TSUKUBA J. MATH.
Vol. 24 No. 2 (2000), 257–278

ON HYPOELLIPTICITY FOR A CLASS OF PSEUDO-DIFFERENTIAL OPERATORS

By

Nobuo NAKAZAWA

1. Introduction and Results

We shall study hypoellipticity for a class of pseudo-differential operators which includes the operator $-a(x)\Delta+1$ with $a(x)\geq 0$ as a typical example. We shall use the Weyl symbols and the Weyl calculus in this paper. For the Weyl calculus we refer to Hörmander [2]. Let $p(x,\xi)\in S^{m}(\equiv S_{1,0}^{m}(\boldsymbol{R}^{2n}))$, i.e., $|p_{(\beta)}^{(\alpha)}(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$ for $(x,\xi)\in\boldsymbol{R}^{2n}$ and any multi-indices α and β, where $x=(x_{1},\ldots,x_{n})\in\boldsymbol{R}^{n}$, $\xi=(\xi_{1},\ldots,\xi_{n})\in\boldsymbol{R}^{n}$, $\langle\xi\rangle=\sqrt{1+|\xi|^{2}}$, $|\xi|=\sqrt{\sum_{j=1}^{n}|\xi_{j}|^{2}}$, $p_{(\beta)}^{(\alpha)}(x,\xi)=\partial_{\xi}^{\alpha}D_{x}^{\beta}p(x,\xi)$ and $D_{x}=(D_{1},\ldots,D_{n})\equiv-i\partial_{x}=-i(\partial/\partial x_{1},\ldots,\partial/\partial x_{n})$. We define for $u\in\mathscr{S}$

$$Pu\equiv p^{w}(x,D)u=(2\pi)^{-n}\int\left(\int e^{-i(x-y)\cdot\xi}p\left(\frac{x+y}{2},\xi\right)u(y)\,dy\right)d\xi,$$

where $x\cdot\xi=x_{1}\xi_{1}+\cdots+x_{n}\xi_{n}$ and $\mathscr{S}$ denotes the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}^{n}$. We call the symbol $p(x,\xi)$ the Weyl symbol of P and write $\sigma_{w}(P)(x,\xi)=p(x,\xi)$. For pseudo-differential operators we also refer to Kumano-go [5] and Shubin [7].

For simplicity we denote $p^{w}(x,D)$ and $\sigma_{w}(P)(x,\xi)$ by $p(x,D)$ and $\sigma(P)(x,\xi)$ respectively, in this paper.

DEFINITION 1.1. *Let $x^{0}\in\boldsymbol{R}^{n}$. We say that P is hypoelliptic at x^{0} if there exists a neighborhood U of x^{0} such that*

$$U\cap\operatorname{sing\,supp}Pu=U\cap\operatorname{sing\,supp}u\quad\text{for }u\in H_{-\infty},$$

where sing supp u *denotes the singular support of u, $H_{-\infty}=\bigcup_{s}H_{s}$ and H_{s} denotes the Sobolev space of order $s\in\boldsymbol{R}$.*

Received August 4, 1999
Revised December 8, 1999

We impose the following conditions on $p(x,\xi)$:

(A-0) The symbol $p(x,\xi)$ can be written in the form

$$p(x,\xi)=p_m(x,\xi)+p_{m-1}(x,\xi)+p_{m-2}(x,\xi)+p_{m-3}(x,\xi),$$

where $p_{m-j}(x,\xi)\in S^{m-j}$ $(0\le j\le 3)$ and $p_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ in ξ for $|\xi|\ge 1$ $(0\le j\le 2)$.

(A-1) There exist a neighborhood U of 0 in $\boldsymbol{R}^n$ and $C>0$ such that

$$s(x,\xi)\equiv p_m(x,\xi)+\operatorname{Re}p_{m-1}(x,\xi)+\operatorname{Re}p_{m-2}(x,\xi)\ge -C\langle\xi\rangle^{m-3}$$

for $(x,\xi)\in U\times\boldsymbol{R}^n$.

(A-2) There exist a neighborhood U of 0, constants $c_0>0$ and $C_0\in\boldsymbol{R}$ such that

$$\operatorname{Re}(p(x,D)u,u)\ge c_0\|u\|^2_{m/2-1}-C_0\|u\|^2_{m/2-2}$$

for $u\in C_0^\infty(U)$, where $(u,v)=\int u(x)\overline{v(x)}\,dx$ and $\|u\|_s=(\langle D\rangle^s u,\langle D\rangle^s u)^{1/2}$.

(A-3) There exist a neighborhood U of 0 and $r\in\boldsymbol{Z}$ with $0\le r\le n$ such that

$$p_m(x,\xi)\neq 0\quad\text{if } x\in U,\quad |\xi|=1\quad\text{and } x'=(x_1,\dots,x_r)\neq 0,$$

where we consider $x'=0$ if $r=0$.

(A-4) There exists a neighborhood U of 0 such that for any $\nu>0$ there is a constant $C_\nu>0$ satisfying

$$\text{(i)}\qquad \sum_{\substack{|\alpha|+|\beta|=2\\ \alpha'=0}}^{n}(\log\langle\xi\rangle)^{|\alpha|}|p^{(\alpha)}_{m(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}\le \nu s(x,\xi)+C_\nu\langle\xi\rangle^{m-3},$$

$$\text{(ii)}\qquad \log\langle\xi\rangle|\operatorname{Im}p_{m-1}(x,\xi)|\langle\xi\rangle^{-1}+\sum_{\substack{|\alpha|+|\beta|=1\\ \alpha'=0}}^{n}(\log\langle\xi\rangle)^{|\alpha|}|\operatorname{Im}p^{(\alpha)}_{m-1(\beta)}(x,\xi)|\langle\xi\rangle^{-|\beta|}$$

$$\le \nu s(x,\xi)+C_\nu\langle\xi\rangle^{m-3}$$

if $x\in U$ and $|\xi|\ge 1$, where $\alpha'=(\alpha_1,\dots,\alpha_r)$.

We note that (A-3) is always valid if $r=0$. Now we can state our main theorem.

THEOREM 1.2. *Under* (A-0)–(A-4), $p(x,D)$ *is hypoelliptic at* $x=0$.

Now we mention several known results relating to the above theorem.

RESULT 1. Hörmander [1] constructed a local parametrix at 0 of the operator

$$L_1 = a(x)(-\Delta)^m + (-\Delta)^{m'},$$

where $m, m' \in \boldsymbol{Z}_+ (= \boldsymbol{N} \cup \{0\})$ and $m > m'$, under the following conditions:

(B-1) $a(x) \in C^\infty$ and $a(x) \geq 0$.

(B-2) In a neighborhood of 0

$$|D_x^\beta a(x)| \leq M_\beta a(x)^{1-\tau|\beta|} \quad (1 - \tau|\beta| \geq 0, 0 < \tau < \{2(m-m')\}^{-1}).$$

Therefore, L_1 is hypoelliptic at 0 under the above conditions.

RESULT 2. Katsuta [4] showed that the existence of a local parametrix at 0 of the operator

$$L_2 = -a(x)\Delta + 1,$$

when L_2 satisfies (B-1) and the following condition:

(B-3) There exist a neighborhood U of 0, $\delta \in \boldsymbol{R}$ with $0 < \delta < 1/2$ and $M > 0$ such that

$$|\partial_{x_j} a(x)| \leq M a(x)^{1/2+\delta} \quad (x \in U, 1 \leq j \leq n).$$

Consequently L_2 is hypoelliptic at 0 under (B-1) and (B-3).

