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TRIGONAL GORENSTEIN CURVES AND WEIERSTRASS POINTS

By

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Abstract. In this paper we study the Weierstrass points of singular Gorenstein curves. We need to analyze separately the cases in which the trigonal pencil is induced by a line bundle or not, in which the Weierstrass point, P, is a smooth point or not, in which P is a smooth ordinary or total ramification point or not.

0. Introduction

Let Y be an integral Gorenstein projective curve. In this paper we will say that Y is trigonal if there exists a rank 1 torsion free sheaf L on Y with $\deg(L) = 3$ and $h^0(Y, L) \geq 2$, but Y is not hyperelliptic, i.e. there is no line bundle R on Y with $\deg(R) = 2$ and $h^0(Y, R) \geq 2$. In this paper we study the Weierstrass points of trigonal Gorenstein curves. Let Y be an integral trigonal Gorenstein curve with $g := p_a(Y) \geq 5$ and let L the associated trigonal pencil. Since Y is Gorenstein but not hyperelliptic, L is spanned and $h^0(Y, L) = 2$ ([6, Th. A of the Appendix with J. Harris]). Since $g \geq 5$ the sheaf L is unique (see e.g. [1, Lemma 2.6]). Such curves were deeply studied in [17] and [18]. By [18, Th. 3.5] the projective geometry of the canonical model of Y is very different if L is locally free or not. We study the case in which L is not locally free in section 1. We need to study the "vertex" $\boldsymbol{v} \in Y$ (see 1.1, 1.2 and 1.3), the other singular points (if any) of Y (see 1.4) and the smooth Weierstrass points (see 1.5 and 1.6); 1.5 and 1.6 give a complete description of the possible gap sequences of smooth Weierstrass points which are not on the ramification of the projection from the vertex $\boldsymbol{v}$. All the smooth ramification points of the projection from the vertex $\boldsymbol{v}$ are Weierstrass points (Proposition 1.7). For an existence theorem for Gorenstein

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trigonal curves with prescribed singularities and non-locally free trigonal pencil, see Theorem 1.10. In section 2 we study the case in which L is locally free. Among the smooth points we have to distinguish the non-ramification ones, the ordinary ramification ones and the total ramification ones. We summarize our results for smooth Weierstrass points in the following statement proved in 2.1, 2.2 and 2.4.

THEOREM 0.1. *Assume char($\boldsymbol{K}$) = 0. Let Y be an integral projective trigonal curve with a spanned $L \in Pic^3(Y)$ and let $u : Y \to \boldsymbol{P}^1$ be the associated degree 3 pencil. The possible gap sequences of a smooth Weierstrass point P of Y are the same as for smooth trigonal curves of the same genus, i.e. we have:*

(i) *if $P \in Y_{\mathrm{reg}}$ and P is a simple ramification point of u, then the possible gap sequences of P are the ones described in* [5];

(ii) *if $P \in Y_{\mathrm{reg}}$ and P is a total ramification point of u, then there are two possible gap sequences of P (Type I and Type II in the terminology of* [4]);

(iii) *if $P \in Y_{\mathrm{reg}}$ and P is not a ramification point of u, the possible gap sequences of P are the ones described in* [20] *for smooth curves.*

In section three we give a rather complete description of all trigonal Gorenstein curves whose associated minimal degree rational map, u, onto $\boldsymbol{P}^1$ is birational.

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1. Non Locally Free Degree 3 Pencil

Let Y be an integral projective trigonal non-hyperelliptic Gorenstein curve with $g := p_a(Y) \geq 5$. Let $\pi : X \to Y$ be the normalization. In this section we assume that the trigonal pencil is not induced by a line bundle, i.e. that it is induced by a spanned degree 3 torsion free but not locally free rank 1 sheaf. By [18, Th. 3.5] these curves are exactly the trigonal Gorenstein curves whose canonical model, say $Y \subset \boldsymbol{P}^{g-1}$, is contained in a cone, T, over a rational normal curve C in a hyperplane, H, of $\boldsymbol{P}^{g-1}$. Call $\boldsymbol{v}$ the vertex of T. Hence $\boldsymbol{v} \in (\boldsymbol{P}^{g-1} \backslash H)$. By [18, Th. 3.5] $\boldsymbol{v} \in \mathrm{Sing}(Y)$. Call $f : Y \backslash \{\boldsymbol{v}\} \to C$ and $\phi : \boldsymbol{P}^{g-1} \backslash \{\boldsymbol{v}\} \to H$ the projections from $\boldsymbol{v}$. Since $\deg(f) = 2$ and $\deg(C) = g - 2$, Y has multiplicity two at $\boldsymbol{v}$. To analyze the Weierstrass points on $Y \backslash \{\boldsymbol{v}\}$ we will use that C is a rational normal curve in H and that a rational normal curve has no osculating point, i.e. for every $P \in C$ the osculating hyperplane $M \subset H$ of C at P has order of contact $g - 2$ with C at P (Bezout theorem). Furthermore, since any line D of T through $\boldsymbol{v}$ intersects Y, outside $\boldsymbol{v}$, in a scheme of length at most two, every

