delta_A \otimes H)$$

$$= \varepsilon_A \otimes \varepsilon_H \otimes \eta_A \otimes \eta_H.$$

The first equality follows from the H-comodule structure on A, the bialgebra condition on H and the naturality of the braiding, the second one by equality 2.1.2, the Hopf algebra condition on H and the naturality of the braiding, the third one by the weak action condition for φ_A and the naturality of the braiding; the fourth one by the cocycle condition of σ and the naturality of the braiding. The fifth and the sixth one are a consequence of the equality 2.1.2, the Hopf algebra condition on H and the naturality of the braiding, the seventh ones follows from the equality

$$\mu_A \circ (\varphi_A \otimes A) \circ (H \otimes c_{A,A}) \circ (H \otimes \sigma \otimes \sigma^{-1}) \circ (H \otimes H \otimes \lambda_H \otimes \lambda_H \otimes H)$$

$$\circ (H \otimes \delta_H \otimes \delta_H) \circ (H \otimes \delta_H) \circ \delta_H = \varepsilon_H \otimes \eta_A.$$

Finally, the eighth one follows by the H-comodule condition on A and the invertibility of id_A in $Reg(A, A)$. □

3.7. We point out that the dual results of 3.3 and 3.6 can be derived just by working within the opposite category. For example, in this situation, we work with a left H-module algebra (A, φ_A), a left weak coaction $r_A : A \to H \otimes A$ and a morphism $\tau : A \to H \otimes H$ verifying:

a) $\mu_{H\otimes H\otimes H} \circ (H \otimes \tau \otimes H \otimes \delta_H) \circ (r_A \otimes \tau) \circ \delta_A$
$= (\mu_{H\otimes H} \otimes H) \circ (H \otimes H \otimes \delta_H \otimes H) \circ (\tau \otimes \tau) \circ \delta_A$

b) $(\mu_{H\otimes H} \otimes A) \circ (H \otimes H \otimes \tau \otimes A) \circ (H \otimes H \otimes c_{A,A}) \circ (H \otimes r_A \otimes A) \circ$
$(r_A \otimes A) \circ \delta_A$
$= (\mu_{H\otimes H} \otimes A) \circ (\tau \otimes \delta_H \otimes A) \circ (A \otimes r_A) \circ \delta_A$

c) $(\varepsilon_H \otimes H) \circ \tau = (H \otimes \varepsilon_H) \circ \tau = \eta_H \otimes \varepsilon_A$

Thus, we obtain the next proposition:

PROPOSITION 3.8. *Let H be a bialgebra in $(\mathscr{C}, c)$ and let A be an algebra coalgebra such that $\eta_A \otimes \eta_A = \delta_A \circ \eta_A$ and $\varepsilon_A \circ \mu_A = \varepsilon_A \otimes \varepsilon_A$. If (A, φ_A) is a left H-module algebra verifying that $\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A$, r_A is a left weak coaction satisfying the condition $r_A \circ \eta_A = \eta_H \otimes \eta_A$ and $\tau : A \to H \otimes H$ is a morphism satisfying the conditions of* 3.7 *and such that $\tau \circ \eta_A = \eta_H \otimes \eta_H$, then*

$$A\, {}^{\tau}\!\!\propto H = (A \otimes H, \eta_{A\sharp H}, \mu_{A\sharp H}, \varepsilon_{A\propto_\tau H}, \delta_{A\propto_\tau H}), \text{ being } \varepsilon_{A\propto_\tau H} = \varepsilon_A \otimes \varepsilon_H \text{ and}$$

$$\delta_{A\propto_\tau H} = (A \otimes \mu_H \otimes A \otimes H) \circ (A \otimes H \otimes c_{A,H} \otimes H) \circ (A \otimes r_A \otimes \mu_{H\otimes H})$$
$$\circ\, (\delta_A \otimes \tau \otimes \delta_H) \circ (\delta_A \otimes H)$$

is a bialgebra iff we have the next conditions:

