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ANALYTIC REGULARITY OF SOLUTIONS TO THE CAUCHY PROBLEM FOR DEGENERATE PARABOLIC EQUATIONS

Dedicated to Professor Kunihiko Kajitani on his sixtieth birthday

By

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1. Introduction

It is known that the solution of the Cauchy problem for heat equation is analytic with respect to the space variable x for $t>0$, if the initial value belongs to $L^2(\boldsymbol{R}_x^n)$. The purpose of this paper is to show that the solution of the Cauchy problem for degenerate parabolic equations is analytic with respect to the space variable x for $t>0$, if the initial values are in Gevrey classes.

We shall consider the following Cauchy problem for degenerate parabolic equations on $(0,T)\times\boldsymbol{R}^n$ $(T>0)$,

$$\begin{cases} P(t,x,\partial_t,D_x)u(t,x)=f(t,x), & (t,x)\in(0,T)\in\boldsymbol{R}^n, \\ \partial_t^j u(0,x)=u_j(x), & x\in\boldsymbol{R}^n,\ j=0,\dots,m-1, \end{cases} \tag{1.1}$$

where $D_x=-i\partial_x$ and

$$P(t,x,\partial_t,D_x)=\partial_t^m+\sum_{j=1}^{m}\sum_{\alpha:finite}a_{j\alpha}(t,x)D_x^\alpha\partial_t^{m-j}. \tag{1.2}$$

We assume that P is degenerate at $t=0$, namely, the coefficients $a_{j\alpha}(t,x)$ satisfy

$$a_{j\alpha}(t,x)=t^{\sigma(j\alpha)}b_{j\alpha}(t,x),$$

where $\sigma(j\alpha)$ are nonnegative integers and $b_{j\alpha}(t,x)$ belong to $C^\infty([0,T];\gamma^{\langle s_0\rangle}(\boldsymbol{R}^n))$. Denote by $\gamma^{\langle s\rangle}(\boldsymbol{R}^n)$ the Gevrey class with exponent s (>0), that is, the set of all

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functions $a(x)$ defined on $\boldsymbol{R}^n$ such that for any $A>0$ there is a constant $C_A>0$ fulfilling

$$\sup_{x\in\boldsymbol{R}^n}|D_x^\alpha a(x)|\le C_A A^{|\alpha|}|\alpha|!^s$$

for every multi-index $\alpha\in\boldsymbol{N}^n$, where $\boldsymbol{N}=\{0,1,2,\dots\}$.

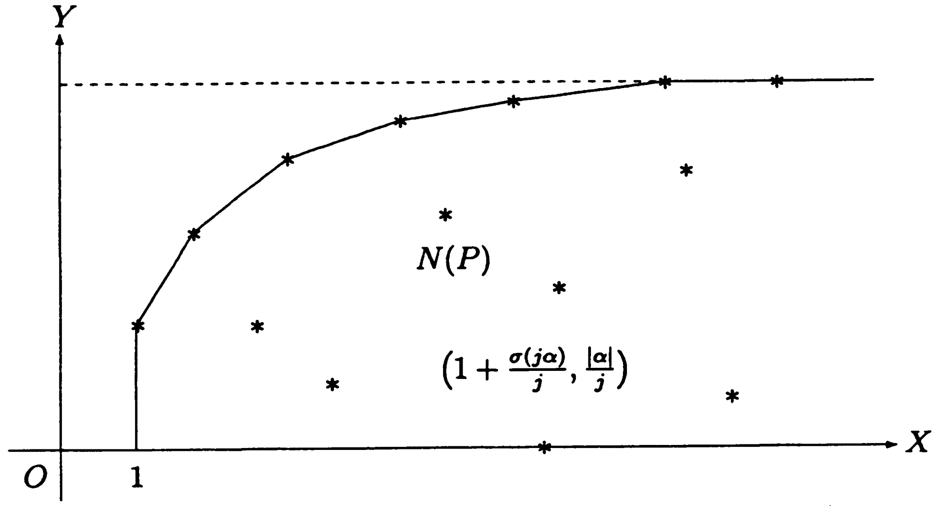
M. Mikami [6] mentions sufficient conditions to be wellposed in H^∞ by using Newton's polygon when the coefficients of P are independent of the space variable x. We shall introduce Newton's polygon associated with degenerate parabolic equation (1.2) (see S. Gindikin-L. R. Volevich [2]).

DEFINITION 1.1. *Let* $\tau(P)=\{(j,\alpha)\in\boldsymbol{N}^{n+1};b_{j\alpha}(0,x)\not\equiv0\}$ *and* $v(P)=\{(1+\sigma(j\alpha)/j,|\alpha|/j)\in\boldsymbol{R}_+^2;\ (j,\alpha)\in\tau(P)\}$, *where* $\boldsymbol{R}_+^2=[0,\infty)\times[0,\infty)$. *Denote by* $N(P)$ *the smallest convex polygon in* $\boldsymbol{R}_+^2$ *possessing following properties,*

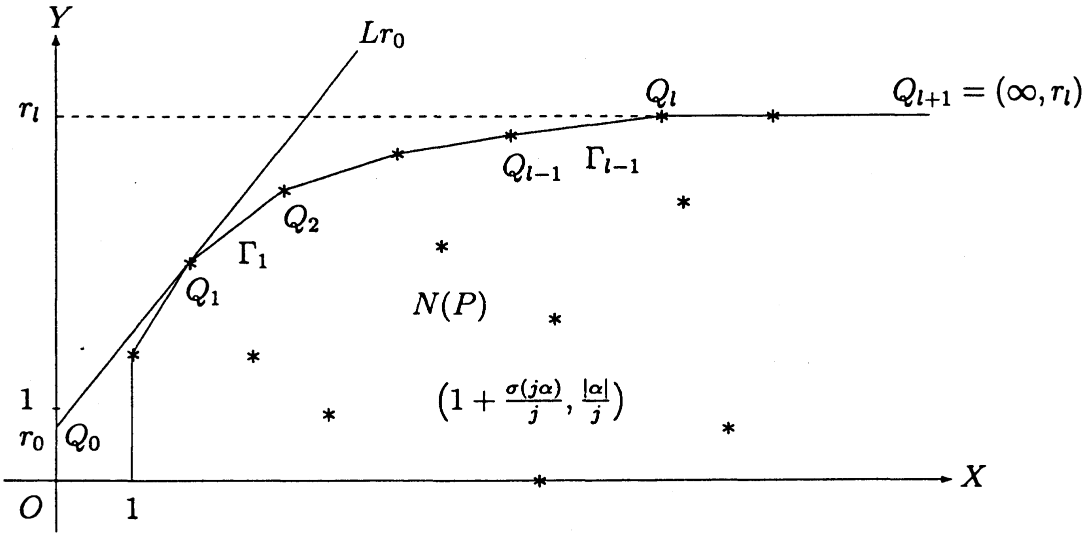
(i) $v(P)\subset N(P)$,

(ii) *if* $(q,r)\in\boldsymbol{R}_+^2$, $(q',r')\in N(P)$, $q'\le q$ *and* $r\le r'$, *then* $(q,r)\in N(P)$.

$N(P)$ is called Newton's polygon associated with P, and $N(P)$ is represented in the following figure where, for example, $*=(1+\sigma(j\alpha)/j,|\alpha|/j)$ lie on it.



