

THE FIRST VARIATION FORMULA FOR WEYL STRUCTURES

By

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Abstract. The purpose of this paper is to determine explicitly the Euler Lagrange equations of our conformal gauge invariant functional on the space of all Weyl structures.

1. Introduction

The geometry of Weyl structures has its classical roots in the work of H. Weyl and is now a very active research area, having close connections to conformal geometry (especially Einstein-Weyl geometry), contact geometry, gauge theory and gravitational theory. In recent years, Alexandrov and Ivanov [1] generalized the vanishing theorem of Bochner type on compact manifolds admitting a Weyl structure whose Ricci tensor satisfies certain positively condition. Katagiri [16] showed that for a conformal connection, the symmetric part of the Ricci curvature determines the full Ricci curvature. Calderbank [7] studied the Jones-Tod correspondence between self-dual four-manifolds with symmetry and Einstein-Weyl three-manifolds with an abelian monopole using Weyl derivatives, Weyl-Lie derivatives and conformal submersions. And Kamada [15] showed that a compact almost Hermitian-Einstein-Weyl four-manifold with non-negative conformal scalar curvature must be Hermitian-Einstein-Weyl. Torres del Castillo and Pérez-Pérez [23] studied that the coupled gravitational and neutrino field perturbations of the exact solution of the Einstein-Weyl equations are determined by a set of four first-order ordinary differential equations determines the conservation factors between a gravitational and neutrino waves. Since Weyl geometry and related fields are so rich, we consider some variational problem on the space \mathfrak{W} of all Weyl structures as follows.

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In [10] we introduced a conformal gauge invariant functional for Weyl structures and studied some properties about it. The purpose of this paper is to determine explicitly the Euler-Lagrange equations of our conformal gauge invariant functional on \mathfrak{W} (cf. Theorem 2.2). Especially, in dimension four, our Euler-Lagrange equations have very simple forms which are the mixtures of a Yang-Mills equation for a Weyl connection and a gravitational field equation characterized by two symmetric 2-tensor fields \check{R}^D and g (Corollary 2.3). This study leads naturally to the notion of Yang-Mills theory in affine geometry, which is studied in [8] related to the Einstein-Weyl structures.

2. Statement of Main Results

In this paper, we always assume that M is an n -dimensional compact connected orientable C^∞ manifold and $n \geq 4$. The Weyl structure (g, D) on M is described by a pair $(g, \omega) \in \mathfrak{M} \times A^1(M)$ such that $Dg = \omega \otimes g$. A manifold equipped with a Weyl structure is called a Weyl manifold. Let \mathfrak{M} be the space of all Riemannian metrics on M , \mathfrak{C} , the space of all torsion-free affine connections on M and \mathfrak{W} , the space of all Weyl structures on M . We consider the following conformal gauge invariant functional

$$C_n : \mathfrak{M} \times \mathfrak{C} \rightarrow \mathbf{R}, \quad (g, D) \mapsto \int_M |R^D|_g^{n/2} v_g = \int_M \langle R^D, R^D \rangle_g^{n/4} v_g,$$

where R^D is the curvature tensor of a Weyl connection D , defined by $R^D(X, Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z$, $|\bullet|_g$ is the norm induced by g and v_g is the volume form with respect to g .

Here, we recall that there exists a natural isomorphism from $\mathfrak{M} \times A^1(M)$ to \mathfrak{W} (cf. [10, Lemma 2.4]). Namely, for any $(g, \omega) \in \mathfrak{M} \times A^1(M)$, using the Levi-Civita connection ∇ , we can define the corresponding Weyl connection D by

$$D_X Y = \nabla_X Y + \frac{1}{2} \{g(X, Y)\omega^\sharp - \omega(Y)X - \omega(X)Y\},$$

where ω^\sharp denotes the dual vector field of a 1-form $\omega \in A^1(M)$ with respect to g . From this identification, we can study the restricted functional on the space of all Weyl structures \mathfrak{W} , denoted by

$$C_n^W := C_n|_{\mathfrak{W}} : \mathfrak{W} \cong \mathfrak{M} \times A^1(M) \rightarrow \mathbf{R}.$$

Then, we considered the following variational problem for Weyl structures (cf. [10]). Fix $(g, \omega) \in \mathfrak{M} \times A^1(M) \cong \mathfrak{W}$ and consider a smooth deformation of

Riemannian metrics $g_t \in \mathfrak{M}$ and 1-forms $\omega_t \in A^1(M)$ such that $g_0 = g$ and $\omega_0 = \omega$.

In this situation, we proved the following theorem.

THEOREM 2.1 (Theorem 3.7 in [10]). *Let (M, g, D) be an n -dimensional compact Weyl manifold. Then, a couple (g, D) in \mathfrak{B} is a critical point of the functional $C_n^W : \mathfrak{B} \rightarrow \mathbf{R}$ if and only if it satisfies*

$$\begin{cases} 2|R^D|_g^{(n-4)/2} \left(\tilde{R}^D - \frac{1}{n} |R^D|_g^2 g \right) - ((\nabla - \mathfrak{d}^\nabla)^* + \omega)(\delta^D(|R^D|_g^{(n-4)/2} R^D)) \\ \quad = 0, \\ \langle \eta^\sharp \otimes g - \text{Id} \otimes \eta - \eta \otimes \text{Id}, \delta^D(|R^D|_g^{(n-4)/2} R^D) \rangle_g \\ \quad = 0 \text{ for any } \eta \in A^1(M), \end{cases} \quad (2.1)$$

where $\tilde{R}^D(X, Y) := \sum_{i,j,k=1}^n g(R^D(X, e_i)e_j, e_k)g(R^D(Y, e_i)e_j, e_k)$, $\{e_i\}_{i=1}^n$ is orthonormal local frame field with respect to g , \mathfrak{d}^∇ denotes the Codazzi operator (cf. Definition 3.3), $\eta := (d/dt)|_{t=0}\omega_t \in A^1(M)$, η^\sharp is the dual vector field of a 1-form η with respect to g and Id is the identity transformation on $\Gamma(TM)$.