$P \in \mathrm{Sing}(Y)$ with $P \neq \boldsymbol{v}$ has multiplicity two. Since $T\backslash\{\boldsymbol{v}\}$ is a smooth surface, every $P \in \mathrm{Sing}(Y)$ with $P \neq \boldsymbol{v}$ is a planar singularity of Y. We recall that a planar curve singularity of multiplicity two is either a tacnode or a generalized cusp. If Y_P is the partial normalization of Y at P, set $\delta(P, Y) = p_a(Y) - p_a(Y_P)$. Let $\alpha : S \to T$ be the blowing-up of T at $\boldsymbol{v}$. Set $\boldsymbol{h} := \alpha^{-1}(\boldsymbol{v})$. S is a smooth surface isomorphic to the Hirzebruch surface F_{g-2} and we will take as basis of $\mathrm{Pic}(S) \cong \boldsymbol{Z}^2$ the curve $\boldsymbol{h} \cong \boldsymbol{P}^1$ and a fiber, F, of the ruling $u : S \to \boldsymbol{P}^1$ of S. Hence $\boldsymbol{h}^2 = 2 - g$, $\boldsymbol{h} \cdot F = 1$ and $F^2 = 0$. We will often use the additive notation for the divisors on S. We have $\alpha^*(\boldsymbol{O}_T(1)) = \boldsymbol{h} + (g-2)F$. Let Y' be the strict transform of Y in S. We assume that the pencil $|F|$ does not induce a degree 1 map from X onto $\boldsymbol{P}^1$; this case will be studied in 3.2. We have $Y' \in |2\boldsymbol{h} + xF|$ for some x; since $\deg(Y) = 2g-2$ we obtain $x = 2g-2$. Set $v := u|Y'$. Viceversa, for any integral $Y' \in |2\boldsymbol{h} + (2g-2)F|$ the curve $\alpha(Y') \subset T \subset \boldsymbol{P}^{g-1}$ is a canonical Gorenstein curve by [18, Formula 3.1] (with d instead of $d-1$), and the proof of [18, Th. 3.2] for $m = 0$. We have $\omega_S \cong \boldsymbol{O}_S(-2\boldsymbol{h} - gF)$. Hence by the adjunction formula we have $\omega_{Y'} \cong \boldsymbol{O}_S((g-2)F) \mid Y'$. Thus $p_a(Y') = g-1$.

(**1.1**) Here and in 1.2 and 1.3 we will analyze the vertex $\boldsymbol{v} \in T$. Since $Y' \in |2\boldsymbol{h} + (2g-2)F|$, we have $Y' \cdot \boldsymbol{h} = 2$. Hence either Y' intersects transversally $\boldsymbol{h}$ at two points or $\mathrm{card}(Y' \cap \boldsymbol{h}) = 1$. In the latter case either Y' is tangent to $\boldsymbol{h}$ at one point, Q, of Y'_{reg} or $Y' \cap \boldsymbol{h} = \{Q\}$, Y' has a planar double point at Q (tacnode or cusp, perhaps not ordinary) and the tangent of $\boldsymbol{h}$ at Q is not in the tangent cone of Y' at Q. Now assume $Y' \cap \boldsymbol{h} = \{Q', Q''\}$ with $Q' \neq Q''$ and set $D' := \alpha(u^{-1}(v(Q'))$ and $D'' := \alpha(u^{-1}(v(Q''))$. Hence D' and D'' are lines. We have $D' \neq D''$, $\boldsymbol{v}$ is an ordinary node of Y and $D' \cup D''$ is the tangent cone to Y at $\boldsymbol{v}$. Now we will analyze the situation for a general curve, i.e. for a curve $Y = \alpha(Y')$ with Y' general element of $|2\boldsymbol{h} + (2g-2)F|$. Hence Y has an ordinary node at $\boldsymbol{v}$. By [7, Prop. 3.5], $\boldsymbol{v}$ is a Weierstrass point of Y with weight $w(\boldsymbol{v}) \geq g(g-1)$. The non-negative integer $E(\boldsymbol{v}) := w(\boldsymbol{v}) - g(g-1)$ is called the extraweight of $\boldsymbol{v}$ and it is the real measure of how much $\boldsymbol{v}$ is a Weierstrass point of Y, not just how singular is Y at $\boldsymbol{v}$. By [7, Prop. 5.5], it is possible to compute $E(\boldsymbol{v})$ looking at the gap sequences of all points of $\pi^{-1}(\boldsymbol{v})$ with respect to a suitable linear system, V, on X with $V \cong \boldsymbol{P}(\pi^*(H^0(Y, \omega_Y)))$. The next result will show that for every Y with two ordinary branches at $\boldsymbol{v}$ and such that the fibers of the ruling α are not tangent to Y' at Q' or Q'' we have $E(\boldsymbol{v}) = 0$ and thus $w(\boldsymbol{v}) = g(g-1)$ attains the minimal a priori possible value. Notice that for a general $Y' \in |2\boldsymbol{h} + (2g-2)F|$ the canonical curve $\alpha(Y')$ has an ordinary node at $\boldsymbol{v}$ and it is smooth ouside $\boldsymbol{v}$. Furthermore, $Y' \cong X$, none of the branches of Y at $\boldsymbol{v}$ is tangent to a line of T and $V \cong \boldsymbol{P}(\pi^*(H^0(Y, \omega_Y)))$ is just the restriction to Y' of $|\boldsymbol{h} + (g-2)F|$.