RESULT 3. We showed in [6] that L_2 is hypoelliptic at 0 under (B-1) and the condition

(B-4) there exists a neighborhood U of 0 such that $\partial_x^\alpha a(x) = 0$ if $x \in U$, $a(x) = 0$ and $|\alpha| = 2$.

Concerning the above results, it is easy to see that (B-2) implies (B-3) and that (B-3) does (B-4) under the assumption (B-1). Furthermore, if L_2 satisfies (B-1) and (B-4), then L_2 satisfies (A-0)–(A-4). This follows from Propositions 4.1 and 4.2 in Section 4 (see Section 4).

In addition, (A-1) and (A-2) are satisfied if the following conditions are satisfied (see Proposition 4.1 below):

(A-1)$'$ there exists a neighborhood U of 0 such that

$$p_m(x,\xi) \geq 0, \quad \mathrm{Re}\, p_{m-1}(x,\xi) \geq 0, \quad \mathrm{Re}\, p_{m-2}(x,\xi) > 0$$

for $x \in U$ and $|\xi| = 1$.

(A-2)$'$ $p_{m(\beta)}(0,\xi) = 0$ for any $\xi \in \boldsymbol{R}^n$ with $|\xi| = 1$ and $\beta \in \boldsymbol{Z}_+^n$ with $|\beta| \leq 2$ if $p_m(0,\xi^0) = 0$ for some $\xi^0 \in \boldsymbol{R}^n$ with $|\xi^0| = 1$.

The plan of this paper is as follows. In Section 2, we give a general criterion of hypoellipticity which is a simple variant of criteria given in Kajitani and Wakabayashi [3] and Wakabayashi and Suzuki [8]. We also reduce the operator $p(x,D) \in S^m$ to $\tilde{p}(x,D) \in S^2$. In Sectition 3, we complete the proof of Theorem 1.2. Finally in Sectition 4, we give some remarks and examples.

The author wishes to thank Professors S. Wakabayashi and M. Suzuki for their valuable advice and encourgement.

2. Preliminaries

In this section, we shall give propositions for the proof of Theorem 1.2 and reduce the problem for $p(x,D)$ to the problem for $\tilde{p}(x,D) = \langle D\rangle^{-m/2+1} p(x,D)\langle D\rangle^{-m/2+1}$.

First we assume that $p(x,\xi) \in S^m$ and that $p(x,\xi)$ satisfies (A-3). Let $x^0 = (0, x^{0\prime\prime}) \in U$, and choose $\varphi(x'') \in C_0^\infty(\boldsymbol{R}^{n-r})$ so that

$$\varphi(x'') = \begin{cases} |x'' - x^{0\prime\prime}|^2 & (|x'' - x^{0\prime\prime}| \leq 1), \\ 2 & (|x'' - x^{0\prime\prime}| \geq 2), \end{cases}$$

where $x'' = (x_{r+1}, \dots, x_n) \in \boldsymbol{R}^{n-r}$. Here we consider $x^0 = 0$ and $\varphi(x'') \equiv 0$ if $r = n$.

Define

$$\begin{aligned} \Lambda(x'',\xi) &= \Lambda_\delta(x'',\xi; s, a, N) \\ &= (-s + a\varphi(x''))\log\langle\xi\rangle + N\log(1 + \delta|\xi|^2), \end{aligned}$$

$$p_\Lambda(x,D) = (e^{-\Lambda})(x'',D) p(x,D)(e^{\Lambda})(x'',D).$$

The following proposition is essentially due to Kajitani and Wakabayashi [3] and Wakabayashi and Suzuki [8].

PROPOSITION 2.1. *Assume that there exist a neighborhood* U_0 *of* x^0, $l_1, l_2, l_3 \in \boldsymbol{R}$, $a_0 \geq 0$, N_0, $s_0 \in \boldsymbol{R}$ *and* $\chi(x') \in C_0^\infty(\boldsymbol{R}^r)$ *satisfying* $\chi(x') = 1$ *near* 0 *so that for any* $a \geq a_0$, $N \geq N_0$, $s \geq s_0$ *there are constants* $\delta_0 > 0$ *and* $C > 0$ *such that*

$$\|u\|_{l_1} \le C(\|p_\Lambda(x,D)u\|_{l_2} + \|u\|_{l_1-1} + \|(1-\chi)u\|_{l_3}), \tag{2.1}$$

for $u \in C_0^\infty(U_0)$ *if* $0 < \delta \le \delta_0$. *Here we consider* $\chi(x') \equiv 1$ *if* $r = 0$. *Then* $p(x,D)$ *is hypoelliptic at* x^0 *namely,* $x^0 \notin \operatorname{sing\,supp} u$ *if* $u \in H_{-\infty}$ *and* $x^0 \notin \operatorname{sing\,supp} p(x,D)u$.

Proof. Let $u \in H_{-\infty}$. Then there exists a constant $s' \in \boldsymbol{R}$ such that $u \in H_{s'}$. Assume that $x^0 \notin \operatorname{sing\,supp} p(x,D)u$. For simplicity we assume that $r \le n-1$. Then there is a neighborhood $U_1 = U_1' \times U_1''$ of x^0 such that

$$U_1 \subset\subset U \cap U_0 \cap \{x = (x', x'') \in \boldsymbol{R}^n; |x'' - x^{0\prime\prime}| \le 1\},$$

$$\operatorname{sing\,supp} p(x,D)u \cap \bar{U}_1 = \emptyset.$$

where $A \subset\subset B$ means that $\bar{A}$ is compact and included in the interior of B. Choose a neighborhood $U_2 = U_2' \times U_2''$ of x^0, $\Psi_1(x') \in C_0^\infty(U_1')$ and $\Psi_2(x'') \in C_0^\infty(U_1'')$ so that

$$U_2 \subset\subset U,$$

$$\Psi_1(x') = 1 \quad \text{in } U_2',$$

$$\Psi_2(x'') = 1 \quad \text{in } U_2''.$$

Here we consider $\Psi_1(x') \equiv 1$ if $r = 0$. Then there is a positive constant ε such that

$$\varphi(x'') = |x'' - x^{0\prime\prime}| \ge 2\varepsilon \quad \text{for } x'' \in U_1'' \backslash U_2''.$$

Fix $\tau > s'$ and choose $a > 0$, N, $s \in \boldsymbol{R}$ so that $a \ge a_0$, $N \ge N_0$, $s \ge s_0$ and

$$\begin{cases} 2a\varepsilon - s \ge l_2 + m - 1 - s', \\ \tau \le l_1 + s - a\varepsilon, \\ 2N \ge s - s' + \max\{l_1 - 1, l_2 + m, l_3\}. \end{cases} \tag{2.2}$$

It follows from the symbol calculus that there exists a symbol $q(x'', \xi)$ $(\equiv q(x'', \xi; \delta)) \in C([0,1]; S^0)$ satisfying

$$(e^\Lambda)(x'', D)(e^{-\Lambda})(x'', D)q(x'', D) - I \in S^{-\infty} \quad \text{uniformly in } \delta \in [0,1]. \tag{2.3}$$