Here, we use a standard notation of the codifferential $\delta^D := (d^D)^*$ instead of our notation $\tilde{D}^* = (d^D)^*$ in [10], where $\tilde{D} := d^D : A^1(\text{End}(TM)) \rightarrow A^2(\text{End}(TM))$ is the exterior derivative induced by a connection D and the codifferential δ^D is defined as follows.

$$\delta^D : A^2(\text{End}(TM)) \rightarrow A^1(\text{End}(TM)), \quad P \mapsto \delta^D P,$$

$$(\delta^D P)(X)Y := - \sum_{i=1}^n (\bar{D}_{e_i} P)(e_i, X)Y,$$

where \bar{D} is the conjugate connection uniquely determined by D as follows (cf. [8]):

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, \bar{D}_X Z),$$

for any C^∞ vector fields X, Y, Z on M .

Our main theorem is the following. Notation used here will be explained in the next section.

THEOREM 2.2 (cf. Theorem 4.4). *Let (M, g, D) be an n -dimensional compact Weyl manifold. Then, a couple (g, D) in \mathfrak{B} is a critical point of the functional*

$C_n^W : \mathfrak{B} \rightarrow \mathbf{R}$ if and only if it satisfies

$$\begin{cases} 2|R^D|_g^{(n-4)/2} \left(\check{R}^D - \frac{1}{n} |R^D|_g^2 g \right) - C_3^1(\operatorname{div}_g \delta^D Q^D) + C_2^1(\nabla \delta^D Q^D) \\ \quad + C_3^1(\nabla \delta^D Q^D) - \omega(\delta^D Q^D) = 0, \\ C_2^1(\delta^D Q^D) - C_2^1((\delta^D Q^D)^T) - (C_3^2(\delta^D Q^D(\bullet)))^\# = 0, \end{cases}$$

where $Q^D := |R^D|_g^{(n-4)/2} R^D \in A^2(\operatorname{End}(TM))$, C_j^i denotes the contraction of the i -th index and the j -th index for a $(1, 2)$ -typed tensor field and α^T is the transposed map of $\alpha \in A^1(\operatorname{End}(TM))$ with respect to g (cf. Definition 3.2).

Especially, in dimension four, we have the following simple form.

COROLLARY 2.3 (cf. Corollary 4.5). *Let (M, g, D) be a four-dimensional compact Weyl manifold. Then, a couple (g, D) in \mathfrak{B} is a critical point of the functional $C_4^W : \mathfrak{B} \rightarrow \mathbf{R}$ if and only if it satisfies*

$$\begin{cases} 2 \left(\check{R}^D - \frac{1}{4} |R^D|_g^2 g \right) - C_3^1(\operatorname{div}_g \delta^D R^D) + C_2^1(\nabla \delta^D R^D) \\ \quad + C_3^1(\nabla \delta^D R^D) - \omega(\delta^D R^D) = 0, \\ C_2^1(\delta^D R^D) - C_2^1((\delta^D R^D)^T) - (C_3^2(\delta^D R^D(\bullet)))^\# = 0. \end{cases}$$

In dimension four, our conformal gauge invariant functional $C_4^W : \mathfrak{B} \rightarrow \mathbf{R}$ coincides with the functional introduced by Pedersen et al. [19]. In their paper, they treated about the relation between Einstein-Weyl structures and topological invariants. But, they did not point out the Euler-Lagrange equations explicitly. Our result in dimension four reveals them completely. The obtained Euler-Lagrange equations are the mixtures of a Yang-Mills equation of a Weyl connection D with respect to g and a gravitational field equation characterized by symmetric 2-tensor fields \check{R}^D (field strength) and g (gravity). In arbitrary dimension, the corresponding Euler-Lagrange equations are regarded as the conformal generalization of a four dimensional case.

3. Preliminaries

In this section, we give all materials needed later. Let (M, g) be a smooth, connected, orientable, compact Riemannian manifold without boundary ($\dim M \geq 4$), \mathfrak{M} , the space of all Riemannian metrics on M , \mathfrak{C} , the space of all

torsion-free affine connections on M and \mathfrak{W} , the space of all Weyl structures on M . A couple $(g, D) \in \mathfrak{W} \times \mathfrak{C}$ is a Weyl structure if there exists a 1-form $\omega \in A^1(M)$ such that $Dg = \omega \otimes g$, that is, for $X, Y, Z \in \mathcal{X}(M)$,

$$(D_X g)(Y, Z) = \omega(X)g(Y, Z).$$

In some Lemmas, we use a global inner product on M denoted by (\bullet, \bullet) , that is, $(\bullet, \bullet) := \int_M \langle \bullet, \bullet \rangle_g v_g$.

For an endomorphism $f \in \Gamma(\text{End}(TM))$, we can uniquely determine the $(0, 2)$ -tensor field $\overset{\circ}{f} \in \Gamma(T^*M \otimes T^*M)$ using a Riemannian metric g as follows (cf. [3, p. 22]).

$$\overset{\circ}{f}(X, Y) := \langle X, f(Y) \rangle_g, \quad X, Y \in \mathcal{X}(M).$$

Here and in the sequel, we will identify an endomorphism f with the corresponding $(0, 2)$ -tensor field $\overset{\circ}{f}$ and use the same notation f for both of them.