PROPOSITION 1.2. *In the set-up of* 1.1 *assume* $Y' \cap \boldsymbol{h} = \{Q', Q''\}$ *with* $Q' \neq Q''$ *and that the fibers of the ruling* $u : S \to \boldsymbol{P}^1$ *are not tangent to* Y' *at* Q' *or* Q''. *Then we have* $E(\boldsymbol{v}) = 0$, *i.e.* $w(\boldsymbol{v}) = g(g-1)$. *In particular, for a general* $Y' \in |2\boldsymbol{h} + (2g-2)F|$ *the curve* $\alpha(Y')$ *has extraweight* $E(\boldsymbol{v}) = 0$ *at* v.

PROOF. It is sufficient to check that there is no $D \in |\boldsymbol{h} + (g-2)F|$ with D containing either Q' or Q'' with multiplicity at least g. Fix $D \in |\boldsymbol{h} + (g-2)F|$ with $Q' \in D$. Since $Q' \in \boldsymbol{h}$, $(\boldsymbol{h} + (g-2)F) \cdot \boldsymbol{h} = 0$ and $\boldsymbol{h}$ is irreducible, D contains $\boldsymbol{h}$. Hence D is union of $\boldsymbol{h}$ and $g-2$ fibers. Since Y' is transversal both to $\boldsymbol{h}$ and to the fiber of the ruling through Q', we easily conclude that for general Y' no such D has order of contact at least g with Y' at Q' or at Q''.

REMARK 1.3. Assume that Y has two ordinary branches at $\boldsymbol{v}$ exactly γ of them $(1 \leq \gamma \leq 2)$ have a line of T as tangent at $\boldsymbol{v}$. The proof of 1.2 shows the inequality $E(\boldsymbol{v}) \geq \gamma$.

REMARK 1.4. Fix $P \in \mathrm{Sing}(Y)$, $P \neq \boldsymbol{v}$, and set $\delta := \delta(P, Y) > 0$. By [7, Prop. 3.5], P is a Weierstrass point of Y with weight $w(P) \geq g(g-1)\delta$.

PROPOSITION 1.5. *Fix an integer* z *with* $g \leq z \leq 2g-2$, $P \in Y_{\mathrm{reg}}$ *and a hyperplane* M *with* $P \in M$ *and* $\boldsymbol{v} \notin M$. *Assume* $i(Y, M; P) = z$, *i.e. assume that the scheme* $M \cap Y$ *contains the Cartier divisor* zP *of* Y *but not the Cartier divisor* $(z+1)P$. *Then* P *is a Weierstrass point of* Y *and the sequence of non gaps of* P *is given by the integers* i *with* $g \leq i \leq z$ *and by the integers* $j \geq z+2$.

PROOF. By construction M contains an osculating linear subspace to Y at P. Since $\boldsymbol{v} \notin M$ and the embedding of C in H has no ramification point, the $(g-3)$-dimensional osculating space $V(g-3)$ to Y has contact order $g-2$ with Y at P. Hence every integer i with $1 \leq i \leq g-1$ is a gap for P. Hence M is the osculating hyperplane to Y at P. Since $z \geq g$, P is a Weierstrass point of Y. The assumption on the scheme $Y \cap M$ implies that all integers i with $g \leq i \leq z$ are non gaps for P, while by the geometric form of Riemann-Roch we have $h^1(Y, \boldsymbol{O}_Y((z+1)P)) = 0$. Hence $z+1$ is a gap, while every integer $j \geq z+2$ is a non gap, proving 1.5.

THEOREM 1.6. *Fix integers* g, z *with* $g \geq 5$ *and* $g \leq z \leq 2g-2$. *If* $\mathrm{char}(\boldsymbol{K}) > 0$ *assume* $z < 2g-2$. *Then there exists a pair* (Y, P) *with* $Y \subset T$ *an integral trigonal curve with* $p_a(Y) = g$, $\mathit{Sing}(Y) = \{\boldsymbol{v}\}$, *an ordinary double point at* $\boldsymbol{v}$ *and* $P \in Y_{\mathrm{reg}}$,

P is a Weierstrass point of Y and the sequence of non gaps of P is given by the integers i with $g \leq i \leq z$ and by the integers $j \geq z+2$.

PROOF. Fix $P \in (T\backslash\{v\})$ and set $Q := \pi^{-1}(P) \in S$. Let F be the fiber of the ruling of S passing through Q. Let A be a zero-dimensional subscheme of S with $A_{\mathrm{red}} = \{Q\}$, length$(A) = z+1$ and such that the scheme-theoretical intersection $A \cap F$ is Q with its reduced structure. Thus A is curvilinear and it contains a unique length z subscheme; call it Z.

Claim: We have $h^1(S, \boldsymbol{I}_A(2\boldsymbol{h}+(2g-2)F)) = h^1(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F)) = 0$.