We have

$$\begin{aligned} &p(x,D)(\Psi_1(x')\Psi_2(x'')u(x)) \\ &\quad = \Psi_1(x')\Psi_2(x'')p(x,D)u(x) + [p(x,D), \Psi_1(x')\Psi_2(x'')]u(x), \end{aligned} \tag{2.4}$$

where $[A,B] = AB - BA$. Operating $(e^{-\Lambda})(x'', D)$ to the both sides of (2.4) we have

$$
\begin{aligned}
&p_\Lambda(x,D)(e^{-\Lambda})(x'',D)q(x'',D)(\Psi_1(x')\Psi_2(x'')u(x)) \\
&\quad = (e^{-\Lambda})(x'',D)(\Psi_1(x')\Psi_2(x'')p(x,D)u(x)) \\
&\qquad + (e^{-\Lambda})(x'',D)([p(x,D),\Psi_1(x')\Psi_2(x'')]u(x)) \\
&\qquad + (e^{-\Lambda})(x'',D)p(x,D)((e^{\Lambda})(x'',D)e^{-\Lambda}(x'',D)q(x'',D)-I) \\
&\qquad \times (\Psi_1(x')\Psi_2(x'')u(x)) \\
&\quad \equiv f_1+f_2+f_3.
\end{aligned}
$$

Put $v_\delta = (e^{-\Lambda})(x'',D)q(x'',D)(\Psi_1(x')\Psi_2(x'')u(x))$. Then we have

$$p_\Lambda(x,D)v_\delta(x) = f_1+f_2+f_3.$$

Since $\Psi_1(x')\Psi_2(x'')p(x,D)u(x) \in H_\infty$, there is a constant C such that

$$\|f_1\|_{l_2} \le C \quad \text{for } 0\le\delta\le 1.$$

Here and after the constants do not depend on δ unless stated. By (2.3) we have

$$\|f_3\|_{l_2} \le C \quad \text{for } 0\le\delta\le 1.$$

As for f_2, we know that

$$
\begin{aligned}
&[p(x,D),\Psi_1(x')\Psi_2(x'')]u(x) \\
&\quad = [p(x,D),\Psi_1(x')]\Psi_2(x'')u(x) + \Psi_1(x')[p(x,D),\Psi_2(x'')]u(x),
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{supp}\sigma([p(x,D),\Psi_1(x')]\Psi_2(x'')) \subset\subset (U_1'\backslash U_2')\times U_1''\times \boldsymbol{R}^n \bmod S^{-\infty}, \\
&\operatorname{supp}\sigma(\Psi_1(x')[p(x,D),\Psi_2(x'')]) \subset\subset U_1'\times (U_1''\backslash U_2'')\times \boldsymbol{R}^n \bmod S^{-\infty}.
\end{aligned}
$$

In virtue of (A-3), we have

$$u\in C^\infty \quad \text{in } (U_1'\backslash U_2')\times U_1''.$$

Therefore there exists a constant C such that

$$\|(e^{-\Lambda})(x'',D)[p(x,D),\Psi_1(x')]\Psi_2(x'')u\|_{l_2} \le C \quad \text{for } 0\le\delta\le 1.$$

For $x''\in U_1''\backslash U_2''$

$$|e^{-\Lambda(x'',\xi)}| \le \langle\xi\rangle^{s-2a\varepsilon} \quad \text{for } 0\le\delta\le 1.$$

Then by (2.2) we obtain, with some $C > 0$,

$$\|(e^{-\Lambda})(x'', D)\Psi_1(x')[p(x, D), \Psi_2(x')]u\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

Therefore there is a constant C such that

$$\|f_2\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

Hence, we have

$$\|p_\Lambda(x, D)v_\delta\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

Let $\Psi \in C_0^\infty(U_0)$ satisfy

$$\Psi(x) = 1 \quad \text{in } U_1.$$

Then

$$\|p_\Lambda(x, D)(\Psi(x)v_\delta(x))\|_{l_2} \leq C \quad \text{for } 0 \leq \delta \leq 1.$$

If $0 < \delta \leq 1$ then

$$\Psi(x)v_\delta(x) \in H_{s'-s+2N} \subset H_{\max\{l_1-1, l_2+m, l_3\}}.$$

Therefore by using an inequality (2.1) with $u = \Psi v_\delta$, we have

$$\|\Psi v_\delta\|_{l_1} \leq C(\|p_\Lambda(x, D)\Psi v_\delta\|_{l_2} + \|\Psi v_\delta\|_{l_1-1} + \|(1 - \chi(x'))\Psi v_\delta\|_{l_3}),$$

for $0 < \delta \leq \delta_0$. Since $\Psi(x')\Psi(x'')u(x)$ belongs to C^∞ in $\{x' \neq 0\}$, we have

$$\|(1 - \chi(x'))\Psi v_\delta\|_{l_3} \leq C' \quad \text{for } 0 \leq \delta \leq 1$$

with some $C' > 0$. We can find a constant C'' so that

$$C\|\Psi v_\delta\|_{l_1-1} \leq \frac{1}{2}\|\Psi v_\delta\|_{l_1} + C''\|u\|_{s'}.$$

Then we obtain, with another constant C,

$$\|\Psi v_\delta\|_{l_1} \leq C \quad \text{for } 0 < \delta \leq \delta_0.$$

Therefore, we have

$$\|v_\delta\|_{l_1} \leq C \quad \text{for } 0 < \delta \leq \delta_0,$$

modifying C if necessary. This means that $\{v_\delta\}$ is bounded in a Hilbert space H_{l_1}. So we can see that there exists a subsequence which converges weakly in H_{l_1}. Therefore we have

$$v_0 = (e^{-\Lambda_0})(x'', D)q(x'', D; 0)\Psi_1(x')\Psi_2(x'')u(x) \in H_{l_1}.$$

Let $U_3(\subset\subset U_2)$ be a neighborhood of 0 satisfying

$$\varphi(x'') < \varepsilon \quad \text{for } x \in U_3.$$

Then

$$e^{\Lambda_0(x'',\xi)} \leq \langle\xi\rangle^{-s+a\varepsilon} \leq \langle\xi\rangle^{l_1-\tau}$$

for $x \in U_3$. Since $(e^{\Lambda_0})(x'',D)v_0 - \Psi_1(x')\Psi_2(x'')u(x) \in H_\infty$, we have

$$u(x) \in H^\tau \quad \text{in } U_3,$$

which implies that

$$x^0 \notin \operatorname{sing\,supp} u.$$

This completes the proof of Proposition 2.1. □

Next, we shall give the reduction as mentioned before. In addition to (A-3), we assume that $p(x,\xi)$ satisfies (A-0)–(A-2) and (A-4). Put

$$\tilde{p}(x,D) = \langle D\rangle^{-m/2+1}p(x,D)\langle D\rangle^{-m/2+1},$$

$$a_2(x,\xi) = \langle\xi\rangle^{-m+2}p_m(x,\xi),$$

$$a_1(x,\xi) = \langle\xi\rangle^{-m+2}p_{m-1}(x,\xi),$$

$$a_0(x,\xi) = \langle\xi\rangle^{-m+2}p_{m-2}(x,\xi) - \frac{1}{4}\sum_{j,k=1}^{n}(\partial_{\xi_j}\partial_{\xi_k}\langle\xi\rangle^{-m/2+1})(\partial_{x_j}\partial_{x_k}p_m(x,\xi))\langle\xi\rangle^{-m/2+1}$$

$$+\frac{1}{4}\sum_{j,k=1}^{n}(\partial_{\xi_j}\langle\xi\rangle^{-m/2+1})(\partial_{x_j}\partial_{x_k}p_m(x,\xi))(\partial_{\xi_k}\langle\xi\rangle^{-m/2+1})$$

and

$$a(x,\xi) = a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi),$$

$$b(x,\xi) = \tilde{p}(x,\xi) - a(x,\xi).$$

Then we have $a(x,\xi) \in S^2$, $b(x,\xi) \in S^{-1}$ and

$$\tilde{p}(x,D) = a(x,D) + b(x,D).$$

Since $\langle D\rangle$ is elliptic, $p(x,D)$ is hypoelliptic at 0 if $\tilde{p}(x,D)$ is hypoelliptic at 0.