For $\alpha \in A^1(\text{End}(TM))$ and $X \in \mathcal{X}(M)$, $(\alpha(X))^T \in \Gamma(\text{End}(TM))$ means the transposed map of $\alpha(X) \in \Gamma(\text{End}(TM))$ with respect to g , namely, for $Y, Z \in \mathcal{X}(M)$,

$$g((\alpha(X))^T Y, Z) = g(Y, \alpha(X)Z).$$

Then, we can write the equation (2.2) of Theorem 2.1 in the following form.

PROPOSITION 3.1. *For $\alpha = \delta^D(|R^D|_g^{(n-4)/2} R^D) \in A^1(\text{End}(TM))$, the equation (2.2) is equivalent to the following equation:*

$$\sum_{i=1}^n \alpha(e_i)e_i - \sum_{i=1}^n (\alpha(e_i))^T(e_i) - \sum_{i,j=1}^n \langle e_j, \alpha(e_i)e_j \rangle_g e_i = 0. \quad (3.1)$$

PROOF. Since $\eta^\sharp = \sum_{i=1}^n \eta(e_i)e_i \in \mathcal{X}(M)$, we have

$$\begin{aligned} & \langle \eta^\sharp \otimes g - \text{Id} \otimes \eta - \eta \otimes \text{Id}, \alpha \rangle_g \\ &= \sum_{i,j=1}^n \langle g(e_i, e_j)\eta^\sharp - \eta(e_j)e_i - \eta(e_i)e_j, \alpha(e_i)e_j \rangle_g \\ &= \sum_{i=1}^n \langle \eta^\sharp, \alpha(e_i)e_i \rangle_g - \sum_{i=1}^n \langle e_i, \alpha(e_i)\eta^\sharp \rangle_g - \sum_{i=1}^n \langle e_i, \alpha(\eta^\sharp)e_i \rangle_g \\ &= \sum_{i=1}^n \langle \eta^\sharp, \alpha(e_i)e_i \rangle_g - \sum_{i=1}^n \langle \eta^\sharp, (\alpha(e_i))^T e_i \rangle_g - \sum_{i=1}^n \langle e_i, \alpha(\eta^\sharp)e_i \rangle_g. \end{aligned}$$

Here, we note that

$$\sum_{i=1}^n \langle e_i, \alpha(\eta^\sharp) e_i \rangle_g = \sum_{i,j=1}^n \langle e_i, \alpha(\eta(e_j) e_j) e_i \rangle_g = \sum_{i,j=1}^n \eta(e_j) \langle e_i, \alpha(e_j) e_i \rangle_g. \quad (3.2)$$

On the other hand, we also have the following equation.

$$\left\langle \eta^\sharp, \sum_{i,j=1}^n \langle e_j, \alpha(e_i) e_j \rangle_g e_i \right\rangle_g = \sum_{i,j=1}^n \langle e_i, \alpha(e_i) e_j \rangle_g \langle \eta^\sharp, e_i \rangle_g = \sum_{i,j=1}^n \eta(e_i) \langle e_j, \alpha(e_i) e_j \rangle_g. \quad (3.3)$$

By (3.2) and (3.3), we have

$$\sum_{i=1}^n \langle e_i, \alpha(\eta^\sharp) e_i \rangle_g = \left\langle \eta^\sharp, \sum_{i,j=1}^n \langle e_j, \alpha(e_i) e_j \rangle_g e_i \right\rangle_g,$$

which completes the proof. \square

In order to make the equation (2.1) of Theorem 2.1 in the final form, we recall the covariant derivative of an $\text{End}(TM)$ -valued 1-form. Let α be an $\text{End}(TM)$ -valued 1-form and ∇ the Levi-Civita connection of g . $(\nabla_X \alpha)(Y) \in \Gamma(\text{End}(TM))$ is defined by

$$(\nabla_X \alpha)(Y)Z := \nabla_X(\alpha(Y)Z) - \alpha(\nabla_X Y)Z - \alpha(Y)\nabla_X Z. \quad (3.4)$$

Then, we have tensorial properties of $\nabla \alpha$ as follows.

For $\alpha \in A^1(\text{End}(TM))$, a map $\nabla \alpha : (X, Y, Z) \mapsto (\nabla_X \alpha)(Y)Z$ is a tensor field. Namely, for $X, Y, Z \in \mathcal{X}(M)$ and $f \in C^\infty(M)$, we have

$$(\nabla_{fX} \alpha)(Y)Z = (\nabla_X \alpha)(fY)Z = (\nabla_X \alpha)(Y)fZ = f(\nabla_X \alpha)(Y)Z.$$

To state our main results, we define the following contractions for $(1,2)$ -typed tensors $\alpha, \nabla_X \alpha \in A^1(\text{End}(TM))$.