Proof of the Claim: Since F is transversal to any smooth curve containing A (i.e. the scheme $A \cap F$ is reduced), Z is the residual scheme of A with respect to F and for every integer u with $1 \leq u \leq z$ the residual scheme of A (resp. Z) with respect to uF has length $z+1-u$ (resp. $z-u$). Since $Q \notin \boldsymbol{h}$, A is transversal to F and $2g-2 \geq \text{length}(A)-1$, we obtain $h^0(S, \boldsymbol{I}_A(2\boldsymbol{h}+(2g-2)F)) = h^0(S, \boldsymbol{O}_S(2\boldsymbol{h}+(2g-2)F)) - \text{length}(A)$ and $h^0(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F)) = h^0(S, \boldsymbol{O}_S(2\boldsymbol{h}+(2g-2)F)) - z$. Since $h^1(S, \boldsymbol{O}_S(2\boldsymbol{h}+(2g-2)F)) = 0$, we obtain the claim.

Set $W := \boldsymbol{P}(H^0(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F)))$. As in the proof of the Claim we obtain $h^1(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F)) \leq 2$, i.e. $h^0(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-3)F)) < h^0(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F))$. Thus the linear system W has no fiber of the ruling as a base component. Since $h^0(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F)) = h^0(S, \boldsymbol{O}_S(2\boldsymbol{h}+(2g-2)F)) - z = 3g+3-z > 3g+1-z = h^0(S, \boldsymbol{O}_S(\boldsymbol{h}+(2g-2)F)) - z = h^0(S, \boldsymbol{I}_Z(\boldsymbol{h}+(2g-2)F))$, $\boldsymbol{h}$ is not a base component of W. Take a general $X \in W$. Since $h^0(S, \boldsymbol{I}_A(2\boldsymbol{h}+(2g-2)F)) < h^0(S, \boldsymbol{I}_Z(2\boldsymbol{h}+(2g-2)F))$, A is not contained in X. Hence by 1.5 it is sufficient to show that X is smooth. Since W contains the reducible element $2\boldsymbol{h}+(2g-3)F$, W has no base points outside $\boldsymbol{h} \cup F$. Since $(2\boldsymbol{h}+(2g-2)F) \cdot \boldsymbol{h} = 0$ and $\boldsymbol{h}$ is smooth an rational, we have $\boldsymbol{O}_{\boldsymbol{h}}(2\boldsymbol{h}+(2g-2)F) \cong \boldsymbol{O}_{\boldsymbol{h}}$. Since $\boldsymbol{h}$ is not a base component of W, this implies that no point of $\boldsymbol{h}$ is a base point of W. Since $(2\boldsymbol{h}+(2g-2)F) \cdot F = 2$, F is not a component of X and $Q \in F \cap X$, either X is smooth along F or X is singular at Q and $X \cap (F\backslash\{Q\}) = \varnothing$. Assume $Q \in \text{Sing}(X)$. Hence $Q \in \text{Sing}(X')$ for every $X' \in W$ by the generality of X. Take a general $Q' \in F$. Every $X' \in W$ with $Q' \in X'$ contains F because $(2\boldsymbol{h}+(2g-2)F) \cdot F = 2$ and X' has intersection multiplicity at least two with F at Q'. Call Z' the residual scheme of Z with respect to F. Since $h^0(S, \boldsymbol{I}_{Z'}(\boldsymbol{h}+(2g-3)F)) > h^0(S, \boldsymbol{I}_Z(\boldsymbol{h}+(2g-2)F)) - 1 = h^0(S, \boldsymbol{I}_{Z\cup\{Q'\}}(\boldsymbol{h}+(2g-2)F))$ (remember that F is not a base component of W), we obtain a contradiction. Hence X is smooth along F and W has no base points outside F. If $\mathrm{char}(\boldsymbol{K}) = 0$ the curve X is smooth by Bertini's theorem. If $\mathrm{char}(\boldsymbol{K}) > 0$ to apply Bertini's theorem it is necessary to check that W separates also tangent vectors outside F. Fix $Q'' \in S \backslash (\boldsymbol{h} \cup F)$ and let F'' be the fiber of the ruling of S containing Q''. Since $z < 2g - 2$, $2\boldsymbol{h} \cup (2g-3)F \cup F'' \in W$. Hence W separates the tangent vectors outside $F \cup \boldsymbol{h}$, except perhaps the "vertical" ones, i.e. the one tangent to the fibers of the ruling. Since the morphism γ associated to W is étale along F, γ is étale in a neighborhood Ω of F. Since $\dim(S \backslash \Omega) \leq 1$, this is sufficient to apply the classical dimensional count proof of Bertini's theorem and obtain the smoothness of a general $X \in W$.

PROPOSITION 1.7. *Let $P \in Y_{\mathrm{reg}}$ be a ramification point for the projection $u : Y \backslash \{v\} \to C \subset H \cong \boldsymbol{P}^{g-2}$. Then P is a Weierstrass point.*

PROOF. Set $Q := u(P)$. Let M be the osculating hyperplane of C at Q. Since C is a rational normal curve of H, M intersects C only at Q and with multiplicity $g-2$. Set $N := \langle \{v\} \cup M \rangle$. Thus N is a hyperplane of $\boldsymbol{P}^{g-1}$ intersecting Y at P with multiplicity at least $2g-4$. By the geometric form of Riemann-Roch the Cartier divisor $(2g-4)P$ is a special divisor on Y. Hence P is a Weierstrass point of Y.