By (A-4) there is a constant C' such that

$$\operatorname{Re} a(x,\xi) \geq \langle\xi\rangle^{-m+2}\left(s(x,\xi) - C'\sum_{j,k=1}^{n}\{\nu s(x,\xi) + C_\nu\langle\xi\rangle^{m-3}\}\right),$$

$$\geq \langle\xi\rangle^{-m+2}\{(1-C'\nu)s(x,\xi) - C'C_\nu\langle\xi\rangle^{m-3}\}$$

if $\nu > 0$ and $x \in U$. We choose $\nu > 0$ so that $C'\nu \le 1/2$. Then we have

$$\operatorname{Re} a(x,\xi) \ge \tfrac{1}{2}\langle\xi\rangle^{-m+2} s(x,\xi) - C''\langle\xi\rangle^{-1}.$$

By virtue of (A-1), we see that $a(x,\xi)$ satisfies the following:

($\tilde{\mathrm{A}}$-1) There exist a neighborhood U of 0 and a constant C such that

$$\operatorname{Re} a(x,\xi) \ge -C\langle\xi\rangle^{-1} \quad (x \in U).$$

By the definition of $a(x,\xi)$, we have for $u \in C_0^\infty(U)$

$$\begin{aligned}\operatorname{Re}(a(x,D)u,u) &= \operatorname{Re}(\langle D\rangle^{-m/2+1} p(x,D)\langle D\rangle^{-m/2+1}u,u) - \operatorname{Re}(b(x,D)u,u)\\ &= \operatorname{Re}(p(x,D)\langle D\rangle^{-m/2+1}u, \langle D\rangle^{-m/2+1}u) - \operatorname{Re}(b(x,D)u,u). \qquad (2.5)\end{aligned}$$

Let U_1 be a neighborhood of 0 satisfying $U_1 \subset\subset U$, and choose $\chi \in C_0^\infty(U)$ so that $\chi(x) = 1$ near $\overline{U_1}$. Then for each s there exists $C_s > 0$ such that

$$\|(1-\chi)\langle D\rangle^{-m/2+1}u\|_s \le C_s\|u\|_{-1} \quad \text{for } u \in C_0^\infty(U_1).$$

Assume that $u \in C_0^\infty(U_1)$, and put $v = \chi\langle D\rangle^{-m/2+1}u$. Then we have

$$\begin{aligned}\operatorname{Re}(a(x,D)u,u) &\ge \operatorname{Re}(p(x,D)u,u) - \operatorname{Re}(b(x,D)u,u) - C(\|u\|_{-1}^2 + \|v\|_{m/2-2}^2)\\ &\ge c_0\|v\|_{m/2-1}^2 - C'(\|u\|_{-1/2}^2 + \|v\|_{m/2-2}^2)\\ &\ge \frac{c_0}{2}\|u\|_0^2 - c_0\|(1-\chi)\langle D\rangle^{-m/2+1}u\|_{m/2-1}^2 - C''\|u\|_{-1/2}^2\\ &\ge \frac{c_0}{2}\|u\|_0^2 - C_0\|u\|_{-1/2}^2.\end{aligned}$$

Therefore the following condition is satisfied:

($\tilde{\mathrm{A}}$-2) There exist a neighborhood U_1 of 0 and constants $c_0 > 0$ and C_0 such that

$$\operatorname{Re}(a(x,D)u,u) \ge \frac{c_0}{2}\|u\|_0^2 - C_0\|u\|_{-1/2}^2 \quad \text{for } u \in C_0^\infty(U_1).$$

By (A-3) we see that

($\tilde{\mathrm{A}}$-3) there exists a neighborhood U of 0 such that

$$a_2^0(x,\xi) \ne 0 \quad \text{if } x = (x',x'') \in U, \quad |\xi| = 1 \text{ and } x' \ne 0,$$

where

$$a_2^0(x,\xi) = |\xi|^{-m+2} p_m(x,\xi) \quad \text{for } |\xi| \ge 1.$$

Next we consider (A-4). Let $|\alpha| + |\beta| = 2$ and $\alpha' = 0$. Then

$$
\begin{aligned}
&(\log\langle\xi\rangle)^{|\alpha|}|a_{2(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-|\beta|} \\
&\quad = (\log\langle\xi\rangle)^{|\alpha|}|\partial_\xi^\alpha(\langle\xi\rangle^{-m+2}p_{m(\beta)}(x,\xi))|\langle\xi\rangle^{-|\beta|} \\
&\quad \le \langle\xi\rangle^{-m+2}(\log\langle\xi\rangle)^{|\alpha|}p_{m(\beta)}^{(\alpha)}(x,\xi)\langle\xi\rangle^{-|\beta|} \\
&\qquad + \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ \alpha^1>0}} \frac{\alpha!}{\alpha^1!\alpha^2!}(\log\langle\xi\rangle)^{|\alpha|}|\partial_\xi^{\alpha^1}\langle\xi\rangle^{-m+2}|\,|p_{m(\beta)}^{(\alpha^2)}(x,\xi)|\langle\xi\rangle^{-|\beta|} \\
&\quad \le \nu\langle\xi\rangle^{-m+2}s(x,\xi) + C_\nu\langle\xi\rangle^{-1} + C_1(\log\langle\xi\rangle)^{|\alpha|}\sqrt{p_m(x,\xi)}\langle\xi\rangle^{-m/2} \\
&\qquad + C_2(\log\langle\xi\rangle)^2|p_m(x,\xi)|\langle\xi\rangle^{-m} \qquad (2.6)
\end{aligned}
$$

if $\nu > 0$, where C_1 and C_2 are some positive constants. Note that

$$
\begin{aligned}
(\log\langle\xi\rangle)^{|\alpha|}\sqrt{p_m(x,\xi)}\langle\xi\rangle^{-m/2} &= \sqrt{p_m(x,\xi)\langle\xi\rangle^{-m+2\varepsilon}}\langle\xi\rangle^{-\varepsilon}(\log\langle\xi\rangle)^{|\alpha|} \\
&\le \frac{1}{2}(p_m(x,\xi)\langle\xi\rangle^{-m+2\varepsilon} + \langle\xi\rangle^{-2\varepsilon}(\log\langle\xi\rangle)^{2|\alpha|}) \\
&\le \frac{1}{2}s(x,\xi)\langle\xi\rangle^{-m+2\varepsilon} + C'\langle\xi\rangle^{-1+2\varepsilon} \\
&\quad + \frac{1}{2}\langle\xi\rangle^{-2\varepsilon}(\log\langle\xi\rangle)^{2|\alpha|}
\end{aligned}
$$

for $\varepsilon > 0$. Let $\varepsilon = 1/3$. Then we have

$$(\log\langle\xi\rangle)^{|\alpha|}\sqrt{p_m(x,\xi)}\langle\xi\rangle^{-m/2} \le \frac{1}{2}s(x,\xi)\langle\xi\rangle^{-m+2} + C''\langle\xi\rangle^{-1/3}.$$