Here, as the figure below, we introduce some notation for Newton's polygon $N(P)$. For a number $r_0>0$, let L_{r_0} be the line passing the point $Q_0=(0,r_0)$ which is tangent to $N(P)$. Denote by $Q_1=(1+q_1,r_1)$ the vertex of $N(P)$ such that $q_1\ge q$ and $r_1\ge r$ hold if $(1+q,r)$ belongs to $N(P)$ and Lr_0. And denote by $Q_i=(1+q_i,r_i)$, $i=1,\dots,l$ the vertices of $N(P)$ indexed in the clockwise direction beginning with Q_1. We note that Q_i can be determined by r_0. Conversely, r_0 can be chosen for Q_i which is defined beforehand.



For such picked $Q_i = (1 + q_i, r_i)$, put $q_0 = -1$, $q_{l+1} = \infty$ and $r_{l+1} = r_l$ for the sake of convenience, and let σ_i $(i = 0, \dots, l)$ stand for the slopes of the sides Q_iQ_{i+1}, i.e., $\sigma_i = (r_{i+1} - r_i)/(q_{i+1} - q_i)$, $i = 0, \dots, l$. It is evident to get the following inequalities by the property (ii) of Newton's polygon $N(P)$

$$0 \le q_1 < \cdots < q_l, \quad r_0 < r_1 \cdots < r_l, \quad \sigma_0 > \sigma_1 > \cdots > \sigma_l = 0.$$

Moreover denote by Γ_i the sides joining the two vertices Q_i, Q_{i+1} for $i = 1, \dots,$ $l - 1$, i.e., $\Gamma_i = \overline{Q_iQ_{i+1}}$. Besides let $\Gamma = \bigcup_{i=1}^{l-1} \Gamma_i$ if $l \ge 2$ and $\Gamma = Q_1$ if $l = 1$. For Γ we shall define the principal part of P as follows.

$$P_\Gamma(t, x, \lambda, \xi) = \lambda^m + \sum_{(1+\sigma(j\alpha)/j, |\alpha|/j) \in \Gamma} t^{\sigma(j\alpha)} b_{j\alpha}(0, x) \xi^\alpha \lambda^{m-j} \tag{1.3}$$

for $t \ge 0$, $x, \xi \in \boldsymbol{R}^n$ and $\lambda \in \boldsymbol{C}$. Further we define a weight function associated with $N(P)$ as follows.

$$w_\Gamma(t, \xi) = \sum_{i=1}^{l} t^{q_i} |\xi|^{r_i}. \tag{1.4}$$

Definition 1.2. *The operator P is said to be Γ-parabolic at $t = 0$ if there exist constants $C_0 > 0$ and $\delta_0 > 0$ such that P_Γ satisfies the inequality below*

$$|P_\Gamma(t, x, \lambda, \xi)| \ge C_0(|\lambda| + w_\Gamma)^m \tag{1.5}$$

for $t \ge 0$, $x, \xi \in \boldsymbol{R}^n$ and $\lambda \in \boldsymbol{C}$ with $\operatorname{Re} \lambda > -\delta_0 w_\Gamma$.

We shall introduce the function spaces in which we consider the Cauchy problem (1.1). For $s > 0$ and $\rho > 0$, denote by $H_\rho^{\langle s\rangle}$ the set of functions whose element $u \in L^2(\boldsymbol{R}^n)$ such that

$$e^{\rho|\xi|^{1/s}}\hat{u}(\xi) \in L^2(\boldsymbol{R}_\xi^n), \tag{1.6}$$

where $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$ and $\hat{u}$ means the Fourier transform of u. Denote by $H^{\langle s\rangle}$ the set of functions of $u(x) \in L^2(\boldsymbol{R}_x^n)$ satisfying (1.6) for any ρ (> 0). Denote $H^{\langle\infty\rangle} = H^\infty = \bigcap_{s\geq 0} H^s$, where $H^s = \left\{f(x) \in \mathscr{S}'; \langle\xi\rangle^s\hat{f}(\xi) \in L^2(\boldsymbol{R}_\xi^n), \langle\xi\rangle = \sqrt{1+|\xi|^2}\right\}$. We know the following theorem on the wellposedness in $H^{\langle s\rangle}$ of the Cauchy problem (1.1).

THEOREM 1.3 (Theorem 3 in K. Kajitani-M. Mikami [3]). *Assume that the differential operator P with the coefficients $b_{j\alpha}(t,x)$ in $C^\infty([0,T];\gamma^{\langle s_0\rangle}(\boldsymbol{R}^n))$ is degenerate and Γ-parabolic at $t=0$. Then there is $T>0$ such that the Cauchy problem* (1.1) *is wellposed in $H^{\langle s\rangle}$ for s satisfying $1 \leq s_0 \leq s \leq r_0^{-1}$ if $r_0 > 0$ and $1 \leq s_0 \leq s \leq \infty$ if $r_0 = 0$, that is, for any $u_j(x) \in H^{\langle s\rangle}$ and $f(t,x) \in C^\infty([0,T];H^{\langle s\rangle})$ there exists a unique solution $u(t,x) \in C^\infty([0,T];H^{\langle s\rangle})$ of the Cauchy problem* (1.1).

We need the above theorem of version of pseudo-differential operators in order to prove our main Theorem 1.4 below. Denote by the symbol class $\gamma^\kappa S^m$ $(\kappa > 0, m \in \boldsymbol{R})$ the set of all functions $p(x,\xi) \in C^\infty(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ such that for any $r > 0$ there exists a constant $C_r > 0$ satisfying

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_r r^{|\alpha+\beta|}|\alpha+\beta|!^\kappa\langle\xi\rangle^{m-|\alpha|},$$

for every multi-indices $\alpha, \beta \in \boldsymbol{N}^n$, where $a_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta a(x,\xi)$. Theorem 1.3 is valid for the following pseudo-differential operator

$$P(t,x,\lambda,\xi) = \lambda^m + \sum_{j=1}^m \sum_{\alpha:\mathit{finite}} t^{\sigma(j\alpha)}b_{j\alpha}(t,x)\xi^\alpha\lambda^{m-j} + \sum_{j=1}^m \sum_{k:\mathit{finite}} t^{\sigma(jk)}b_{jk}(t,x,\xi)\langle\xi\rangle^k\lambda^{m-j},$$

where $\sigma(jk)$ are nonnegative integers, $b_{jk}(t,x,\xi)$ belong to $C^\infty([0,T];\gamma^{s_0}S^{d_{jk}})$, $\sigma(jk)$ and d_{jk} satisfy

$$\bigcup_{i=1}^m \bigcup_{k:\mathit{finite}} \left(1 + \frac{\sigma(jk)}{j}, \frac{d_{jk}}{j}\right) \subset\subset N(P).$$

Our main result is the following

THEOREM 1.4. *Assume that $1 = s_0 < s \leq r_0^{-1}$ if $r_0 > 0$ and $1 = s_0 < s \leq \infty$ if $r_0 = 0$ moreover the differential operator P with the coefficients $b_{j\alpha}(t,x)$ belong-*

ing to $C^\infty([0,T];\gamma^{\langle 1\rangle}(\boldsymbol{R}^n))$ *is degenerate and* Γ*-parabolic at* $t=0$. *Then for any* $u_j(x)\in H^{\langle s\rangle}$ *and* $f(t,x)\in C^\infty([0,T];H^{\langle 1\rangle})$ *there exists a unique solution* $u(t,x)\in C^\infty(([0,T];H^{\langle 1\rangle}_{\varepsilon t^\delta})$ $(\varepsilon>0)$ *of the Cauchy problem* (1.1), *where*

$$\delta=1+\max_{\substack{0\le i\le l-1\\ 1\le j\le m}}\left\{\left[j\left(\frac{1-r_i}{\sigma_i}+q_i+1\right)\right],\max_{\substack{1\le h\le j-1\\ \alpha:\text{finite}}}\left[\frac{|\alpha|+j-h-jr_i}{\sigma_i}+jq_i-\sigma(h\alpha)+j-h\right]\right\} \tag{1.7}$$

Here, denote by $[p]$ *the maximal integer not greater than* p.