a) $\mu_{H\otimes H} \circ (\delta_H \otimes \tau) = \mu_{H\otimes H} \circ (\tau \otimes \delta_H) \circ (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)$
b) $\tau \circ \mu_A = \mu_{H\otimes H} \circ (\tau \otimes \tau)$
c) $r_A \circ \mu_A = \mu_{H\otimes A} \circ (\mu_H \otimes A \otimes H \otimes \varphi_A) \circ (H \otimes c_{A,H} \otimes c_{H,H} \otimes A)$
$\circ\, (r_A \otimes \tau \otimes r_A) \circ (\delta_A \otimes A)$
d) $\delta_A \circ \varphi_A = (\varphi_A \otimes \varphi_A) \circ \delta_{H\otimes A}$
e) $\delta_A \circ \mu_A = (\mu_A \otimes \mu_A) \circ (A \otimes \varphi_A \otimes A \otimes A) \circ (A \otimes \mu_H \otimes c_{A,A} \otimes \varphi_A)$
$\circ\, (A \otimes H \otimes c_{A,H} \otimes c_{H,A} \otimes A) \circ (A \otimes r_A \otimes \tau \otimes \delta_A) \circ (\delta_A \otimes A \otimes A)$
$\circ\, (\delta_A \otimes A)$
f) $(\mu_A \otimes H) \circ (A \otimes \varphi_A \otimes H) \circ (A \otimes H \otimes c_{H,A}) \circ (A \otimes \delta_H \otimes A)$
$= (\mu_A \otimes H) \circ (A \otimes \varphi_A \otimes H) \circ (A \otimes H \otimes c_{H,A})$
$\circ\, (A \otimes (\mu_{H\otimes H} \circ (\tau \otimes \delta_H)) \otimes A) \circ (\delta_A \otimes H \otimes A)$
g) $(\mu_H \otimes \varphi_A) \circ (H \otimes c_{H,H} \otimes A) \circ (\delta_H \otimes r_A) = (\mu_H \otimes A) \circ (H \otimes c_{A,H})$
$\circ\, ((r_A \circ \varphi_A) \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)$.

Moreover, if H is a Hopf algebra with invertible antipode λ_H and id_A has inverse λ_A in $Reg(A, A)$, we have that $A\, {}^{\tau}\!\!\propto H$ is a Hopf algebra in $(\mathscr{C}, c)$ with antipode

$$\lambda_{A\,{}^{\tau}\!\propto H} = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \circ (\lambda_H \otimes \lambda_A) \circ (\mu_H \otimes A)$$
$$\circ\, (\mu_H \otimes c_{A,H}) \circ (l \otimes r_A \otimes H) \circ (\delta_A \otimes H)$$

where $l = \mu_H \circ c_{H,H}^{-1} \circ (H \otimes \lambda_H^{-1}) \circ \tau^{-1}$.

3.9. In the last part of this paper we study the relations between the Hopf algebra $A\, {}_{\sigma}\!\bowtie H$ and the Hopf algebra $A \overset{\times}{\sharp}_{\sigma} H$ obtained by Wang, Jiao and Zhao in [11] using the condition of σ-antipode.

Definition 3.10. Let H be a bialgebra, A an algebra and $\sigma : H \otimes H \to A$, $g : H \to H$ be a morphisms in $\mathscr{C}$. The morphism g is called a σ antipode of H if:

$$(\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (g \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_A \otimes \eta_H$$
$$= (\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (H \otimes g) \circ \delta_H$$

Observe that if g is a σ-antipode then g is necessarily an ordinary antipode.

3.11. Obviously if H is a Hopf algebra with antipode λ_H and $\sigma = \varepsilon_H \otimes \varepsilon_H \otimes \eta_A$ we can consider λ_H as a σ-antipode. In the next proposition we will find conditions for the antipode λ_H to be a σ-antipode.

Proposition 3.12. *Let H be a Hopf algebra with antipode λ_H, A an algebra and $\sigma : H \otimes H \to A$ a morphism in $\mathscr{C}$. Then, λ_H is a σ-antipode of H if and only if*

$$\sigma \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_A = \sigma \circ (H \otimes \lambda_H) \circ \delta_H$$

Proof. Note that:

i) $(\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (\lambda_H \otimes H) \circ \delta_H$
$= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (\lambda_H \otimes (\sigma \circ (\lambda_H \otimes H) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H$

ii) $(\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (H \otimes \lambda_H) \circ \delta_H = (\sigma \circ (H \otimes \lambda_H) \circ \delta_H) \otimes \eta_H$

Then if $\sigma \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_A = \sigma \circ (H \otimes \lambda_H) \circ \delta_H$ we obtain that λ_H is a σ-antipode. Conversely, if λ_H is a σ-antipode of H composing in i) with δ_H and $(\varepsilon_H \otimes A) \circ (\mu_H \otimes A) \circ (H \otimes c^{-1}_{H,A})$ we obtain $\sigma \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_A$. Trivially ii) implies that $\sigma \circ (H \otimes \lambda_H) \circ \delta_H = \varepsilon_H \otimes \eta_A$. □