DEFINITION 3.2. We define the contraction C_j^i by, for $\alpha \in A^1(\text{End}(TM))$,

1. $C_2^1(\alpha) := \sum_{i=1}^n \alpha(e_i) e_i \in \mathcal{X}(M)$,
2. $C_2^1(\alpha^T) := \sum_{i=1}^n (\alpha(e_i))^T e_i \in \mathcal{X}(M)$,
3. $C_2^1(\nabla \alpha)Z := \sum_{i=1}^n (\nabla_{e_i} \alpha)(e_i)Z = \sum_{i=1}^n \{ \nabla_{e_i}(\alpha(e_i)Z) - \alpha(\nabla_{e_i} e_i)Z - \alpha(e_i)\nabla_{e_i} Z \} \in \mathcal{X}(M)$,
4. $C_3^1(\nabla \alpha)Y := \sum_{i=1}^n (\nabla_{e_i} \alpha)(Y)e_i = \sum_{i=1}^n \{ \nabla_{e_i}(\alpha(Y)e_i) - \alpha(\nabla_{e_i} Y)e_i - \alpha(Y)\nabla_{e_i} e_i \} \in \mathcal{X}(M)$.

These definitions are independent of the choice of basis $\{e_i\}_{i=1}^n$. Here, $\alpha^T \in A^1(\text{End}(TM))$ is defined by

$$\alpha^T : \mathcal{X}(M) \rightarrow \text{End}(TM), \quad X \mapsto \alpha^T(X) := (\alpha(X))^T.$$

Then, for $f \in C^\infty(M)$, we have $\alpha^T(fX) = f\alpha^T(X)$. So, the contraction $C_2^1(\alpha^T) \in \mathcal{X}(M)$ is well-defined.

To prove Proposition 4.1, using the contraction of $\alpha \in A^1(\text{End}(TM))$ we prepare the following 1-form $C_3^2(\alpha(\bullet)) \in A^1(M)$ defined by

$$X \mapsto C_3^2(\alpha(X)) := \sum_{j=1}^n \langle \alpha(X)e_j, e_j \rangle_g, \quad (3.5)$$

which is independent of the choice of basis $\{e_i\}_{i=1}^n$.

Next, we express the formal adjoint of $\nabla - \mathfrak{d}^\nabla$ with respect to g using the Codazzi operator and its representation by the Levi-Civita connection of g . Here, we use the same notation in our paper [10]. This operator is significant for our studies and its special properties lead to our main theorem. Now, we recall the Codazzi operator \mathfrak{d}^∇ with respect to the Levi-Civita connection ∇ (cf. [5, p. 20], [21, p. 103], [10, p. 557, Definition 3.3]).

DEFINITION 3.3. For any symmetric 2-tensor field $h \in S^2(M)$, we define the Codazzi operator \mathfrak{d}^∇ as

$$(\mathfrak{d}^\nabla h)(X, Y, Z) := (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z).$$

For simplicity, we set $A := \nabla - \mathfrak{d}^\nabla$. Then, for any $h \in S^2(M)$, we have

$$(Ah)(X, Y, Z) = -(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) + (\nabla_Z h)(X, Y).$$

Here, for a $(0, 3)$ -tensor Ah , we introduce three differential operators as follows.

$$\begin{cases} (A_1 h)(X, Y, Z) := -(\nabla_X h)(Y, Z), \\ (A_2 h)(X, Y, Z) := (\nabla_Y h)(X, Z), \\ (A_3 h)(X, Y, Z) := (\nabla_Z h)(X, Y). \end{cases}$$

We notice the decomposition $A = A_1 + A_2 + A_3$ and have the following formula.

LEMMA 3.4. For any 3-tensor field $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, we have

$$\begin{aligned}
(1) \quad (A_1^* \beta)(X, Y) &= \sum_{i=1}^n (\nabla_{e_i} \beta)(e_i, X, Y), \\
(2) \quad (A_2^* \beta)(X, Y) &= - \sum_{i=1}^n (\nabla_{e_i} \beta)(X, e_i, Y), \\
(3) \quad (A_3^* \beta)(X, Y) &= - \sum_{i=1}^n (\nabla_{e_i} \beta)(X, Y, e_i),
\end{aligned}$$

where A_1^* , A_2^* and A_3^* are the formal adjoints of A_1 , A_2 and A_3 with respect to g , respectively.

PROOF. Since the proofs of the properties (2) and (3) are similar to (1), we prove only the property (1). For any 3-tensor field $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, we set

$$(A_1^* \beta)(X, Y) := \sum_{i=1}^n (\nabla_{e_i} \beta)(e_i, X, Y). \quad (3.6)$$

Then, we will show the following equation.

$$(A_1 h, \beta) = (h, A_1^* \beta),$$

where (\bullet, \bullet) denotes a global inner product on M .

For this purpose, we calculate

$$\begin{aligned}
&\langle A_1 h, \beta \rangle_g - \langle h, A_1^* \beta \rangle_g \\
&= - \sum_{i,j,k=1}^n (\nabla_{e_i} h)(e_j, e_k) \beta(e_i, e_j, e_k) - \sum_{i,j=1}^n h(e_i, e_j) \left(\sum_{k=1}^n (\nabla_{e_k} \beta)(e_k, e_i, e_j) \right) \\
&= - \sum_{i,j,k=1}^n \{ (e_i(h(e_j, e_k))) - h(\nabla_{e_i} e_j, e_k) - h(e_j, \nabla_{e_i} e_k) \} \beta(e_i, e_j, e_k) \\
&\quad + h(e_i, e_j) (e_k(\beta(e_k, e_i, e_j)) - \beta(\nabla_{e_k} e_k, e_i, e_j) \\
&\quad - \beta(e_k, \nabla_{e_k} e_i, e_j) - \beta(e_k, e_i, \nabla_{e_k} e_j)) \}. \quad (3.7)
\end{aligned}$$