Here, we should mention that $H^{\langle 1\rangle}$ is contained in the set of analytic functions by Lemma 2.2. Consequently Theorem 1.4 shows the analytic regularity of solutions to the Cauchy problem (1.1). To prove Theorem 1.4 we transform the operator P by exponential mapping and apply Theorem 1.3 to the transformed operator. This proof is given in the section 3. We shall consider the following Cauchy problem with the coefficients depending only on t and $2l>k$ $(2l,k\in\boldsymbol{N})$

$$\begin{cases}\partial_t u(t,x)+t^{2l}a(t)D_x^2u(t,x)+t^kb(t)D_xu(t,x)=0, & (t,x)\in(0,T)\times\boldsymbol{R},\\ u(0,x)=u_0(x).\end{cases} \tag{1.8}$$

The following simple example satisfies the conditions in Theorem 1.4 but we can show easily the analyticity of solutions without Theorem 1.3. In the section 2 we shall explain this fact.

EXAMPLE 1.5. *Let* $1<s\le(2l-k)/(2l-2k-1)$. Assume that $\mathrm{Re}\,a(t)\ge C>0$ then for any $u_0(x)\in H^{\langle s\rangle}$ there is a unique solution $u(t,x)\in C^\infty((0,T];H^{\langle 1\rangle}_{\varepsilon t^{k+1}})$ $(\varepsilon>0)$ of the Cauchy problem (1.8). Consequently for any $\varepsilon>0$ there is $C_\varepsilon>0$ such that

$$|D_x^\alpha u(t,x)|\le C_\varepsilon(\varepsilon t^{k+1})^{-|\alpha|}|\alpha|! \tag{1.9}$$

for $0<t\le T$, $x\in\boldsymbol{R}$.

2. An Example

We shall investigate the case of (1.8) and prove the statement in Example 1.5 in this section. First of all, we transform $u(t,x)$ into $v(t,x)$ by

$$v(t,x)=e^{\varepsilon t^{k+1}\langle D_x\rangle}u(t,x)=(2\pi)^{-1}\int_{\boldsymbol{R}}e^{ix\cdot\xi+\varepsilon t^{k+1}\langle\xi\rangle}\hat{u}(t,\xi)\,d\xi. \tag{2.1}$$

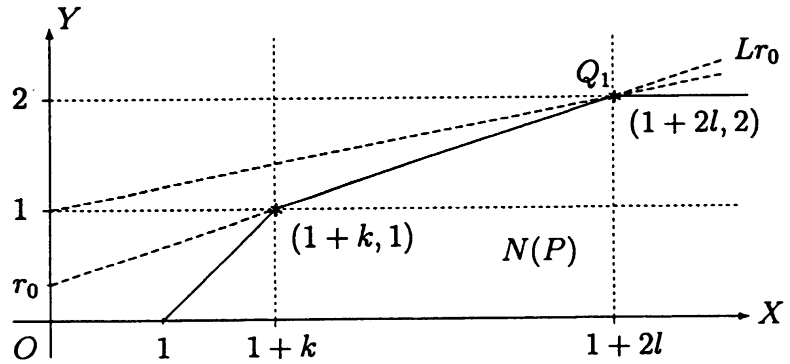
Then, since

$$\partial_t v = \varepsilon(k+1)t^k\langle D_x\rangle e^{\varepsilon t^{k+1}\langle D_x\rangle}u + e^{\varepsilon t^{k+1}\langle D_x\rangle}\partial_t u$$
$$= -t^{2l}a(t)D_x^2 v - t^k\{b(t)D_x - \varepsilon(k+1)\langle D_x\rangle\}v,$$

(1.8) is transformed into the Cauchy problem below

$$\begin{cases} \partial_t v(t,x) + t^{2l}a(t)D_x^2 v(t,x) + t^k\{b(t)D_x - \varepsilon(k+1)\langle D_x\rangle\}v(t,x) = 0, \\ v(0,x) = u_0(x). \end{cases} \tag{2.2}$$

The Cauchy problem (2.2) obviously has same degenerate form at $t=0$ as (1.8). Now we shall observe Newton's polygon associated with (2.2). In this case, Newton's polygon associated with (2.2) consist of the two points $(1+k,1)$ and $(1+2l,2)$ on $\boldsymbol{R}_+^2$. Putting $Q_1 = (1+2l,2)$, we can take $r_0 = (2l-2k-1)/(2l-k)$. The operator P for (1.8) is Γ-parabolic at $t=0$ since $\operatorname{Re} a > 0$. The Newton's polygon associated with (2.2) is represented in following picture.



Then, we get

LEMMA 2.1. *Let* $1 < s \le (2l-k)/(2l-2k-1)$. *Assume that* $\operatorname{Re} a(t) > 0$ *on the Cauchy problem* (2.2). *Then for any* $u_0(x) \in C^\infty([0,T];H^{\langle s\rangle})$ *there exists an unique solution* $v(t,x) \in C^\infty([0,T];H^{\langle s\rangle})$ *on* $x \in \boldsymbol{R}$.

Though this lemma is proved in [3], we shall give a simple proof. We transform $v(t,x)$ into $w(t,x)$ by

$$w(t,x) = e^{-\Lambda(t,D_x)}v(t,x) = (2\pi)^{-1}\int_{\boldsymbol{R}} e^{ix\cdot\xi - \Lambda(t,\xi)\langle\xi\rangle}\hat{u}(t,\xi)\,d\xi,$$

where

$$\Lambda(t,\xi) = -\frac{\sigma M}{r_0}\{(t+\langle\xi\rangle_h^{-\sigma})^{-r_0/\sigma} + \langle\xi\rangle_h^{1/s}\}, \quad \sigma = \frac{1}{2l-k}, \tag{2.3}$$

$\langle\xi\rangle_h = \sqrt{h^2+|\xi|^2}$ and $h \ge 1$, $M > 0$ are some large parameters. Putting

$$\Lambda_t(t,\xi) = \partial_t\Lambda(t,\xi) = M(t+\langle\xi\rangle_h^{-\sigma})^{-1-r_0/\sigma}, \tag{2.4}$$

we can write

$$\begin{aligned}\partial_t w &= -\Lambda_t(t,D_x)e^{-\Lambda(t,D_x)}v + e^{-\Lambda(t,D_x)}\partial_t v \\ &= -\Lambda_t(t,D_x)w + t^{2l}a(t)D_x^2 w + t^k\{b(t)D_x - \varepsilon(k+1)\langle D_x\rangle\}w.\end{aligned}$$