3.13. Suppose that H is a Hopf algebra with invertible antipode λ_H, A is an algebra and $\sigma : H \otimes H \to A$ is a morphism in $\mathscr{C}$. If we denote by σ' the morphism $\sigma' = \sigma \circ c^{-1}_{H,H}$ we have that:

i) $(\sigma \otimes \mu_H) \circ (c^{-1}_{H,H} \otimes c^{-1}_{H,H}) \circ (H \otimes c^{-1}_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\lambda^{-1}_H \otimes H) \circ \delta_H$
$= (\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ c^{-1}_{H,H} \circ (\lambda^{-1}_H \otimes H) \circ \delta_H$
$= (\sigma \otimes \mu_H) \circ \delta_{H\otimes H} \circ (\lambda_H \otimes H) \circ \delta_H \circ \lambda^{-1}_H$

and

i) $(\sigma\otimes\mu_H)\circ(c_{H,H}^{-1}\otimes c_{H,H}^{-1})\circ(H\otimes c_{H,H}^{-1}\otimes H)\circ(\delta_H\otimes\delta_H)\circ(H\otimes\lambda_H^{-1})\circ\delta_H$
$= (\sigma\otimes\mu_H)\circ\delta_{H\otimes H}\circ c_{H,H}^{-1}\circ(H\otimes\lambda_H^{-1})\circ\delta_H$
$= (\sigma\otimes\mu_H)\circ\delta_{H\otimes H}\circ(H\otimes\lambda_H)\circ\delta_H\circ\lambda_H^{-1}$

Then, using these equalities it is easy to prove the next result:

PROPOSITION 3.14. *Let H be a Hopf algebra with invertible antipode λ_H, A an algebra and $\sigma : H\otimes H\to A$ be a morphism in $\mathscr{C}$. The morphism λ_H is a σ-antipode of H if and only if λ_H^{-1} is a σ'-antipode of H^{op}.*

As a consequence we obtain the next corollary:

COROLLARY 3.15. *Let H be a Hopf algebra with invertible antipode λ_H, A an algebra and $\sigma : H\otimes H\to A$ be a morphism in $\mathscr{C}$. If λ_H is a σ-antipode of H the morphism γ defined in* 3.6 *is $\gamma=\varepsilon_H\otimes\eta_A$.*

PROOF. From proposition 3.14 we know that λ_H^{-1} is a σ'-antipode of H^{op} and then by 3.12 we obtain that $\gamma=\varepsilon_H\otimes\eta_A$. □

REMARK 3.16. Therefore in the conditions of last corollary the Hopf algebra $A\,{}_{\sigma}\!\bowtie H$ and the Hopf algebra $A \mathbin{\sharp_{\sigma}^{\times}} H$ obtained by Wang, Jiao and Zhao in [11] are equal because $\lambda_{A \sharp_{\sigma}^{\times} H}=\lambda_{A_{\sigma}\bowtie H}$.

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References

[1] J. N. Alonso Alvarez, J. M. Fernández Vilaboa, R. González Rodríguez, On the (Co)Commutativity Class of a Hopf Algebra and Crossed Products in a Braided category, (Preprint).
[2] Y. Doi, M. Takeuchi, Multiplication Alteration by Two Cocycles. —The Quantum Version—, Comm. in Algebra, **22** (1994), 5715–5732.
[3] A. Joyal, S. Street, Braided monoidal categories, Macquarie Univ. Reports 860081, 1986.
[4] A. Joyal, S. Street, Braided tensor categories, Advances in Math. **102** (1993), 20–78.
[5] S. Majid, Transmutation Theory and Rank for Quantum Braided Groups, Math. Proc. Camb. Phil. Soc. **113** (1993), 45–70.
[6] S. Majid, Cross Products by Braided Groups and Bosonization, Journal of Algebra **163** (1994), 165–190.

[7] S. Majid, Algebras and Hopf Algebras in Braided Categories, Advances in Hopf Algebras, Lecture Notes in Pure and Applied Math. (Blue Series), vol. 158, Marcel Dekker Inc., 1994, pp. 55–105.
[8] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, 1995.
[9] D. E. Radford, The structure of Hopf Algebras with a Projection, J. of Algebra **92** (1985), 322–347.
[10] Shuan-hong Wang, Jin-qi Li, On Twisted Smash Products for Bimodule Algebras and the Drinfeld Double, Comm. in Algebra **26** (1998), 2435–2444.
[11] Shuan-hong Wang, Zhengming Jiao, Wenzheng Zhao, Hopf Algebra Structures on Crossed Products, Comm. in Algebra **26** (1998), 1293–1303.

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