Therefore for any $\nu > 0$ there are constants C_ν and C'_ν such that

$$
\begin{aligned}
(\log\langle\xi\rangle)^{|\alpha|}|a_{2(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-|\beta|} &\le \nu s(x,\xi)\langle\xi\rangle^{-m+2} + C_\nu\langle\xi\rangle^{-1/3} \\
&\le 2\nu\,\mathrm{Re}\,a(x,\xi) + C'_\nu\langle\xi\rangle^{-1/3}.
\end{aligned}
$$

Similarly we can deal with (ii) in (A-4). Then we have the following:

($\tilde{\text{A}}$-4) There is a constant $C_\nu > 0$ such that

(i) $$\sum_{\substack{|\alpha|+|\beta|=2 \\ \alpha'=0}} (\log\langle\xi\rangle)^{|\alpha|}|a_{2(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-|\beta|} \le \nu\,\mathrm{Re}\,a(x,\xi) + C_\nu\langle\xi\rangle^{-1/3},$$

(ii) $$\sum_{\substack{|\alpha|+|\beta|=1 \\ \alpha'=0}} (\log\langle\xi\rangle)^{|\alpha|}|\operatorname{Im} a_{1(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-|\beta|} \le \nu \operatorname{Re} a(x,\xi) + C_\nu\langle\xi\rangle^{-1/3}$$

if $x \in U$.

Therefore, in order to prove Theorem 1.2 it suffices to show that $\tilde{p}(x,D)$ is hypoelliptic at 0 under ($\tilde{\mathrm{A}}$-1)–($\tilde{\mathrm{A}}$-4).

We need a simple variant of the Fefferman-Phong inequality to prove Theorem 1.2.

PROPOSITION 2.2. *Let* $q(x,\xi) \in S^2$ *satisfy*

$$|q_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{2-|\alpha|} \quad \textit{for} \ (x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n.$$

Let U *and* U_1 *be open sets in* $\boldsymbol{R}^n$ *satisfying* $U_1 \subset\subset U$. *If* $q(x,\xi) \ge 0$ *for* $x \in U$, *then there exists a constant* $C \equiv C(\{C_{\alpha,\beta}\}, U, U_1)$ *such that*

$$(q(x,D)u,u) \ge -C\|u\|_0^2 \quad \textit{for} \ u \in C_0^\infty(U_1). \tag{2.7}$$

PROOF. We choose a cut-off function $\chi \in C_0^\infty(\boldsymbol{R}^n)$ so that

$$\chi(x) = 1 \quad \text{in a neighborhood of } \overline{U_1}.$$

Then

$$(q(x,D)u,u) = (\chi(x)q(x,D)u,u) + ((1-\chi(x))q(x,D)u,u).$$

Since $\chi(x)q(x,\xi) \ge 0$, we can apply the Fefferman-Phong inequality. So there exists a constant C such that

$$(\chi(x)q(x,D)u,u) \ge -C\|u\|_0^2 \quad \text{for } u \in C_0^\infty(U_1).$$

On the other hand,

$$((1-\chi(x))q(x,D)u,u) = 0 \quad \text{for } u \in C_0^\infty(U_1),$$

since $\chi = 1$ in a neighborhood of $\overline{U_1}$ and $u \in C_0^\infty(U_1)$. Therefore we obtain the estimate (2.7). □

3. Proof of Theorem 1.2

In this section, we shall show that $\tilde{p}(x,D)$ is hypoelliptic at 0 applying Proposition 2.1. Put

$$\tilde{p}_\Lambda(x,D) = (e^{-\Lambda})(x'',D)\tilde{p}(x,D)(e^{\Lambda})(x'',D).$$

Then we can write

$$\begin{aligned}\tilde{p}_\Lambda(x,\xi) &= e^{-\Lambda(x'',\xi)}\sharp(a(x,\xi)+b(x,\xi))\sharp e^{\Lambda(x'',\xi)}\\ &= a(x,\xi)+b(x,\xi)+i\{\Lambda,a\}(x,\xi)-\frac{1}{2}(\operatorname{Hess} a)(-H_\Lambda)\\ &\quad+\frac{1}{2}\sum_{j,k=r+1}^{n}(\Lambda_{x_jx_k}\Lambda_{\xi_k}-\Lambda_{x_j\xi_k}\Lambda_{x_k})a_{\xi_j}(x,\xi)\\ &\quad+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=r+1}^{n}(\Lambda_{\xi_j\xi_k}\Lambda_{x_k}-\Lambda_{\xi_jx_k}\Lambda_{\xi_k})a_{x_j}(x,\xi)\\ &\quad+\frac{1}{4}\sum_{j=r+1}^{n}\{\Lambda_{\xi_j},\Lambda_{x_j}\}a(x,\xi)+r_1(x,\xi),\end{aligned}$$

where $a(x,\xi)\sharp b(x,\xi)=\sigma(a(x,D)b(x,D))(x,\xi)$, $\{a,b\}(x,\xi)=\sum_{j=1}^{n}\{a_{\xi_j}(x,\xi)b_{x_j}(x,\xi)-a_{x_j}(x,\xi)b_{\xi_j}(x,\xi)\}$, $(\operatorname{Hess} a)(x,\xi)$ stands for the Hessian matrix of $a(x,\xi)$, $(\operatorname{Hess} a)(\delta z)={}^t\delta z(\operatorname{Hess} a)(x,\xi)\delta z$, H_Λ does the Hamilton vector field of $\Lambda(x,\xi)$, $\Lambda_{x_j}(x,\xi)=(\partial/\partial x_j)\Lambda(x,\xi)$, $\Lambda_{x_jx_k}(x,\xi)=\partial^2/(\partial x_j\partial x_k)\Lambda(x,\xi)$ and $r_1(x,\xi)\in\bigcap_{\varepsilon>0}S^{-1+\varepsilon}$.