Therefore, we have

$$\begin{cases}\partial_t\hat{w}(t,\xi) = [-\Lambda_t(t,\xi) - t^{2l}a(t)D_x^2 - t^k\{b(t)D_x - \varepsilon(k+1)\langle D_x\rangle\}]\hat{w}(t,\xi), \\ \hat{w}(0,\xi) = e^{-\Lambda(0,\xi)}\hat{u}_0(\xi).\end{cases} \tag{2.5}$$

In advance, by noting $r_0 \le s^{-1} w(0,\cdot) \in L^2$ holds because of $u_0 \in H^{\langle s\rangle}$. It is enough for the proof of Lemma 2.1 to derive $w(t,\cdot) \in L^2$. We show $w(t,\cdot) \in L^2$ by using that if $(d/dt)|\hat{w}(t,\xi)|^2 = 2\,\mathrm{Re}\,\hat{w}'\bar{\hat{w}} \le 0$ holds then $|\hat{w}(t,\xi)|^2 \le |\hat{w}(0,\xi)|^2$. Therefore, our task is to find some condition satisfying

$$-2\,\mathrm{Re}\,\hat{w}'\bar{\hat{w}} = \Lambda_t(t,\xi) + t^{2l}\,\mathrm{Re}\,a(t)\xi^2 + t^k\{\mathrm{Re}\,b(t)\xi - \varepsilon(k+1)\langle\xi\rangle\} \ge 0, \tag{2.6}$$

Noting $\Lambda_t(t,\xi) \ge 0$, we have

$$\begin{aligned}&t^{2l}\,\mathrm{Re}\,a(t)\xi^2 + t^k\{\mathrm{Re}\,b(t)\xi - \varepsilon(k+1)\langle\xi\rangle\} \\ &\quad\ge t^{2l}\,\mathrm{Re}\,a(t)\xi^2 - t^k\{|\mathrm{Re}\,b(t)| + \varepsilon(k+1)\}\langle\xi\rangle \ge t^{2l}\,\mathrm{Re}\,a(t)\xi^2 - C_\varepsilon t^k\langle\xi\rangle.\end{aligned}$$

Hence, (2.6) holds if $t^{2l-k} \ge C_\varepsilon\langle\xi\rangle/\xi^2\,\mathrm{Re}\,a(t)$ since $2l \ge k$. On the other hand, $t^{2l-k} \le C_\varepsilon\langle\xi\rangle/\xi^2\,\mathrm{Re}\,a(t)$ implies

$$t \le C_\varepsilon\langle\xi\rangle^{-1/(2l-k)}, \tag{2.7}$$

because

$$\frac{C_\varepsilon\langle\xi\rangle}{\xi^2\,\mathrm{Re}\,a(t)} \ge \frac{C_\varepsilon\langle\xi\rangle}{|\xi^2 a(t)|} \ge C'_\varepsilon\langle\xi\rangle^{-1}.$$

By the way, by $s > 1$ and $\mathrm{Re}\,a(t) > 0$, (2.6) is estimated as

$$\begin{aligned}&\Lambda_t(t,\xi) + t^{2l}\,\mathrm{Re}\,a(t)\xi^2 + t^k\{\mathrm{Re}\,b(t)\xi - \varepsilon(k+1)\langle\xi\rangle\} \\ &\quad\ge \Lambda_t(t,\xi) - t^k\{\mathrm{Re}\,b(t)\xi + \varepsilon(k+1)\langle\xi\rangle\} \ge \Lambda_t(t,\xi) - C_\varepsilon t^k\langle\xi\rangle.\end{aligned}$$

Noting (2.4) and (2.7) we get (2.6) from the above if M satisfies

$$C\langle\xi\rangle_h^{1-\sigma(1+k)-r_0} \le M.$$

In fact we can choose $M > 0$ since $1 - \sigma(1+k) - r_0 \le 0$. It follows from (2.3) that $r_0 \ge (2l-2k-1)/(2l-k)$ implies $w(t,\cdot) \in L^2(\boldsymbol{R})$. Hence Lemma 2.1, namely $v(t,x) \in H^{\langle s \rangle}$, is verified.

Then transformation (2.1) turns out $v(t,x) = e^{\varepsilon t^{k+1}\langle D_x \rangle} u(t,x) \in L^2(\boldsymbol{R}^n_x)$ on account of $H^{\langle s \rangle} \subset L^2(\boldsymbol{R}^n_x)$. Finally, we can see $u(t,x) \in H^{\langle 1 \rangle}_{\varepsilon t^{k+1}}$. We can easily show the following fact.

LEMMA 2.2. *If $u(x)$ belongs to $H^{\langle s \rangle}$, then $u(x)$ is in Gevrey class with exponent s, that is, for any $\rho > 0$ there exists a constant $C_\rho > 0$*

$$\sup_{x \in \boldsymbol{R}^n} |D_x^\alpha u(x)| \le C_\rho \rho^{-|\alpha|} |\alpha|!^s,$$

for every multi-index $\alpha \in \boldsymbol{N}^n$. In particular, $u(x)$ is a real analytic function if $s = 1$.

Using this lemma, we have (1.8).

3. Proof of Theorem 1.4

We shall prove Theorem 1.4 in this section. First of all, we shall transform $u(t,x)$ in (1.1) into $v(t,x)$ by

$$v(t,x) = e^{\varepsilon t^\delta \langle D_x \rangle} u(t,x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{ix\cdot\xi + \varepsilon t^\delta \langle \xi \rangle} \hat{u}(t,\xi)\, d\xi, \tag{3.1}$$

similar to (2.1) for some nonnegative integer δ. Then we remark that

$$\partial_t^k u(t,x) = e^{\Omega(t,D_x)} (\partial_t + \Omega_t)^k v(t,x), \quad k = 1, \dots, m,$$

where

$$\Omega(t,D_x) = -\varepsilon t^\delta \langle D_x \rangle, \quad \Omega_t(t,D_x) = \partial_t \Omega(t,D_x) = -\varepsilon\delta t^{\delta-1} \langle D_x \rangle. \tag{3.2}$$

Therefore P of (1.2) can be rewritten such that

$$\begin{aligned}
&P(t,x,\partial_t,D_x)u(t,x) \\
&= \left(\partial_t^m + \sum_{j=1}^m \sum_{\alpha:\mathit{finite}} t^{\sigma(j\alpha)} b_{j\alpha}(t,x) D_x^\alpha \partial_t^{m-j} \right) (e^{\Omega(t,D_x)} v(t,x)) \\
&= e^{\Omega(t,D_x)} (\partial_t + \Omega_t)^m v(t,x) + \sum_{j=1}^m \sum_{\alpha:\mathit{finite}} t^{\sigma(j\alpha)} b_{j\alpha}(t,x) D_x^\alpha e^{\Omega(t,D_x)} (\partial_t + \Omega_t)^{m-j} v(t,x) \\
&= e^{\Omega(t,D_x)} \left\{ (\partial_t + \Omega_t)^m + \sum_{j=1}^m \sum_{\alpha:\mathit{finite}} t^{\sigma(j\alpha)} e^{-\Omega(t,D_x)} b_{j\alpha}(t,x) e^{\Omega(t,D_x)} D_x^\alpha (\partial_t + \Omega_t)^{m-j} \right\} v(t,x).
\end{aligned}$$

Hence, we have the following new Cauchy problem for degenerate parabolic equations on $(0,T)\times\boldsymbol{R}^n$ $(T>0)$

$$\begin{cases} \bar{P}(t,x,\partial_t,D_x)v(t,x)=e^{-\Omega(t,D_x)}f(t,x), & (t,x)\in(0,T)\times\boldsymbol{R}^n, \\ (\partial_t+\Omega_t)^j v(0,x)=u_j(x), & x\in\boldsymbol{R}^n,\ j=0,\dots,m-1, \end{cases} \tag{3.3}$$

where

$$\bar{P}(t,x,\partial_t,D_x)=(\partial_t+\Omega_t)^m+\sum_{j=1}^{m}\sum_{\alpha:finite}t^{\sigma(j\alpha)}b_{j\alpha\Omega}(t,x,D_x)D_x^\alpha(\partial_t+\Omega_t)^{m-j} \tag{3.4}$$

and $b_{j\alpha\Omega}(t,x,D_x)=e^{-\Omega(t,D_x)}b_{j\alpha}(t,x)e^{\Omega(t,D_x)}$. We shall prove the following fact bellow.