By $\nabla_{e_i} e_j = \sum_{l=1}^n g(\nabla_{e_i} e_j, e_l) e_l$, we have

$$h(\nabla_{e_i} e_j, e_k) = \sum_{l=1}^n g(\nabla_{e_i} e_j, e_l) h(e_l, e_k) = - \sum_{l=1}^n g(e_j, \nabla_{e_i} e_l) h(e_l, e_k). \quad (3.8)$$

By (3.8), we have

$$\begin{aligned}
\sum_{i,j,k=1}^n h(\nabla_{e_i} e_j, e_k) \beta(e_i, e_j, e_k) &= - \sum_{i,k,l=1}^n h(e_l, e_k) \beta\left(e_i, \sum_{j=1}^n g(e_j, \nabla_{e_i} e_l) e_j, e_k\right) \\
&= - \sum_{i,j,k=1}^n h(e_i, e_j) \beta(e_k, \nabla_{e_k} e_i, e_j). \tag{3.9}
\end{aligned}$$

Similarly, we have

$$\sum_{i,j,k=1}^n h(e_j, \nabla_{e_i} e_k) \beta(e_i, e_j, e_k) = - \sum_{i,j,k=1}^n h(e_i, e_j) \beta(e_k, e_i, \nabla_{e_k} e_j). \tag{3.10}$$

Here, we define a 1-form $\kappa \in A^1(M)$ by the contraction of a symmetric 2-tensor field $h \in S^2(M)$ and a 3-tensor field $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, namely,

$$\kappa(X) := \sum_{i,j=1}^n h(e_i, e_j) \beta(X, e_i, e_j).$$

Then, we have

$$\begin{aligned}
\operatorname{div}_g \kappa^\sharp &= \sum_{k=1}^n g(e_k, \nabla_{e_k} \kappa^\sharp) \\
&= \sum_{k=1}^n \left\{ e_k \left(\sum_{i,j=1}^n h(e_i, e_j) \beta(e_k, e_i, e_j) \right) - \sum_{i,j=1}^n h(e_i, e_j) \beta(\nabla_{e_k} e_k, e_i, e_j) \right\} \\
&= \sum_{i,j,k=1}^n e_i (h(e_j, e_k)) \beta(e_i, e_j, e_k) + \sum_{i,j,k=1}^n h(e_i, e_j) e_k (\beta(e_k, e_i, e_j)) \\
&\quad - \sum_{i,j,k=1}^n h(e_i, e_j) \beta(\nabla_{e_k} e_k, e_i, e_j). \tag{3.11}
\end{aligned}$$

By (3.6), (3.7) and (3.11), we have

$$\langle A_1 h, \beta \rangle_g - \langle h, A_1^* \beta \rangle_g = -\operatorname{div}_g \kappa^\sharp. \tag{3.12}$$

Integrating on M the both hand sides of (3.12) and applying Green's theorem, we have

$$(A_1 h, \beta) = (h, A_1^* \beta),$$

which completes the proof. \square

From Lemma 3.4 and the decomposition of A , we have the following formula.

PROPOSITION 3.5. *For any 3-tensor field $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, we have*

$$(A^*\beta)(X, Y) = \sum_{i=1}^n \{(\nabla_{e_i}\beta)(e_i, X, Y) - (\nabla_{e_i}\beta)(X, e_i, Y) - (\nabla_{e_i}\beta)(X, Y, e_i)\},$$

where A^* is the formal adjoint of the differential operator $A := \nabla - \mathfrak{d}^\nabla$ with respect to g .

In order to apply Proposition 3.5 to $\alpha = \delta^D(|R^D|_g^{(n-4)/2} R^D) \in A^1(\text{End}(TM))$, we need [10, Lemma 3.4]. Then, Ah is given by

$$(Ah)(X, Y, Z) = ((\nabla - \mathfrak{d}^\nabla)h)(X, Y, Z) = 2g(X, \gamma_Y Z),$$

where $\gamma := (d/dt)|_{t=0} \nabla^{g_t} \in A^1(\text{End}(TM))$ and ∇^{g_t} is the Levi-Civita connection corresponding to a smooth deformation of Riemannian metric $g_t \in \mathfrak{M}$.

For $\alpha \in A^1(\text{End}(TM))$, we can set

$$\beta(X, Y, Z) = g(X, \alpha(Y)Z). \quad (3.13)$$

Here, we introduce the divergence of an $\text{End}(TM)$ -valued 1-form to express the formal adjoint of the Codazzi operator.

DEFINITION 3.6. For $\alpha \in A^1(\text{End}(TM))$, $\text{div}_g \alpha \in A^1(\text{End}(TM))$, the divergence of α with respect to g is defined by

$$(\text{div}_g \alpha)(X)Y := \sum_{i=1}^n g(e_i, (\nabla_{e_i}\alpha)(X)Y). \quad (3.14)$$

PROPOSITION 3.7. *Let $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$ be a 3-tensor field and $\alpha \in A^1(\text{End}(TM))$ an $\text{End}(TM)$ -valued 1-form. Then, we have*

$$((\nabla - \mathfrak{d}^\nabla)^*\beta)(X, Y) = (\text{div}_g \alpha)(X)Y - \sum_{i=1}^n g(X, \nabla_{e_i}(\alpha(e_i)Y + \alpha(Y)e_i)), \quad (3.15)$$

where $\alpha = \delta^D(|R^D|_g^{(n-4)/2} R^D)$ and $\beta(X, Y, Z) = g(X, \alpha(Y)Z)$.