(**1.8**) Here we consider the case of a smooth ramification point. Fix $P \in Y_{\mathrm{reg}}$ such that the line $\langle \{v, P\} \rangle$ is the tangent line of Y at P. Set $Q := \alpha^{-1}(P)$. Since $F \cdot Y' = 2$, the fiber $u^{-1}(v(Q))$ intersects Y' at Q with multiplicity 2. Thus for every integer $t \geq 1$ the Cartier divisor $2tQ$ of Y' is the scheme-theoretic intersection of Y' with the divisor $u^{-1}(tv(Q))$ of S. Hence we see that $2t+3$ is a gap for all integers t with $0 \leq t \leq g-3$. Since also 1 and 2 are gaps and there are exactly g gaps, the semi-group of non gaps to Y and P is given by the integers $2j+2$, $1 \leq j \leq g-3$ and the integers $z \geq 2g-2$. In full generality this was noticed by the referee of a previous version of this paper. The same referee continued with the following observations. This is remarkable because in the smooth case such a gap sequence may occur only on bielliptic curves ([3]). This may be explained in the following way, at least if Y has an ordinary double point at v and hance $p_a(Y') = g-1$ and the hyperelliptic pencil is induced by $|F|$. Set $\{Q', Q''\} := Y' \cap \boldsymbol{h}$. Consider the morphism $\phi : Y' \to \boldsymbol{P}^3$ induced by $|3f|$. Hence $\phi(Y')$ is a rational normal curve. Consider the line $L := \langle (Q'), \phi(Q'') \rangle$ and the osculating plane V of $\phi(Y')$ at $\phi(P)$. Set $\{Z\} := L \cap V$. The image of $\phi(Y')$ by the projection with center Z defines a nodal plane cubic, R', and we obtain a morphism $\phi' : Y' \to R'$; this corresponds to $|6P|$ on Y. Hence this is a kind of bielliptic structure on Y. The case considered in Theorem 1.6 corresponds to case (c) of Lemma 0.2 in [3] for the integer $z-g$. In the case of smooth bielliptic curves all ramification points are Weierstrass points but there are exactly two gap sequences for such ramification points (see e.g. the introduction of [3]).

REMARK 1.9. The referee of a previous version of this paper remarked that the arguments of 1.8 show that if $P \in Y_{\mathrm{reg}} \backslash \{v\}$ is not a ramification point of u, then all integers t with $1 \le t \le g-1$ are always gaps for P.

Now we will prove the existence of Gorenstein trigonal curves whose trigonal pencil is not induced by a line bundle and with prescribed singularities outside the vertex v of the minimal degree cone $T \subset \boldsymbol{P}^{g-1}$ with $Y \subset T$. By [18, Th. 3.2] any such curve is associated to an affine curve $\{f(x,y)=0\} \subset \boldsymbol{A}^2$ with $f(x,y) = c_2(x)y^2 + c_1(x)y + c_0(x)$ with c_0, c_1 and c_2 polynomials, $\deg(c_2) \le 2$, $\deg(c_1) \le g$, $\deg(c_0) \le 2g-2$ and such that equality holds for at least one degree and $f(x,y)$ is irreducible; if $c_2 \equiv 0$, then the base point has multiplicity bigger than two. Viceversa, any such polynomial gives the canonical model of a trigonal Gorenstein curve $Y \subset T$ with non-locally free trigonal pencil. To obtain the following existence theorem it will be sufficient to take the very particular case $c_1 \equiv 0$.

THEOREM 1.10. *Fix an integer $m \ge 0$ and positive integers $g, k, \delta_1, \dots, \delta_k$ with $\sum_{1 \le i \le k} 2\delta_i + m \le 2g-4$. For every integer i with $1 \le i \le k$ take a label "tacnode with invariant δ_i" or "cusp with invariant δ_i"; assume that exactly m labels say "cusp!". Then there exists an integral genus g Gorenstein canonical curve $Y \subset T$ with exactly k singular points, say $P_1, \dots, P_k$, each P_i tacnode with invariant δ_i or cusp with invariant δ_i according to its label. Furthermore, the set of all such curves, Y, has an irreducible component, Γ, of dimension at least $2g-3-\sum_{1 \le i \le k} 2\delta_i - m$ whose general member has an ordinary double point at the vertex $v \in T$.*

PROOF. Fix k distinct numbers $x_1, \dots, x_k$. We take $c_1 \equiv 0$, i.e. we take Y corresponding to an irreducible polynomial $f(x,y) = c_2(x)y^2 + c_1(x)y + c_0(x)$ and as P_i the point corresponding to $(x_i, 0) \in \boldsymbol{A}^2$. It is sufficient to take $c_0(x)$ of degree $2g-2$ and with x_i root of multiplicity $2\delta_i$ if P_i has as label "tacnode with invariant δ_i" and with x_i root of multiplicity $2\delta_i + 1$ if P_i has as label "cusp with invariant δ_i". For fixed $x_1, \dots, x_k$ and fixed $c_1 \equiv 0$ the set of all such c_0, c_2 has codimension $\sum_{1 \le i \le k} 2\delta_i + m$ in the vector space of all (c_0, c_2) with $\deg(c_0) \le 2g-2$ and $\deg(c_2) \le 2$. Since $h^0(\boldsymbol{P}^1, \boldsymbol{O}_{\boldsymbol{P}^1}(2)) + h^0(\boldsymbol{P}^1, \boldsymbol{O}_{\boldsymbol{P}^1}(2g-2)) - 1 - \dim(\mathrm{Aut}(\boldsymbol{P}^1)) = 2g-3$, moving $x_1, \dots, x_k$ we obtain the existence of the component Γ with $\dim(\Gamma) \ge 2g-3-\sum_{1 \le i \le k} 2\delta_i - m$. The last assertion is easy taking $x_1, \dots, x_k$ general and then, for fixed $x_1, \dots, x_k$, Y' sufficiently general (see 1.1); here we use $\dim(\Gamma) \neq 0$.