Let

$$A=(\operatorname{Re} a)(x,D)+C_0\langle D\rangle^{-1},$$

where C_0 is the constant in ($\tilde{\text{A}}$-2). Let U and U_1 be neighborhoods of 0 which appeared in ($\tilde{\text{A}}$-1)–($\tilde{\text{A}}$-4). We may assume that $U_1\subset\subset U$. Then by ($\tilde{\text{A}}$-2) we have

$$(Au,u)\geq\frac{c_0}{2}\|u\|_0^2\quad\text{for } u\in C_0^\infty(U_1). \tag{3.1}$$

Further, we have

$$\begin{aligned}\operatorname{Re}(\tilde{p}_\Lambda(x,D)u,u) &\geq (Au,u)-C\|u\|_{-1/4}^2\\ &\quad-\frac{1}{4}\sum_{j=r+1}^{n}|(\operatorname{Op}(\{\Lambda_{\xi_j},\Lambda_{x_j}\}a_2)u,u)|\\ &\quad-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=r+1}^{n}|\operatorname{Op}((\Lambda_{\xi_j\xi_k}\Lambda_{x_j}-\Lambda_{\xi_jx_k}\Lambda_{\xi_k})a_{x_j})u,u)|\\ &\quad-\frac{1}{2}\sum_{j,k=r+1}^{n}|\operatorname{Op}((\Lambda_{x_jx_k}\Lambda_{\xi_j}-\Lambda_{x_j\xi_k}\Lambda_{x_k})a_{\xi_j})u,u)|\\ &\quad-\left|\left(\operatorname{Op}\left(\{\Lambda,\operatorname{Im} a_1\}+\frac{1}{2}(\operatorname{Hess} a)(-H_\Lambda)\right)u,u\right)\right|\\ &\equiv (Au,u)-C\|u\|_{-1/4}^2-\frac{1}{4}I_1-\frac{1}{2}I_2-\frac{1}{2}I_3-I_4,\end{aligned}\tag{3.2}$$

where $\mathrm{Op}(q)$ denotes the pseudo-differential operator with the Weyl symbol $q(x,\xi)$.

As for I_1, Schwarz' inequality shows that

$$|(Au,v)| \le (Au,u)^{1/2}(Av,v)^{1/2}$$

for $u,v \in C_0^\infty(U_1)$. Let $u \in C_0^\infty(U_1)$. Since $\{\Lambda_{\xi_j},\Lambda_{x_j}\} \in S^{-2+\varepsilon}$ $(\varepsilon > 0)$, we obtain

$$\begin{aligned} I_1 &\le \sum_{j=r+1}^{n} |(Au,\mathrm{Op}(\{\Lambda_{\xi_j},\Lambda_{x_j}\})u| + C\|u\|_{-1/4}^2 \\ &\le C(Au,u)^{1/2}\|u\|_{-1/4} + C\|u\|_{-1/4}^2. \end{aligned} \tag{3.3}$$

Therefore for any $\nu > 0$ there is a constant C_ν such that

$$I_1 \le \nu(Au,u) + C_\nu\|u\|_{-1/4}^2.$$

Next we estimate I_2. We can see that

$$I_2 \le C\sum_{j=1}^{n}\sum_{k=r+1}^{n} \|\mathrm{Op}(\langle\xi\rangle^{-1}\,\mathrm{Re}\,a_{x_j})u\|_0\|u\|_{-1/2},$$

and

$$\|\mathrm{Op}(\langle\xi\rangle^{-1}\,\mathrm{Re}\,a_{x_j})u\|_0^2 = (\mathrm{Op}(\langle\xi\rangle^{-1}\,\mathrm{Re}\,a_{x_j})\,\mathrm{Op}(\langle\xi\rangle^{-1}\,\mathrm{Re}\,a_{x_j})u,u).$$

Set

$$c(x,\xi) = (\langle\xi\rangle^{-1}\,\mathrm{Re}\,a_{x_j}(x,\xi))\sharp(\langle\xi\rangle^{-1}\,\mathrm{Re}\,a_{x_j}(x,\xi)).$$

We have

$$c(x,\xi) = \langle\xi\rangle^{-2}(\mathrm{Re}\,a_{x_j}(x,\xi))^2 + r_1(x,\xi),$$

where $r_1(x,\xi) \in S^0$. We choose a constant C so that

$$\mathrm{Re}\,a(x,\xi) + C\langle\xi\rangle^{-1} \ge 0 \quad \text{for } x \in U,$$

that is, C is just the same appeared in ($\tilde{\mathrm{A}}$-1). We write

$$c(x,\xi) = \langle\xi\rangle^{-2}\{(\mathrm{Re}\,a(x,\xi) + C\langle\xi\rangle^{-1})_{x_j}\}^2 + r_1(x,\xi).$$

Therefore

$$|c(x,\xi)| \le C'(\mathrm{Re}\,a(x,\xi) + 1) \quad (x \in U).$$

So we obtain

$$C'(\operatorname{Re} a(x,\xi)+1) \pm c(x,\xi) \geq 0 \quad (x \in U).$$

Applying Proposition 2.2 we have

$$((C' \operatorname{Op}(\operatorname{Re} a) \pm c(x,D))u,u) \geq -C\|u\|_0^2.$$

This shows that

$$|(c(x,D)u,u)| \leq C'(Au,u) + C\|u\|_0^2.$$

So we have

$$I_2 \leq \nu((Au,u) + \|u\|_0^2) + C_\nu \|u\|_{-1/4}^2. \tag{3.4}$$

As for I_3, we have

$$I_3 \leq C \sum_{j,k=r+1}^{n} \|\operatorname{Op}(\operatorname{Re} a_{\xi_j})u\|_0 \|u\|_{-1/2}.$$

Let

$$\tilde{c}(x,\xi) = (\operatorname{Re} a_{\xi_j}(x,\xi)) \sharp (\operatorname{Re} a_{\xi_j}(x,\xi)).$$

Then

$$\begin{aligned}\tilde{c}(x,\xi) &= (\operatorname{Re} a_{\xi_j}(x,\xi))^2 + r_2(x,\xi)\\ &= \{(\operatorname{Re} a(x,\xi) + C\langle\xi\rangle^{-1})_{\xi_j}\}^2 + r_2'(x,\xi),\end{aligned}$$

where $r_2(x,\xi)$, $r_2'(x,\xi) \in S^0$ and the constant C is as in ($\tilde{\mathrm{A}}$-1). Therefore we have

$$|\tilde{c}(x,\xi)| \leq C'(\operatorname{Re} a(x,\xi)+1) \quad (x \in U),$$

for some C'. This gives

$$I_3 \leq \nu((Au,u) + \|u\|_0^2) + C_\nu \|u\|_{-1/4}^2. \tag{3.5}$$

Choose $\Psi(\xi) \in C^\infty(\boldsymbol{R}^n)$ so that

$$\Psi(\xi) = \begin{cases} 1 & (|\xi| \geq 2)\,, \\ 0 & (|\xi| \leq 1). \end{cases}$$

For $0 < \nu \leq 1$ we put

$$q_\nu^{\pm}(x,\xi) = \left(\nu \operatorname{Re} a(x,\xi) + C_\nu\langle\xi\rangle^{-1} \pm \frac{1}{\nu}\{\Lambda, \operatorname{Im} a_1\} \pm \frac{1}{2\nu}(\operatorname{Hess} a)(-H_\Lambda)\right)\Psi(s_\nu \xi),$$

where the s_ν satisfy $0 < s_\nu \leq 1$ and are determined later. By virtue of ($\tilde{\mathrm{A}}$-4) we can choose C_ν so that

$$q_\nu^{\pm}(x,\xi) \geq 0 \quad (x \in U).$$

Therefore we have

$$|q_{\nu(\beta)}^{\pm(\alpha)}(x,\xi)| \leq \langle\xi\rangle^{2-|\alpha|}\left(C_{\alpha,\beta} + C_\nu s_\nu^3 C_{\alpha,\beta} + \frac{s_\nu^{3/2}}{\nu} C_{\alpha,\beta}\right).$$