PROOF. We set $A := \nabla - \mathfrak{d}^\nabla$. From Proposition 3.5, we have

$$\begin{aligned} (Ah, \beta) &= (h, A^*\beta) \\ &= \sum_{i,j,k=1}^n \int_M \langle h(e_j, e_k), (\nabla_{e_i}\beta)(e_i, e_j, e_k) - (\nabla_{e_i}\beta)(e_j, e_i, e_k) \\ &\quad - (\nabla_{e_i}\beta)(e_j, e_k, e_i) \rangle_g v_g. \end{aligned} \quad (3.16)$$

From the definition of a covariant derivative, we have the following 3 equations.

$$\begin{aligned}
(\nabla_{e_i}\beta)(e_i, X, Y) &= e_i(g(e_i, \alpha(X)Y)) - g(e_i, \alpha(\nabla_{e_i}X)Y) - g(e_i, \alpha(X)\nabla_{e_i}Y) \\
&= g(e_i, \nabla_{e_i}(\alpha(X)Y)) - g(e_i, \alpha(\nabla_{e_i}X)Y) - g(e_i, \alpha(X)\nabla_{e_i}Y) \\
&= g(e_i, \nabla_{e_i}(\alpha(X)Y) - \alpha(\nabla_{e_i}X)Y - \alpha(X)\nabla_{e_i}Y) \\
&= g(e_i, (\nabla_{e_i}\alpha)(X)Y), \\
(\nabla_{e_i}\beta)(X, e_i, Y) &= e_i(g(X, \alpha(e_i)Y)) - g(\nabla_{e_i}X, \alpha(e_i)Y) - g(X, \alpha(e_i)\nabla_{e_i}Y) \\
&= g(X, \nabla_{e_i}(\alpha(e_i)Y)) - g(X, \alpha(e_i)\nabla_{e_i}Y), \\
(\nabla_{e_i}\beta)(X, Y, e_i) &= e_i(g(X, \alpha(Y)e_i)) - g(\nabla_{e_i}X, \alpha(Y)e_i) - g(X, \alpha(\nabla_{e_i}Y)e_i) \\
&= g(X, \nabla_{e_i}(\alpha(Y)e_i)) - g(X, \alpha(\nabla_{e_i}Y)e_i).
\end{aligned} \tag{3.17}$$

From (3.14), (3.16) and (3.17), we have

$$\begin{aligned}
(Ah, \beta) &= (h, A^*\beta) \\
&= \sum_{j,k=1}^n \int_M \langle h(e_j, e_k), (\operatorname{div}_g \alpha)(e_j)e_k \rangle_g v_g \\
&\quad - \sum_{j,k=1}^n \int_M \left\langle h(e_j, e_k), g\left(e_j, \sum_{i=1}^n \nabla_{e_i}(\alpha(e_i)e_k)\right) + g\left(e_j, \sum_{i=1}^n \nabla_{e_i}(\alpha(e_k)e_i)\right) \right\rangle_g v_g.
\end{aligned}$$

Thus, we have

$$A^*\beta(X, Y) = (\operatorname{div}_g \alpha)(X)Y - g\left(X, \sum_{i=1}^n \nabla_{e_i}(\alpha(e_i)Y + \alpha(Y)e_i)\right),$$

which completes the proof. \square

Here, we remark that the symmetric 2-tensor $\omega(\delta^D(|R^D|_g^{(n-4)/2}R^D))$ is expressed as follows.

$$\omega(\alpha)(X, Y) = \sum_{i=1}^n \omega(e_i)g(e_i, \alpha(X)Y) = g(\omega^\sharp, \alpha(X)Y) \tag{3.18}$$

where $\alpha = \delta^D(|R^D|_g^{(n-4)/2}R^D) \in A^1(\operatorname{End}(TM))$ and $\omega \in A^1(M)$ denotes the 1-form corresponding to a Weyl connection D . Thus, (3.18) means the following symmetric 2-tensor:

$$\omega(\delta^D(|R^D|_g^{(n-4)/2}R^D))(X, Y) = g(\omega^\sharp, \delta^D(|R^D|_g^{(n-4)/2}R^D)(X)Y). \tag{3.19}$$

4. Proofs

We first give the final form of the equation (2.2) in Theorem 2.1 as follows.

PROPOSITION 4.1. *For $Q^D := |R^D|_g^{(n-4)/2} R^D \in A^2(\text{End}(TM))$, the equation (3.1) is equivalent to the following equation:*

$$C_2^1(\delta^D Q^D) - C_2^1((\delta^D Q^D)^T) - (C_3^2(\delta^D Q^D(\bullet)))^\sharp = 0.$$

PROOF. For $\alpha = \delta^D Q^D$ and (3.5), the dual vector field of a 1-form $C_3^2(\alpha(\bullet)) \in A^1(M)$ is given by

$$(C_3^2(\alpha(\bullet)))^\sharp = \sum_{i=1}^n C_3^2(\alpha(e_i))e_i = \sum_{i,j=1}^n \langle \alpha(e_i)e_j, e_j \rangle_g e_i. \quad (4.1)$$

From Proposition 3.1, Definition 3.2 and the equation (4.1), we obtain the final form of the equation (2.2) of Theorem 2.1. \square

We can make Proposition 3.7 in the following form.

LEMMA 4.2. *For any affine connection D , we have*

$$(\nabla - \mathfrak{d}^\nabla)^* \beta(X, Y) = g \left(X, \sum_{j=1}^n (\text{div}_g \alpha)(e_j) Y e_j - \sum_{i=1}^n \nabla_{e_i} (\alpha(e_i) Y + \alpha(Y) e_i) \right),$$

where $\alpha = \delta^D(|R^D|_g^{(n-4)/2} R^D) \in A^1(\text{End}(TM))$ and $\beta(X, Y, Z) = g(X, \alpha(Y)Z)$.