2. Locally Free Trigonal Pencil

In this section we assume that the trigonal pencil of Y is induced by a spanned $L \in \mathrm{Pic}^3(Y)$. By [18, Th. 3.5], the canonical model of Y lies on a minimal degree surface scroll $S \subset \boldsymbol{P}^{g-1}$, $S \cong F_e$, with $g-e$ even and (for $g \geq 5$)) $(g-4)/3 \leq (g-2-e)/2 \leq (g-2)/2$, i.e. the Maroni invariant $(g-2-e)/2$ of Y is one of the Maroni invariants of smooth genus g trigonal curves. We assume $g \geq 6$. Set $q := p_a(X)$.

(**2.1**) Here we study the gap sequences of an ordinary ramification point, P, of L. Hence $P \in Y_{\mathrm{reg}}$ and there exists $Q \in Y$, $Q \neq P$, with $2P + Q \in |L|$. For the case of smooth trigonal curves, see [4, 5, 13]. Here we do not make any restriction on $\mathrm{char}(\boldsymbol{K})$. Since $2P$ is a Cartier divisor of Y and $L \in \mathrm{Pic}(Y)$, Q is a Cartier divisor of Y, i.e. $Q \in Y_{\mathrm{reg}}$. By [18, Th. 3.5], the canonical model of Y lies on a minimal degree surface scroll $S \subset \boldsymbol{P}^{g-1}$ and the possible Maroni invariants of S are the same as in the smooth case. Hence we may copy [5, §6]. In particular in our situation we have verbatim Theorem 8, Lemma 9, Lemma 10, Lemma 11, Proposition 12, Theorem 13 and Remark 14 of [5]; for Lemma 11 it is used [4, Notation 2.10], which in turns depends on [4, Cor. 2.7], and this is OK in our set-up; for Theorem 13 and Remark 14 we need [4, Lemma 5], which is OK in our set-up dropping the word "smooth", i.e. taking Y only integral.

(**2.2**) Here we study the gap sequences of a total ramification point, P, of L. Hence $P \in Y_{\mathrm{reg}}$ and $3P \in |L|$. Since $(g-4)/3 \leq (g-2-e)/2 \leq (g-2)/2$, we may copy [4] and obtain [4, Lemma 2.12] i.e. that the only possible gap sequences are the ones described in the first page of [4] and called there of Type I or of Type II. Now we assume $\mathrm{char}(\boldsymbol{K}) \neq 2, 3$. Remember that $g \geq 6$ and hence the trigonal pencil is unique ([1, Lemma 2.6]). We will try to follow the notation of [4]; hence m is the last integer with $h^1(Y, L^{\otimes m}) \neq 0$ and $n = g - m - 1$. Call t the number of total ramification points of L and $t(\mathrm{II})$ the number of total ramification points of Type II of L. Since $\pi^*(L)$ induces a g^1_3 on X and $\mathrm{char}(\boldsymbol{K}) \neq 2, 3$, we have $0 \leq t \leq q + 2$ (Riemann-Hurwitz). We have verbatim [4, Prop. 2.14], i.e. P has Type II if and only if it is a base point of $|\omega_Y \otimes L^{\otimes -m}|$. We have [4, Remark 2.15]. Since $\deg(\omega_Y \otimes L^{\otimes -m}) = 3n - g - 1$, from [5, Prop. 2.14], we obtain at once that [4, Th. 2.17], holds i.e. we have the following result.

Proposition 2.3. *We have* $0 \leq t(\mathrm{II}) \leq 3n - g - 1$.

(**2.4**) Here we study the possible gap sequences of the smooth Weierstrass points which are not ramification points. If Y is smooth the corresponding problem was solved in [14] (if $\mathrm{char}(\boldsymbol{K}) = 0$) and then in arbitrary characteristic in [20]. Fix $P \in Y_{\mathrm{reg}}$. By [18, Th. 3.5], the canonical model of Y lies in a minimal degree surface scroll whose possible Maroni invariants are the same as for smooth trigonal curves with the same genus. Hence we may copy [20]. We stress that we consider only Weierstrass points of Y which are smooth points of Y. The proof of [20, Th. 2.5], works verbatim and hence we obtain in arbitrary characteristic the possible gap sequences of the ramification points of $|L|$. The proof of [20, Th. 3.7], works verbatim and gives not only the possible gap sequences of smooth non-ramification Weierstrass points of Y, but also several geometric conditions to determine for a given $P \in Y_{\mathrm{reg}}$ what is its gap sequence.