LEMMA 3.1. *If the operator P of the Cauchy problem* (1.1) *is Γ-parabolic at $t=0$, then $\bar{P}$, which is transformed P by* (3.1) *with*

$$\delta=1+\max_{\substack{0\le i\le l-1\\1\le j\le m}}\left\{\left[j\left(\frac{1-r_i}{\sigma_i}+q_i+1\right)\right],\ \max_{\substack{1\le h\le j-1\\\alpha:finite}}\left[\frac{|\alpha|+j-h-jr_i}{\sigma_i}+jq_i-\sigma(h\alpha)+j-h\right]\right\} \tag{3.5}$$

is also Γ-parabolic at $t=0$.

We need the following lemma in order to prove the above lemma.

LEMMA 3.2. *We have*

$$\sigma((\partial_t+\Omega_t)^j)(t,\lambda,\xi)=\begin{cases}\lambda+\Omega_t(t,\xi), & j=1,\\ (\lambda+\Omega_t(t,\xi))^2+d_2^{(2)}(t,\xi), & j=2,\\ (\lambda+\Omega_t(t,\xi))^3+d_2^{(3)}(t,\xi)(\lambda+\Omega_t(t,\xi))+d_3^{(3)}(t,\xi), & j=3,\\ (\lambda+\Omega_t(t,\xi))^j+\displaystyle\sum_{l=2}^{j}d_l^{(j)}(t,\xi)(\lambda+\Omega_t(t,\xi))^{j-l}, & j\ge 4,\end{cases}$$

where, for some constants C_2^j, C_3^j and some polynomials $C_i^{(j)}(t)$

$$d_2^{(j)}=C_2^j t^{\delta-2}\langle\xi\rangle,\quad d_3^{(j)}=C_3^j t^{\delta-3}\langle\xi\rangle\quad\text{and}\quad d_l^{(j)}=t^{\delta-l}\sum_{i=1}^{[l/2]}C_i^{(j)}(t)\langle\xi\rangle^i,\quad 4\le l\le j. \tag{3.7}$$

Here, $\sigma((\partial_t+\Omega_t)^j)$ stands for the symbol of $(\partial_t+\Omega_t)^j$ and denote by $[p]$ the maximal integer not greater than p.

PROOF. We use induction on j. The claim is trivial for $j = 1, 2, 3, 4$ such that

$$\sigma(\partial_t + \Omega_t) = \lambda + \Omega_t,$$

$$\sigma((\partial_t + \Omega_t)^2) = (\lambda + \Omega_t)^2 + \Omega_{tt}(t, \xi),$$

$$\sigma((\partial_t + \Omega_t)^3) = (\lambda + \Omega_t)^3 + 3\Omega_{tt}(t, \xi)(\lambda + \Omega_t) + \Omega_{ttt}(t, \xi),$$

$$\begin{aligned}\sigma((\partial_t + \Omega_t)^4) &= (\lambda + \Omega_t)(\sigma((\partial_t + \Omega_t)^3)) + \partial_t(\sigma((\partial_t + \Omega_t)^3)) \\ &= (\lambda + \Omega_t)^4 + 3\Omega_{tt}(\lambda + \Omega_t)^2 + \Omega_{ttt}(\lambda + \Omega_t) \\ &\quad + 3\Omega_{tt}(\lambda + \Omega_t)^2 + 3\Omega_{ttt}(\lambda + \Omega_t) + 3\Omega_{tt}^2 + \Omega_{tttt}.\end{aligned}$$

Next, assume (3.7) is true for $j - 1$ $(j \geq 5)$. Then

$$\begin{aligned}&\sigma((\partial_t + \Omega_t)^j)(t, \lambda, \xi) \\ &= (\lambda + \Omega_t)(\sigma((\partial_t + \Omega_t)^{j-1})) + \partial_t(\sigma((\partial_t + \Omega_t)^{j-1})) \\ &= (\lambda + \Omega_t)\left\{(\lambda + \Omega_t)^{j-1} + \sum_{i=2}^{j-1} d_i^{(j-1)}(\lambda + \Omega_t)^{j-1-i}\right\} \\ &\quad + \partial_t\left\{(\lambda + \Omega_t)^{j-1} + \sum_{i=2}^{j-1} d_i^{(j-1)}(\lambda + \Omega_t)^{j-1-i}\right\} \\ &= (\lambda + \Omega_t)^j + \sum_{i=2}^{j-1} d_i^{(j-1)}(\lambda + \Omega_t)^{j-i} + (j-1)(\lambda + \Omega_t)^{j-2}\Omega_{tt} \\ &\quad + \sum_{i=2}^{j-1} \partial_t d_i^{(j-1)}(\lambda + \Omega_t)^{j-1-i} + \sum_{i=2}^{j-1}(j-1-i) d_i^{(j-1)}(\lambda + \Omega_t)^{j-2-i}\Omega_{tt} \\ &= (\lambda + \Omega_t)^j + d_2^{(j-1)}(\lambda + \Omega_t)^{j-2} + d_3^{(j-1)}(\lambda + \Omega_t)^{j-3} + \sum_{i=4}^{j-1} d_i^{(j-1)}(\lambda + \Omega_t)^{j-i} \\ &\quad + (j-1)(\lambda + \Omega_t)^{j-2}\Omega_{tt} + \partial_t d_2^{(j-1)}(\lambda + \Omega_t)^{j-3} + \sum_{i=4}^{j-1} \partial_t d_{i-1}^{(j-1)}(\lambda + \Omega_t)^{j-i} \\ &\quad + \partial_t d_{j-1}^{(j-1)} + \sum_{i=4}^{j-1}(j+1-i) d_{i-2}^{(j-1)}(\lambda + \Omega_t)^{j-i}\Omega_{tt} + d_{j-2}^{(j-1)}\Omega_{tt} \\ &= (\lambda + \Omega_t)^j + \{(j-1)\Omega_{tt} + d_2^{(j-1)}\}(\lambda + \Omega_t)^{j-2} + \{d_3^{(j-1)} + \partial_t d_2^{(j-1)}\}(\lambda + \Omega_t)^{j-3} \\ &\quad + \sum_{i=4}^{j-1}\{d_i^{(j-1)} + \partial_t d_{i-1}^{(j-1)} + (j+1-i)\Omega_{tt} d_{i-2}^{(j-1)}\}(\lambda + \Omega_t)^{j-i} + \partial_t d_{j-1}^{(j-1)} + d_{j-2}^{(j-1)}\Omega_{tt}.\end{aligned}$$

Thus putting

$$d_2^{(j)} = (j-1)\Omega_{tt} + d_2^{(j-1)},$$

$$d_3^{(j)} = d_3^{(j-1)} + \partial_t d_2^{(j-1)},$$

$$d_l^{(j)} = d_l^{(j-1)} + \partial_t d_{l-1}^{(j-1)} + (j+1-l)\Omega_{tt} d_{l-2}^{(j-1)}, \quad l = 4, \dots, j-1,$$

$$d_j^{(j)} = \partial_t d_{j-1}^{(j-1)} + d_{j-2}^{(j-1)}\Omega_{tt},$$

we have (3.7) inductively. Q.E.D.