Now we choose s_ν so that $C_\nu s_\nu^3 \leq 1$, $\frac{1}{\nu} s_\nu^{3/2} \leq 1$. Then

$$|q_{\nu(\beta)}^{\pm(\alpha)}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{2-|\alpha|},$$

where $C_{\alpha,\beta}$ are independent of ν. Therefore by Proposition 2.2 we have

$$(q_\nu^{\pm}(x,D)u,u) \geq -C\|u\|_0^2,$$

where C does not depend on ν. Therefore

$$\begin{aligned}&|(\mathrm{Op}(\{\Lambda, \mathrm{Im}\, a_1\} + \tfrac{1}{2}(\mathrm{Hess}\, a)(-H_\Lambda))u,u)| \\ &\qquad \leq \nu^2(\mathrm{Op}(\mathrm{Re}\, a)u,u) + \nu C_\nu\|u\|_{-1/2}^2 + \nu C\|u\|_0^2 + C\|u\|_{-1}^2.\end{aligned}$$

Thus,

$$I_4 \leq \nu((Au,u) + \|u\|_0^2) + C_\nu\|u\|_{-1/2}^2. \tag{3.6}$$

Consequenyly, by (3.1)–(3.6) we have

$$\mathrm{Re}(\tilde{p}_\Lambda(x,D)u,u) \geq \frac{c_0}{4}\|u\|_0^2 - C\|u\|_{-1/4}^2. \tag{3.7}$$

Schwarz' inequality gives

$$\mathrm{Re}(\tilde{p}_\Lambda(x,D)u,u) \leq C\|\tilde{p}_\Lambda(x,D)u\|_0^2 + \frac{c_0}{8}\|u\|_0^2. \tag{3.8}$$

Therefore in virtue of (3.7) and (3.8), there is a constant C such that

$$\|u\|_0 \leq C(\|\tilde{p}_\Lambda(x,D)u\|_0 + \|u\|_{-1}).$$

Applying Proposition 2.1 with $x^0 \in U_1$, we see that $\tilde{p}(x,D)$ is hypoelliptic at 0. This completes the proof of Theorem 1.2. $\square$

4. Remarks and Examples

In this section we shall first study the conditions which we impose on $p(x,\xi)$. Finally we shall give several examples.

PROPOSITION 4.1. *If* (A-0), (A-1)$'$ *and* (A-2)$'$, *then* (A-1) *and* (A-2) *hold.*

PROOF. It is obvious that (A-1) holds. Without loss of generality, we may assume that

$$p_m(0,\xi) \equiv 0 \quad \text{for } |\xi| \geq 1.$$

By using Taylor expansion and (A-2)$'$, we have

$$p_m(x,\xi) = \sum_{|\beta|=3} \frac{3x^\beta}{\beta!} \int_0^1 (1-\theta)^3 (\partial_x^\beta p_m)(\theta x, \xi)\, d\theta.$$

Changing the variable x to y so that $x = \nu y$ where $0 < \nu \leq 1$, we write

$$v_\nu(y) = u(\nu y) \quad \text{for } u \in C_0^\infty.$$

Let B be a unit ball centered at 0, $\chi(x) \in C_0^\infty(B)$ with $\chi(x) = 1$ in $|x| \leq 2/3$, and choose $0 < \nu_0 \leq 1$ so that $\nu_0 B \subset U$, where U is a neighborhood of 0 in ($\tilde{\text{A}}$-1). For ν with $0 < \nu \leq \nu_0$ we put

$$p_{m,\nu}(x,\xi) = \chi\left(\frac{x}{\nu}\right) p_m(x,\xi) \Psi(\nu\xi),$$

where $\Psi(\xi)$ is the symbol used in Section 3. Then

$$\begin{aligned} p_{m,\nu}(x,D)u|_{x=\nu y} &= (2\pi)^{-n} \int \left(\int e^{i\nu(y-\tilde{y})\cdot\xi} p_{m,\nu}\left(\frac{\nu(y+\tilde{y})}{2}, \xi\right) v_\nu(\tilde{y}) \nu^n \, d\tilde{y} \right) d\xi \\ &= (2\pi)^{-n} \int \left(\int e^{i(y-\tilde{y})\cdot\eta} p_{m,\nu}\left(\frac{\nu(y+\tilde{y})}{2}, \frac{\eta}{\nu}\right) v_\nu(\tilde{y}) \, d\tilde{y} \right) d\eta \\ &\equiv q_\nu(y, D_y) v_\nu(y). \end{aligned}$$

Thus we have

$$\begin{aligned} q_\nu(y,\eta) &= p_{m,\nu}\left(\nu y, \frac{\eta}{\nu}\right) \\ &= \chi(y) p_m\left(\nu y, \frac{\eta}{\nu}\right) \Psi(\eta) \\ &= \nu^{-m} \chi(y) p_m(\nu y, \eta) \Psi(\eta) \\ &= \nu^{-m+3} \sum_{|\beta|=3} \frac{3y^\beta}{\beta!} \chi(y) \int_0^1 (1-\theta)^3 (\partial_x^\beta p_m)(\theta \nu y, \eta)\, d\theta \Psi(\eta), \end{aligned}$$

and set

$$\tilde{q}_\nu(y,\eta) = \nu^{m-3} q_\nu(y,\eta).$$

Then

$$\tilde{q}_\nu(y,\eta) \geq 0 \quad \text{for } (y,\eta) \in \boldsymbol{R}^n \times \boldsymbol{R}^n.$$

Further we have

$$|\tilde{q}^{(\alpha)}_{\nu(\beta)}(y,\eta)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m-|\alpha|},$$

where the $C_{\alpha,\beta}$ are independent of ν. Therefore by Proposition 2.2

$$(\tilde{q}_\nu(y, D_y)v_\nu, v_\nu) \geq -C\|v_\nu\|^2_{m/2-1},$$

for $0 < \nu \leq \nu_0$ if $u \in C_0^\infty(U)$ and $v_\nu(y) = u(\nu y)$. Then

$$\begin{aligned}(\tilde{q}_\nu(y, D_y)v_\nu, v_\nu) &= \nu^{m-3}((p_{m,\nu}(x,D)u|_{x=\nu y}, u(\nu y)) \\ &= \nu^{m-3-n}((p_{m,\nu}(x,D)u, u) \quad \text{for } u \in C_0^\infty(U).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|v_\nu\|^2_{m/2-1} &= (2\pi)^{-n} \int \langle \eta \rangle^{m-2} |\hat{v}_\nu(\eta)|^2 \, d\eta \\ &= (2\pi)^{-n} \int \langle \nu\xi \rangle^{m-2} |\hat{v}_\nu(\nu\xi)|^2 \nu^n \, d\xi \\ &= (2\pi)^{-n} \int \langle \nu\xi \rangle^{m-2} |\hat{u}(\xi)|^2 \nu^{-n} \, d\xi.\end{aligned}$$

Since

$$\sqrt{1+\nu^2|\xi|^2} = \nu\sqrt{(1+|\xi|^2) + \frac{1}{\nu^2} - 1},$$

we have

$$\langle \nu\xi \rangle^{m-2} \leq (2\nu)^{m-2}\left(\langle \xi \rangle^{m-2} + \left(\frac{1}{\nu^2} - 1\right)^{m/2-1}\right) \quad \text{if } m \geq 2.$$