PROOF. We set $\mu := (\nabla - \mathfrak{d}^\nabla)^* \beta \in \Gamma(T^*M \otimes T^*M)$ and define $\theta \in \Gamma(\text{End}(TM))$ by

$$\mu(X, Y) = g(X, \theta(Y)). \quad (4.2)$$

Then, we have

$$\theta(Y) = \sum_{j=1}^n g(e_j, \theta(Y)) e_j = \sum_{j=1}^n \mu(e_j, Y) e_j. \quad (4.3)$$

From Proposition 3.7, we have

$$\begin{aligned} \mu(e_j, Y) &= ((\nabla - \mathfrak{d}^\nabla)^* \beta)(e_j, Y) \\ &= (\text{div}_g \alpha)(e_j) Y - \sum_{i=1}^n g(e_j, \nabla_{e_i} (\alpha(e_i) Y + \alpha(Y) e_i)). \end{aligned} \quad (4.4)$$

By (4.3) and (4.4), we have

$$\begin{aligned}\theta(Y) &= \sum_{j=1}^n \left\{ (\operatorname{div}_g \alpha)(e_j) Y - \sum_{i=1}^n g(e_j, \nabla_{e_i}(\alpha(e_i) Y + \alpha(Y) e_i)) \right\} e_j \\ &= \sum_{j=1}^n ((\operatorname{div}_g \alpha)(e_j) Y) e_j - \sum_{i=1}^n \nabla_{e_i}(\alpha(e_i) Y + \alpha(Y) e_i).\end{aligned}\quad (4.5)$$

By (4.2) and (4.5), we have

$$\begin{aligned}((\nabla - \mathfrak{d}^\nabla)^* \beta)(X, Y) &= g(X, \theta(Y)) \\ &= g\left(X, \sum_{j=1}^n ((\operatorname{div}_g \alpha)(e_j) Y) e_j - \sum_{i=1}^n \nabla_{e_i}(\alpha(e_i) Y + \alpha(Y) e_i)\right),\end{aligned}\quad (4.6)$$

which completes the proof. \square

From Lemma 4.2, we have

PROPOSITION 4.3.

$$(\nabla - \mathfrak{d}^\nabla)^* \beta(X, Y) = \langle X, C_3^1(\operatorname{div}_g \alpha) Y \rangle_g - \langle X, C_2^1(\nabla \alpha) Y \rangle_g - \langle X, C_3^1(\nabla \alpha) Y \rangle, \quad (4.7)$$

where $\alpha = \delta^D(|R^D|_g^{(n-4)/2} R^D) \in A^1(\operatorname{End}(TM))$ and $\beta(X, Y, Z) = g(X, \alpha(Y)Z)$.

PROOF. Both sides of (4.7) are a $(0, 2)$ -tensor field, so taking any point $x_0 \in M$ and the following orthonormal local frame field $\{e_i\}_{i=1}^n$ on its neighborhood U , it suffices to show the equation (4.7) holds at $x_0 \in M$. For given any tangent vectors $X, Y \in T_{x_0}M$, we use the same notations X, Y for vector fields on a neighborhood U of x_0 . Take $X, Y \in \mathcal{X}(U)$ such that $\nabla_W X = \nabla_W Y = 0$ for any vector field $W \in \mathcal{X}(U)$ at x_0 . Moreover, we take an orthonormal local frame field $\{e_i\}_{i=1}^n$ satisfying $\nabla_{e_i} e_j = 0$ at $x_0 \in M$.

Then, from Lemma 4.2 we have the following equation at $x_0 \in M$.

$$\begin{aligned}(\nabla - \mathfrak{d}^\nabla)^* \beta(X, Y) &= g(X, C_3^1(\operatorname{div}_g \alpha) Y) \\ &\quad - g\left(X, \sum_{i=1}^n \nabla_{e_i}(\alpha(e_i) Y)\right) - g\left(X, \sum_{i=1}^n \nabla_{e_i}(\alpha(Y) e_i)\right).\end{aligned}\quad (4.8)$$

By (3.4), it follows that, at $x_0 \in M$,

$$\begin{aligned}\nabla_{e_i}(\alpha(e_i)Y) &= (\nabla_{e_i}\alpha)(e_i)Y + \alpha(\nabla_{e_i}e_i)Y + \alpha(e_i)\nabla_{e_i}Y = (\nabla_{e_i}\alpha)(e_i)Y, \\ \nabla_{e_i}(\alpha(Y)e_i) &= (\nabla_{e_i}\alpha)(Y)e_i + \alpha(\nabla_{e_i}Y)e_i + \alpha(Y)\nabla_{e_i}e_i = (\nabla_{e_i}\alpha)(Y)e_i,\end{aligned}\tag{4.9}$$

since $\nabla_{e_i}e_i = \nabla_{e_i}Y = 0$ at x_0 .

By (4.8) and (4.9), we obtain at x_0 ,

$$\begin{aligned}(\nabla - \mathfrak{d}^\nabla)^*\beta(X, Y) &= g(X, C_3^1(\operatorname{div}_g \alpha)Y) - g\left(X, \sum_{i=1}^n (\nabla_{e_i}\alpha)(e_i)Y\right) - g\left(X, \sum_{i=1}^n (\nabla_{e_i}\alpha)(Y)e_i\right) \\ &= \langle X, C_3^1(\operatorname{div}_g \alpha)Y \rangle_g - \langle X, C_2^1(\nabla\alpha)Y \rangle_g - \langle X, C_3^1(\nabla\alpha)Y \rangle_g,\end{aligned}\tag{4.10}$$

which completes the proof. \square

Thus, we obtain the following results. Here, $Q^D := |R^D|_g^{(n-4)/2} R^D \in A^2(\operatorname{End}(TM))$.