(**2.5**) Here we consider a trigonal Gorenstein non-hyperelliptic curve Y of genus $g \geq 6$ whose trigonal pencil, $|L|$, is induced by a line bundle and study the singular points of Y from the point of view of Weierstrass points. Since S is smooth, Y has only planar singularities. Fix $P \in \mathrm{Sing}(Y)$. Let $F := u^{-1}(v(P))$ be the fiber of the ruling of S containing P. It is easy to check that one of the following cases must occur:

(i) Y has multiplicity 2 at P and F is not in the tangent cone of Y at P;

(ii) Y has multiplicity 2 at P and F is in the tangent cone of Y at P;

(iii) Y has multiplicity 3 at P and F is not in the tangent cone of Y at P.

In cases (i) and (ii) P is either a tacnode with invariant $\delta \geq 1$ or a cusp with invariant $\delta \geq 1$. For every integer $g \geq 5$ and every integer e with $g - e$ even and $0 \leq 3e \leq g + 2$ there exists an integral trigonal curve $Y \subset F_e \subset \boldsymbol{P}^{g-1}$ with a unique singular point of any of the types (i), (ii) and (iii).

To show that all cases discussed in 2.5 may arise we prove the following result; we stress that much better statements may be proved with the same method, just with more cumbersome numerical computations; for an hint of a possible statement for more than one singular point, see 1.9.

PROPOSITION 2.6. *Assume char($\boldsymbol{K}$) $= 0$. Fix integers g, e, δ with $g - e$ even, $\delta > 0$, $0 \leq 3e \leq g + 2$, and $g \geq 3e + 4\delta - 1$. Fix a label "tacnode with invariant δ and F not in its tangent cone", "cusp with invariant δ and F not in its tangent cone" or "ordinary triple point". In the latter case assume $g \geq 3e + 5$. Then there exists an integral Gorenstein curve $Y \subset F_e \subset \boldsymbol{P}^{g-1}$ with a unique singular point, P, whose isomorphism type is the one prescribed by the label and such that $E(P) = 0$.*

PROOF. Fix $P \in F_e$ and a line D contained in the projective tangent space $T_P F_e \subset \boldsymbol{P}^{g-1}$ with $P \in D$ and $D \neq F$, where F is the line of the ruling, π, of F_e

containing P. If $e > 0$ assume $P \notin \boldsymbol{h}$, where $\boldsymbol{h}$ is a minimal degree section of the ruling. Take $\boldsymbol{h}$ and a fiber, f, of the ruling as a basis of $\mathrm{Pic}(F_e) \cong \boldsymbol{Z}^{\oplus 2}$. Fix germs C_i, $1 \le i \le 3$, of curves on F_e such that C_1 has a tacnode with invariant δ at P and D as tangent line at P, C_2 has at P a cusp with invariant δ at P and D as tangent line at P and C_3 has at P an ordinary planar triple point. Fix local (holomorphic or formal) coordinates x, y near P such that C_1 (resp. C_2) has equation $y^2 = x^{2\delta}$ (resp. $y^2 = x^{2\delta+1}$). Let $Z(1)$ be the zero-dimensional subscheme of F_e with $Z(1)_{\mathrm{red}} = \{P\}$ and with $(y^2, yx^{\delta}, x^{2\delta})$ as ideal sheaf. Let $Z(2)$ be the zero-dimensional subscheme of F_e with $Z(1)_{\mathrm{red}} = \{P\}$ and with $(y^2, yx^{\delta+1}, x^{2\delta+1})$ as ideal sheaf. Let $Z(3)$ be the second infinitesimal neighborhood of P in F_e, i.e. take $(\boldsymbol{I}_P)^3$ as ideal sheaf of $Z(3)$. The canonically embedded trigonal curves contained in F_e are in the linear system $|3\boldsymbol{h} + \psi F|$ of F_e with $\psi = g/2 + (3/2)e + 1$ (just use the adjunction formula).

First Claim: We have $h^1(F_e, \boldsymbol{I}_{Z(i)}(3\boldsymbol{h} + \psi F)) = 0$ for $1 \le i \le 3$.

Proof of the First Claim: (a) Here we handle $Z(3)$. We have $h^1(F_e, \boldsymbol{I}_{Z(3)}(3\boldsymbol{h} + \psi F)) \le h^1(F_e, \boldsymbol{O}_{F_e}(3\boldsymbol{h} + (\psi - 3)F))$; we have $h^1(F_e, \boldsymbol{O}_{F_e}(3\boldsymbol{h} + (\psi - 3)F)) = 0$ because $\psi - 3 \ge 3e - 1$ (e.g. use that $\pi_*(\boldsymbol{O}_{F_e}(3\boldsymbol{h}) \cong \boldsymbol{O}_{\boldsymbol{P}^1} \oplus \boldsymbol{O}_{\boldsymbol{P}^1}(-e) \oplus \boldsymbol{O}_{\boldsymbol{P}^1}(-2e)$ and apply the projection formula ([10, Ex. II.5.1])).

(b) Here we handle $Z(1)$ and $Z(2)$. Notice that $Z(1)$ and $Z(2)$ are contained in $(2\delta + 1)F$, $\mathrm{leght}(Z(1) \cap F) = \mathrm{length}(Z(2) \cap F) = 2$, $\deg(\boldsymbol{O}_F(3\boldsymbol{h})) = 3 > 0$ and that $h^1(F_e, \boldsymbol{O}_{F_e}(3\boldsymbol{h} + (\psi - 2\delta - 1)F)) = 0$ because $y \ge 3e + 2\delta$.