Hereafter we begin to prove Lemma 3.1. We can see from (3.4),

$$\sigma(\bar{P})(t,x,\lambda,\xi) = \sigma((\partial_t+\Omega_t)^m) + \sum_{j=1}^{m}\sum_{\alpha:\mathit{finite}} t^{\sigma(j\alpha)} b_{j\alpha\Omega}(t,x,\xi)\xi^\alpha \sigma((\partial_t+\Omega_t)^{m-j}). \quad (3.8)$$

It follows from Lemma 3.2 that

$$\begin{aligned}
&\sigma((\partial_t + \Omega_t(t,\xi))^{m-j}) \\
&= (\lambda + \Omega_t(t,\xi))^{m-j} + \sum_{l=2}^{m-j} d_l^{(m-j)}(t,\xi)(\lambda + \Omega_t(t,\xi))^{m-j-l} \\
&= \lambda^{m-j} + \lambda^{m-j-1}\binom{m-j}{1}\Omega_t(t,\xi) \\
&\quad + \sum_{l=2}^{m-j}\lambda^{m-j-l}\left\{\binom{m-j}{l}\Omega_t(t,\xi)^l + \sum_{i=0}^{l-2}\binom{m-j-2-i}{l-2-i}\Omega_t(t,\xi)^{l-2-i} d_{2+i}^{(m-j)}(t,\xi)\right\} \\
&= \lambda^{m-j} + \sum_{l=1}^{m-j} g_l^{(m-j)}(t,\xi)\lambda^{m-j-l}, \qquad (3.9)
\end{aligned}$$

where

$$g_1^{(m-j)}(t,\xi) = \binom{m-j}{1}\Omega_t(t,\xi)$$

and for $l = 2, \dots, m-j$

$$g_l^{(m-j)}(t,\xi) = \binom{m-j}{l}\Omega_t(t,\xi)^l + \sum_{i=0}^{l-2}\binom{m-j-2-i}{l-2-i}\Omega_t(t,\xi)^{l-2-i} d_{2+i}^{(m-j)}(t,\xi).$$

From (3.2) and (3.3), we can see

$$g_l^{(m-j)}(t,\xi) = t^{\delta-l}\tilde{g}_l^{(m-j)}(t,\xi), \quad l = 1, \dots, m-j, \qquad (3.10)$$

and the order in ξ of $\tilde{g}_l^{(m-j)}(t,\xi)$ is not greater than l. Then from (3.8), (3.9) and (3.10),

$$\sigma(\bar{P})(t,x,\lambda,\xi)$$

$$= \lambda^m + \sum_{j=1}^{m}\left\{g_j^{(m)}(t,\xi) + \sum_{\alpha} t^{\sigma(j\alpha)} b_{j\alpha\Omega}(t,x,\xi)\xi^{\alpha}\right\}\lambda^{m-j}$$

$$+ \sum_{j=1}^{m}\sum_{\alpha} t^{\sigma(j\alpha)} b_{j\alpha\Omega}(t,x,\xi)\xi^{\alpha}\sum_{l=1}^{m-j} g_l^{(m-j)}(t,\xi)\lambda^{m-j-l}$$

$$= \lambda^m + \sum_{j=1}^{m}\lambda^{m-j}\left\{g_j^{(m)}(t,\xi) + \sum_{\alpha} t^{\sigma(j\alpha)} b_{j\alpha\Omega}(t,x,\xi)\xi^{\alpha}\right.$$

$$\left. + \sum_{h=1}^{j-1}\sum_{\alpha} t^{\sigma(h\alpha)} b_{h\alpha\Omega}(t,x,\xi)\xi^{\alpha} g_{j-h}^{(m-h)}(t,\xi)\right\}$$

$$= \lambda^m + \sum_{j=1}^{m}\sum_{\alpha} t^{\sigma(j\alpha)} b_{j\alpha\Omega}(t,x,\xi)\xi^{\alpha}\lambda^{m-j}$$

$$+ \sum_{j=1}^{m}\left\{t^{\delta-j}\tilde{g}_j^{(m)}(t,\xi) + \sum_{h=1}^{j-1}\sum_{\alpha} t^{\sigma(h\alpha)+\delta-(j-h)} b_{h\alpha\Omega}(t,x,\xi)\xi^{\alpha}\tilde{g}_{j-h}^{(m-h)}(t,\xi)\right\}\lambda^{m-j}. \quad (3.11)$$

We choose δ such that

$$\left\{\left(1+\frac{\delta-j}{j},1\right), \bigcup_{h=1}^{j-1}\left(1+\frac{\sigma(h\alpha)+\delta-(j-h)}{j}, \frac{|\alpha|+(j-h)}{j}\right)\right\} \subset\subset N(P).$$

This is clear for $i = l$. Therefore, nonnegative integer δ has to be the minimam integer satisfying

$$\delta > j(1-r_i)\sigma_i^{-1} + jq_i + j$$

and

$$\delta > (|\alpha| + j - h - jr_i)\sigma_i^{-1} + jq_i - \sigma(h\alpha) + j - h \quad h = 1,\dots,j-1.$$

for $i = 0, 1, \dots, l-1$. Hence, we can choose

$$\delta = 1 + \max_{\substack{0\le i\le l-1\\ 1\le j\le m}}\left\{\left[j\left(\frac{1-r_i}{\sigma_i} + q_i + 1\right)\right], \max_{\substack{1\le h\le j-1\\ \alpha:\textit{finite}}}\left[\frac{|\alpha|+j-h-jr_i}{\sigma_i} + jq_i - \sigma(h\alpha) + j - h\right]\right\}. \quad (3.12)$$

Moreover, we need prepare following Lemma 3.3 to complete the proof of Lemma 3.1.

LEMMA 3.3. *Let* $a_{\Omega}(t,x,D_x) = e^{-\Omega(t,D_x)}a(t,x)e^{\Omega(t,D_x)}$ *for* $a(t,x) \in C^{\infty}([0,T];\gamma^{\langle 1\rangle}(\boldsymbol{R}^n))$. *Denote by* $a_{\Omega}(t,x,\xi)$ *the symbol of* $a_{\Omega}(t,x,D_x)$. *Then we can write*

$$a_{\Omega}(t,x,\xi) = a(t,x) + \varepsilon t^{\delta} a_1(t,x,\xi) + r(t,x,\xi), \tag{3.13}$$

where $a_1(t,x,\xi) \in \gamma^1 S^0$ *and* $r(t,x,\xi)$ *satisfying there are constants* $C_a > 0$, $r_a > 0$ *and* $\varepsilon_a > 0$ *such that*

$$|r_{(\beta)}^{(\alpha)}(t,x,\xi)| \leq C_a r_a^{-|\alpha+\beta|}|\alpha+\beta|!e^{-\varepsilon_a\langle\xi\rangle} \tag{3.14}$$

for $0 \leq t \leq T,\ x,\xi \in \boldsymbol{R}^n,\ \alpha,\beta \in \boldsymbol{N}^n$.