THEOREM 4.4. *Let (M, g, D) be an n -dimensional compact Weyl manifold. Then, a couple (g, D) in \mathfrak{W} is a critical point of the functional $C_n^W : \mathfrak{W} \rightarrow \mathbf{R}$ if and only if it satisfies*

$$\begin{cases} 2|R^D|_g^{(n-4)/2} \left(\check{R}^D - \frac{1}{n} |R^D|_g^2 g \right) - C_3^1(\operatorname{div}_g \delta^D Q^D) + C_2^1(\nabla \delta^D Q^D) \\ \quad + C_3^1(\nabla \delta^D Q^D) - \omega(\delta^D Q^D) = 0, \\ C_2^1(\delta^D Q^D) - C_2^1((\delta^D Q^D)^T) - (C_3^2(\delta^D Q^D(\bullet)))^\sharp = 0. \end{cases}$$

PROOF. Take $\alpha = \delta^D(|R^D|_g^{(n-4)/2} R^D) \in A^1(\operatorname{End}(TM))$ and $\beta(X, Y, Z) = g(X, \alpha(Y)Z)$. Applying Propositions 3.1 and 4.1 to the equation (2.2) of Theorem 2.1, we have the second equation of Theorem 4.4.

Applying Proposition 4.3 to the equation (2.1) of Theorem 2.1 and identifying an endomorphism with the corresponding $(0, 2)$ -tensor field, we obtain the first equation of Theorem 4.4.

Hence, we prove our main theorem. \square

Especially, in dimension four, we have the following simple form.

COROLLARY 4.5. *Let (M, g, D) be a four-dimensional compact Weyl manifold. Then, a couple (g, D) in \mathfrak{B} is a critical point of the functional $C_4^W : \mathfrak{B} \rightarrow \mathbb{R}$ if and only if it satisfies*

$$\begin{cases} 2\left(\check{R}^D - \frac{1}{4}|R^D|_g^2 g\right) - C_3^1(\operatorname{div}_g \delta^D R^D) + C_2^1(\nabla \delta^D R^D) \\ \quad + C_3^1(\nabla \delta^D R^D) - \omega(\delta^D R^D) = 0, \\ C_2^1(\delta^D R^D) - C_2^1((\delta^D R^D)^T) - (C_3^2(\delta^D R^D(\bullet)))^\sharp = 0. \end{cases}$$

To explain the meaning of our results, we recall the following property of Einstein metrics (cf. [4, p. 134, 4.72]).

PROPOSITION 4.6. *Let ∇ be the Levi-Civita connection and $SR|_{\mathfrak{M}_1}$, a quadratic functional defined by $SR(g) := \int_M |R^\nabla|_g^2 v_g$ restricted to $\mathfrak{M}_1 := \{g \in \mathfrak{M}; \int_M v_g = 1\}$.*

An Einstein metric g (or more generally, a Riemannian metric with parallel Ricci tensor) is critical for the quadratic functional $SR|_{\mathfrak{M}_1}$ if and only if the curvature R^∇ of ∇ satisfies

$$\check{R}^\nabla - \frac{1}{n}|R^\nabla|_g^2 g = 0. \quad (4.11)$$

In dimension four, if R^D is a Yang-Mills field, namely, $\delta^D R^D = 0$ in our sense (see [8]), then we have

COROLLARY 4.7. *Let (M, g, D) be a four-dimensional compact Weyl manifold and R^D a Yang-Mills field determined by a torsion-free affine connection D . Then, a couple $(g, D) \in \mathfrak{B}$ is critical for our conformal gauge invariant functional $C_4^W : \mathfrak{B} \rightarrow \mathbb{R}$ if and only if it satisfies*

$$\check{R}^D - \frac{1}{4}|R^D|_g^2 g = 0. \quad (4.12)$$

Due to this result, in dimension four, the equation (4.12) can be regarded as a conformal generalization of a gravitational field equation characterized by the equation (4.11). Moreover, we would like to study the following system of equations on a four dimensional compact Weyl manifold;

$$\begin{cases} \delta^D R^D = 0, \\ \check{R}^D - \frac{1}{4}|R^D|_g^2 g = 0. \end{cases} \quad (4.13)$$

According to [10, p. 560, Example 4.1], we have

EXAMPLE 4.8. *Let (M, g) be a four-dimensional Einstein manifold and ∇ the Levi-Civita connection of g . Then, $(g, \nabla) \in \mathfrak{B}$ is a critical point of the functional $C_4^W : \mathfrak{B} \rightarrow \mathbf{R}$ and a solution of the system of equations (4.13).*

In the case of dimension $n \geq 4$, we have the following [10, p. 560, Example 4.2].

EXAMPLE 4.9. *Let (M, g) be an n -dimensional isotropy irreducible homogeneous space with its canonical metric and ∇ the Levi-Civita connection. Then, (g, ∇) is a critical point of the functional $C_n^W : \mathfrak{B} \rightarrow \mathbf{R}$ and a solution of the following system of equations;*

$$\begin{cases} \delta^D R^D = 0, \\ \tilde{R}^D - \frac{1}{n} |R^D|_g^2 g = 0. \end{cases}$$

It would be an interesting problem for us to construct a Yang-Mills-Einstein theory in a category of Weyl geometry (cf. [11]).

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