Our second claim is that a general curve $Y \in |3\boldsymbol{h} + yF|$ with $Z(1) \subset Y$ (resp. $Z(2) \subset Y$, resp. $Z(3) \subset Y$) has at P a tacnode with invariant δ and F not as tangent line (resp. a cusp with invariant δ and F not as tangent line, resp. an ordinary triple point). To check the second claim we will use $h^1(F_e, \boldsymbol{I}_{Z(i)}(3\boldsymbol{h} + yF)) = 0$ for $1 \le i \le 3$ (First Claim) to apply [8, Th. 3.7 (ii)]. For the singularities of C_1, C_2 and C_3 the theory of equianalytic or equisingular deformation coincide and any equisingular deformation is trivial; for instance any germ of planar curve singularity near an ordinary triple point and with the same topological type is an ordinary triple point. The second claim for $Z(1)$ follows from the First Claim, [9, Examples 1 and 2 before Definition 2.12] and [8, Th. 3.7 (ii)]. The second claim for $Z(2)$ follows from the First Claim, [9, Example 3 before Definition 2.12] and [8, Th. 3.7 (ii)]. The second claim for $Z(3)$ follows from the First Claim, [9, Example 2 after Definition 2.3], Lemma 2.4, and [8, Th. 3.7]. For the assertion on $E(P)$, repeat the proof of 1.2; for the tacnode and cusp case, use that we may take D general; for the triple point case use that we may take as tangent cone to Y at P three lines of T_PF_e each of which may be considered as a general line of T_PF_e through P.

3. Birational Trigonal Pencils

Assume $g \geq 2$, Y Gorenstein and that Y has a trigonal complete pencil, $|L|$, whose associated rational map, say $u : X \to \boldsymbol{P}^1$, is either birational or purely inseparable. Hence $X \cong \boldsymbol{P}^1$. If u is not separable, then either $\mathrm{char}(\boldsymbol{K}) = 2$ or $\mathrm{char}(\boldsymbol{K}) = 3$ because $\deg(u) \leq 3$. Call L the associated spanned rank 1 torsion free sheaf on Y with $\deg(L) = 3$. We may assume L spanned and $\deg(L) = 3$, because the case $\deg(L) = 3$ and L not spanned is reduced to the case of a spanned L' with $\deg(L') \leq 2$ which is completely described by [6, Th. A of the Appendix with J. Harris], and [12, Prop. 1.1].

Remark 3.1. We have $\deg(u) = \deg((\pi^*(L)/\mathrm{Tors}(\pi^*(L)))$. In particular $\deg(u) = 3$ if and only if $L \in \mathrm{Pic}^3(Y)$ ([6, Lemma 1 of the Appendix with J. Harris].

(**3.2**) Here we consider the case $\deg(u) = 1$. Here we do not have any restriction on $\mathrm{char}(\boldsymbol{K})$. By [18, Th. 3.5], $Y \subset T \subset \boldsymbol{P}^{g-1}$, T cone with vertex $\boldsymbol{v}$ and as base a rational normal curve, C, of a hyperplane of $\boldsymbol{P}^{g-1}$. Since $\deg(u) = 1$ and $\deg(C) = g-2$, Y has multiplicity g at $\boldsymbol{v}$. Since $\deg(u) = 1$, any two divisors of the pencil must contain $\boldsymbol{v}$ with "multiplicity" 2. With the notation of section one for the blowing-up $\alpha : S \to T$ of T at $\boldsymbol{v}$, we have $S \cong F_{g-2}$ and $Y' \in |\boldsymbol{h} + (2g-2)F|$, where Y' is the strict transform of Y in S. We have $X \cong Y' \cong \boldsymbol{P}^1$ and $Y \backslash \{\boldsymbol{v}\}$ is smooth. Viceversa, for any irreducible $Y' \in |\boldsymbol{h} + (2g-2)F|$ the curve $\alpha(Y') \subset \boldsymbol{P}^{g-1}$ has degree $2g-2$, multiplicity g at $\boldsymbol{v}$ and it is non-degenerate. By [18, Formula 3.1] (with d instead of $d-1$), we have $p_a(\alpha(Y')) = g$. Intersecting $\alpha(Y')$ with a hyperplane we obtain a $(g-1)$-dimensional family of rationally equivalent Cartier divisor of degree $2g-2$. Hence $\boldsymbol{O}_{\alpha(Y')}(1) \cong \omega_{\alpha(Y')}$ and $\alpha(Y')$ is Gorenstein, i.e. $\alpha(Y')$ is a trigonal curve with degree 1 associated rational map. Thus the set of all solutions (i.e. of all trigonal curve with degree 1 associated rational map) is parametrized by an irreducible unirational variety of dimension $\dim(|\boldsymbol{h} + (2g-2)F|)$. Two points in the parameter space differing by an element of $\mathrm{Aut}(\boldsymbol{P}^1)$ corresponds to isomorphic trigonal curves. We do not claim that, up to elements of $\mathrm{Aut}(\boldsymbol{P}^1)$, this is a generically finite-to-one parametrization.

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