This lemma can be obtained by simple modification of the proof in K. Kajitani-S. Wakabayashi [4]. The proof of Lemma 3.1 will be given in Appendix.

In conclusion, the choice of δ as (3.12) and (3.13) enable to be written by

$$\sigma(\tilde{P})(t,x,\lambda,\xi) = \lambda^m + \sum_{j=1}^{m}\sum_{k=1}^{jr_j} t^{\sigma(jk)}\{\tilde{b}_{jk}(t,x,\xi) + r(t,x,\xi)\}\lambda^{m-j}, \tag{3.15}$$

where $\tilde{b}_{jk}(t,x,\xi) \in C^{\infty}([0,T];\gamma^{\langle 1\rangle}S^{jr_j})$ with $\tilde{b}_{jk}(0,x,\xi) = b_{jk}(0,x)$ and $r(t,x,\xi)$ satisfying (3.14). Thus Lemma 3.1 is proved.

Now we prove Theorem 1.4. It follows from Lemma 3.1 that if P satisfies the conditions in Theorem 1.3 then $\tilde{P}$ does so. Hence, we can apply Theorem 1.3 to the Cauchy problem for $\tilde{P}$, and we can obtain the solution $v(t,x)$ of (3.3) in $C^{\infty}([0,T];H^{\langle s\rangle})$. Putting $u(t,x) = e^{-\varepsilon t^{\delta}\langle D_x\rangle}v(t,x)$, we get the solution of the Cauchy problem (1.1) which is contained in $C^{\infty}((0,T];H^{\langle 1\rangle}_{\varepsilon t^{\delta}})$ for any $\varepsilon > 0$ and δ chosen in (3.12). Thus we completed the proof of Theorem 1.4.

Appendix

PROOF OF LEMMA 3.3. The symbol of the product of the operators $e^{-\Omega(t,D_x)}$, $a(t,x)$ and $e^{\Omega(t,D_x)}$ is given by

$$\begin{aligned} a_{\Omega}(t,x,\xi) &= os - \iint_{\boldsymbol{R}^{2n}} e^{-i(x-y)\cdot(\xi-\eta)+\Omega(t,\xi)-\Omega(t,\eta)}a(t,y)\,dy\bar{d}\eta \\ &= os - \iint_{\boldsymbol{R}^{2n}} e^{-iy\cdot\eta-\Omega(t,\xi+\eta)+\Omega(t,\xi)}a(t,x+y)\,dy\bar{d}\eta, \end{aligned} \tag{A.1}$$

where $\bar{d}\eta = (2\pi)^{-n}\, d\eta$. Putting

$$\begin{aligned}-\Omega(t,\xi+\eta)+\Omega(t,\xi) &= \varepsilon t^{\delta}(\langle\xi+\eta\rangle-\langle\xi\rangle)\\ &= \varepsilon t^{\delta}\sum_{j=1}^{n}\eta_j\int_0^1(\xi_j+\theta\eta_j)\langle\xi+\theta\eta\rangle^{-1}\,d\theta\\ &\equiv \varepsilon t^{\delta}\eta\cdot w(\xi,\eta),\end{aligned}$$

(A.1) can be rewrited by using Stokes formula

$$\begin{aligned}a_{\Omega}(t,x,\xi) &= os-\iint_{\boldsymbol{R}^{2n}} e^{-i(y-i\varepsilon t^{\delta}w(\xi,\eta))\cdot\eta}a(t,x+y)\,dy\bar{d}\eta\\ &= os-\int_{\boldsymbol{R}^n}\bar{d}\eta\int_{\boldsymbol{R}^n-i\varepsilon t^{\delta}w(\xi,\eta)} e^{-iz\cdot\eta}a(t,x+z+i\varepsilon t^{\delta}w(\xi,\eta))\,dz\\ &= os-\int_{\boldsymbol{R}^n}\bar{d}\eta\int_{\boldsymbol{R}^n} e^{-iy\cdot\eta}a(t,x+y+i\varepsilon t^{\delta}w(\xi,\eta))\,dy\\ &= os-\int_{\boldsymbol{R}^n}\bar{d}\eta\int_{\boldsymbol{R}^n} e^{-iy\cdot\eta}a(t,x+y+i\varepsilon t^{\delta}w(\xi,\eta))\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle)\,dy\bar{d}\eta\\ &\quad+os-\int_{\boldsymbol{R}^n}\bar{d}\eta\int_{\boldsymbol{R}^n} e^{-iy\cdot\eta}a(t,x+y+i\varepsilon t^{\delta}w(\xi,\eta))(1-\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle))\,dy\bar{d}\eta\\ &\equiv b_1(t,x,\xi)+b_2(t,x,\xi), \end{aligned}\tag{A.2}$$

where $\chi_{N+|\alpha+\beta|}(t)=1$ if $t\le 1/4$, $\chi_{N+|\alpha+\beta|}(t)=0$ if $t\ge 1/2$ and

$$|D_t^{k+j}\chi_{N+|\alpha+\beta|}(t)|\le C_jC^{1+N+|\alpha+\beta|}k!,\tag{A.3}$$

if $k\le N+|\alpha+\beta|$ and $j=1,2,\dots$. Thus, by Taylor expansion, we obtain

$$\begin{aligned}b_1(t,x,\xi) &= os-\sum_{|\gamma|<N}\iint e^{-iy\cdot\eta}a_{(\gamma)}(t,x+i\varepsilon t^{\delta}w(\xi,\eta))(iy)^{\gamma}\gamma!^{-1}\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle)\,dy\bar{d}\eta\\ &\quad+r_N(t,x,\xi),\end{aligned}\tag{A.4}$$

where

$$\begin{aligned}r_N(t,x,\xi) &= os-\sum_{|\gamma|=N}N\gamma!^{-1}\iiint_0^1(1-\theta)^{N-1}e^{-iy\cdot\eta}\partial_{\eta}^{\gamma}\{a_{(\gamma)}(t,x+\theta y+i\varepsilon t^{\delta}w(\xi,\eta))\\ &\quad\times\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle)\}\,dy\bar{d}\eta d\theta.\end{aligned}\tag{A.5}$$

Put for $0 \le j \le N-1$,

$$p_j(t,x,\xi) = os - \sum_{|\gamma|=j} \iint e^{-iy\cdot\eta} a_{(\gamma)}(t, x + i\varepsilon t^{\delta} w(\xi,\eta))(iy)^{\gamma}\gamma!^{-1}\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle)\, dy\bar{d}\eta$$

$$= \sum_{|\gamma|=j} \gamma!^{-1}\partial_{\eta}^{\gamma}\{a_{(\gamma)}(t, x + i\varepsilon t^{\delta} w(\xi,\eta))\}_{\eta=0}. \tag{A.6}$$

Since

$$|\partial_{\eta}^{\alpha} w(\xi,\eta)| \le C_0\rho_0^{-|\alpha|}|\alpha|! \int_0^1 \langle\xi + \theta\eta\rangle^{-|\alpha|}\, d\theta,$$

$$|\partial_{\xi}^{\alpha}\omega(\xi)| \le C_1\rho_0^{-|\alpha|}|\alpha|!\langle\xi\rangle^{-|\alpha|},$$

where $\omega(\xi) = \xi\langle\xi\rangle^{-1}$, we can see $p_j(t,x,\xi) \in C^{\infty}([0,T]; \gamma^1 S^{-j})$. Hence it follows from [1] that there is $p(t,x,\xi)$ which satisfies that there is $r > 0$ such that

$$\left|\partial_{\xi}^{\alpha} D_x^{\beta}\left(p(t,x,\xi) - \sum_{j=0}^{N-1} p_j(t,x,\xi)\right)\right| \le C_p(r\rho_0)^{-N-|\alpha+\beta|}|\alpha+\beta|!N!\langle\xi\rangle^{-N-\alpha} \tag{A.7}$$

for $0 \le t \le T$, $x, \xi \in \boldsymbol{R}^n$, $\alpha, \beta \in \boldsymbol{N}^n$, $N = 1, 2, \ldots$.

We define

$$r(t,x,\xi) = a_{\Omega}(t,x,\xi) - p(t,x,\xi).$$

It follows from (A.2), (A.4) and (A.6) that we have

$$r(t,x,\xi) = \left(\sum_{j=0}^{N-1} p_j(t,x,\xi) - p(t,x,\xi)\right) + r_N(t,x,\xi) + b_2(t,x,\xi)$$

for any positive integer N.

We shall prove that $r(t,x,\xi)$ satisfies (3.14). We now estimate $r_N(t,x,\xi)$. Integration by part gives for $l_0 = 2([n/2]+1)$

$$r_{N}{}_{(\beta)}^{(\alpha)}(t,x,\xi)$$

$$= \sum_{|\gamma|=N} N\gamma!^{-1} \iiint_0^1 (1-\theta)^{N-1} e^{-iy\cdot\eta}\langle\eta\rangle^{-l_0}\langle D_y\rangle^{l_0}\langle y\rangle^{-l_0}\langle D_{\eta}\rangle^{l_0}$$

$$\times\, \partial_{\eta}^{\gamma}\partial_{\xi}^{\alpha}\{a_{(\gamma+\beta)}(t, x + \theta y + i\varepsilon t^{\delta} w(\xi,\eta))\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle)\}\, dy\bar{d}\eta d\theta. \tag{A.8}$$

We put

$$F_1(t,x,\xi;y,\eta) = \langle\eta\rangle^{-l_0}\langle D_y\rangle^{l_0}\langle y\rangle^{-l_0}\langle D_{\eta}\rangle^{l_0}$$

$$\times\, \partial_{\eta}^{\gamma}\partial_{\xi}^{\alpha}\{a_{(\gamma+\beta)}(t, x + \theta y + i\varepsilon t^{\delta} w(\xi,\eta))\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle)\}.$$

Then since $|\eta| \le \langle\xi\rangle/2$, on supp $\chi_{N+|\alpha+\beta|}$, we can estimate

$$|F_1(t,x,\xi;y,\eta)| \le C_1(r\rho_1)^{-2N-|\alpha+\beta|}|\alpha+\beta|!N!\langle y\rangle^{-l_0}\langle\eta\rangle^{-l_0}\langle\xi\rangle^{-N}N!$$

Therefore integrating this term we can estimate from (A.8)

$$|r_N{}^{(\alpha)}_{(\beta)}(t,x,\xi)| \le C_2(r\rho_2)^{-N-|\alpha+\beta|}N!|\alpha+\beta|!\langle\xi\rangle^{-N}. \tag{A.9}$$

As to $b_2(t,x,\xi)$, we have from (A.2)

$$b_2{}^{(\alpha)}_{(\beta)}(t,x,\xi) = \iint e^{-iy\cdot\eta}\langle\eta\rangle^{-l_0}\langle D_y\rangle^{l_0}\langle y\rangle^{-l_0}\langle D_\eta\rangle^{l_0}$$
$$\times\, \partial_\xi^\alpha D_x^\beta\{a(t,x+y+i\varepsilon t^\delta w(\xi,\eta))(1-\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle))\}\,dy\bar{d}\eta. \tag{A.10}$$

We put

$$F_2(t,x,\xi;y,\eta) = \langle\eta\rangle^{-l_0}\langle D_y\rangle^{l_0}\langle y\rangle^{-l_0}\langle D_\eta\rangle^{l_0}$$
$$\times\, \partial_\xi^\alpha D_x^\beta\{a(t,x+y+i\varepsilon t^\delta w(\xi,\eta))(1-\chi_{N+|\alpha+\beta|}(|\eta|/\langle\xi\rangle))\}.$$

Then we get from (A.3)

$$|D_y^\lambda F_2(t,x,\xi;y,\eta)| \le C_3(r\rho_3)^{-|\alpha+\beta+\lambda|}|\alpha+\beta+\lambda|!\langle y\rangle^{-l_0}\langle\eta\rangle^{-l_0}$$

for any $\lambda \in \boldsymbol{N}^n$. It follows that for any λ

$$|F_2(t,x,\xi;\eta)| \le C_4(r\rho_3)^{-|\alpha+\beta|-|\lambda|}|\alpha+\beta+\lambda|!\langle\eta\rangle^{-l_0-|\lambda|},$$

where

$$F_2(t,x,\xi;\eta) = \int e^{-iy\cdot\eta}F_2(t,x,\xi;y,\eta)\,dy.$$

Therefore there is $\varepsilon_0 > 0$ such that

$$|F_2(t,x,\xi;\eta)| \le C_5(r\rho_3)^{-|\alpha+\beta|}|\alpha+\beta|!\langle\eta\rangle^{-l_0}e^{-\varepsilon_0\langle\xi\rangle},$$

because of $|\eta| \ge \langle\xi\rangle/2$ and we get from (A.10)

$$|b_2{}^{(\alpha)}_{(\beta)}(t,x,\xi)| \le C_5(r\rho_3)^{-|\alpha+\beta|}|\alpha+\beta|!e^{-\varepsilon_0\langle\xi\rangle}. \tag{A.11}$$

Hence we get from (A.7), (A.9) and (A.11)

$$|r^{(\alpha)}_{(\beta)}(t,x,\xi)| = \left|D_x^\beta\partial_\xi^\alpha\left\{\left(\sum_{j=0}^{N-1}p_j(t,x,\xi)-p(t,x,\xi)\right)+r_N(t,x,\xi)+b_2(t,x,\xi)\right\}\right|$$
$$\le C_2(r\rho_2)^{-N-|\alpha+\beta|}N!|\alpha+\beta|!\langle\xi\rangle^{-N} + C_5(r\rho_3)^{-|\alpha+\beta|}|\alpha+\beta|!e^{-\varepsilon_0\langle\xi\rangle}.$$

for any positive integer N. Taking the minimum with respect to N, we can see for some $\varepsilon_0 > 0$

$$|r_{(\beta)}^{(\alpha)}(t,x,\xi)| \le C_a r_a^{-|\alpha+\beta|}|\alpha+\beta|!e^{-\varepsilon_0\langle\xi\rangle}.$$

Taylor expansion yields

$$p(t,x,\xi) = a(t,x+i\varepsilon t^{\delta}w(\xi,0)) + \varepsilon t^{\delta}a_1(t,x,\xi) = a(t,x) + \varepsilon t^{\delta}a_1(t,x,\xi)$$

where $a_1(t,x,\xi) \in \gamma^1 S^0$. Hence, we can prove (3.13). Q.E.D.

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