If $m < 2$, then

$$\langle \nu\xi \rangle^{m-2} \leq \nu^{m-2} \langle \xi \rangle^{m-2}.$$

Therefore we obtain

$$\|v_\nu\|_{m/2-1}^2 \leq \begin{cases} \nu^{m-2-n}2^{m-2}\left(\|u\|_{m/2-1}^2 + \left(\frac{1}{\nu^2}-1\right)^{m/2-1}\|u\|_0^2\right) & \text{if } m > 2, \\ \nu^{-n}\|u\|_0^2 & \text{if } m = 2, \\ \nu^{m-2-n}\|u\|_{m/2-1}^2 & \text{if } m < 2. \end{cases}$$

Consequently, we have

$$(p_{m,\nu}(x,D)u,u) \geq -\nu C\|u\|_{m/2-1}^2 - C_\nu\|u\|_{m/2-2}^2.$$

Further, there exists a constant $c > 0$ such that

$$\mathrm{Re}\, p_{m-1}(x,\xi) + \mathrm{Re}\, p_{m-2}(x,\xi) \geq c|\xi|^{m-2},$$

if $x \in U$ and $|\xi| \geq 1$. Then there is a constant C' such that

$$\mathrm{Re}((p_{m-1}+p_{m-2})(x,D)u,u) \geq c\|u\|_{m/2-1}^2 - C'\|u\|_{m/2-2}^2$$

for $u \in C_0^\infty(U)$. Taking ν so that $\nu C < c/2$, we have

$$\begin{aligned} \mathrm{Re}(p(x,D)u,u) &= (p_{m,\nu}(x,D)u,u) + \left(\mathrm{Op}\left(\chi\left(\frac{x}{\nu}\right)p_m(x,\xi)(1-\Psi(\nu\xi))u,u\right)\right. \\ &\quad + \left(\mathrm{Op}\left(\left(1-\chi\left(\frac{x}{\nu}\right)\right)p_m(x,\xi)\right)u,u\right) + \mathrm{Re}((p(x,D)-p_m(x,D))u,u) \\ &\geq \frac{c}{2}\|u\|_{m/2-1}^2 - C_\nu\|u\|_{m/2-2} \quad \text{if } u \in C_0^\infty\left(\frac{\nu}{2}B\right), \end{aligned}$$

since $\mathrm{Op}(\chi(x/\nu)p_m(x,\xi)(1-\Psi(\nu\xi))$ and $\mathrm{Op}((1-\chi(x/\nu))p_m(x,\xi)$ are in $S^{-\infty}$. Therefore we know that (A-2) holds with U replaced by $\nu/2B$. □

Proposition 4.2. *We assume that* (A-0), (A-1)′ *and*

(A-2)″ there exists a neighborhood U of 0 such that

$$\text{(i)} \quad p_{m(\beta)}(x,\xi) = 0 \quad (|\beta| = 2),$$

$$\text{(ii)} \quad \mathrm{Im}\, p_{m-1(\beta)}(x,\xi) = 0 \quad (|\beta| = 1),$$

if $x \in U$, $|\xi| \geq 1$ and $p_m(x,\xi) = 0$.

Then we have for any $\nu > 0$ there is a constant C_ν such that

$$\sum_{|\beta|=2} |p_{m(\beta)}(x,\xi)|\langle\xi\rangle^{-2} + \sum_{|\beta|=1} |\mathrm{Im}\, p_{m-1(\beta)}(x,\xi)|\langle\xi\rangle^{-1} \leq \nu s(x,\xi) + C_\nu\langle\xi\rangle^{m-3},$$

if $x \in U$.

PROOF. Let

$$V = \{(x, \xi) \in U \times S^{n-1}; p_m(x, \xi) = 0\},$$

and

$$I(x, \xi) = \sum_{|\beta|=2} |p_{m(\beta)}(x, \xi)| \langle \xi \rangle^{-2} + \sum_{|\beta|=1} |\mathrm{Im}\, p_{m-1(\beta)}(x, \xi)| \langle \xi \rangle^{-1},$$

where S^{n-1} denotes the $(n-1)$-dimensional unit sphere. Then

$$I(x, \xi) = 0 \quad \text{in } \bar{V}.$$

Let $\nu > 0$ and V_ν be a neighborhood of $\bar{V}$ in $\boldsymbol{R}^n \times S^{n-1}$ satisfying

$$I(x, \xi) \le \nu c \quad \text{for } (x, \xi) \in V_\nu,$$

where

$$c = \min_{\substack{x \in \bar{U} \\ |\xi|=1}} \mathrm{Re}\, p_{m-2}(x, \xi) > 0.$$

Then there is a constant $\hat{c}_\nu > 0$ such that

$$p_m(x, \xi) \ge \hat{c}_\nu \quad \text{for } (x, \xi) \in (\bar{U} \times S^{n-1}) \backslash V_\nu.$$

Therefore

$$p_m(x, \xi) \ge (\hat{c}_\nu |\xi|^2) |\xi|^{m-2},$$

if $|\xi| \ge 1$, $(x, \xi/|\xi|) \in (\bar{U} \times S^{n-1}) \backslash V_\nu$. Hence we have

$$I(x, \xi) \le \nu(p_m(x, \xi) + \mathrm{Re}\, p_{m-1}(x, \xi) + \mathrm{Re}\, p_{m-2}(x, \xi)),$$

if $(x, \xi/|\xi|) \in \bar{U} \times S^{n-1} \backslash V_\nu$ and $|\xi| \gg 1$. This proves Proposition 4.2. $\square$

Thus Proposition 4.1 and 4.2 imply that the operator L_2 defined in Section 1 satisfies (A-0)–(A-4) if it satisfies (B-1) and (B-4). In particular, L_2 is hypoelliptic at 0 under the conditions (B-1) and (B-4).

EXAMPLE 4.3. Let $h_k(x) \in C^\infty$ $(1 \le k \le n)$ satisfy $h_k \ge 0$ and

$$h_{k(\beta)}(x) = 0 \quad \text{if } x \in \boldsymbol{R}^n, \quad h_k(x) = 0 \quad \text{and} \quad |\beta| = 2.$$

We assume that there exist constants $C_{kj} > 0$ and $m_{kj} > 0$ such that for any $k, j = 1, \dots, n$

$$h_k(x) \le C_{kj} h_j(x)^{m_{kj}}.$$

Put

$$p(x,\xi)=\sum_{k=1}^{n}h_k(x)\xi_k^2+1.$$

Then, applying Theorem 1.2 and Proposition 4.2 we can see that $p(x,D)$ is hypoelliptic.

For $\sigma>0$ we put

$$f_\sigma(t)=\begin{cases}\exp\left(-\dfrac{1}{|t|^\sigma}\right) & (t\neq 0),\\ 0 & (t=0).\end{cases}$$

EXAMPLE 4.4. Let $n=2$ and $\sigma>0$. Put

$$p(x,\xi)=x_1^4\xi_1^2+f_\sigma(x_1)\xi_2^2+1.$$

Then $p(x,D)$ is hypoelliptic. Indeed, by Proposition 4.1 $p(x,\xi)$ satisfies (A-0) with $m=2$, (A-1), (A-2) and (A-3) with $r=1$. Note that

$$|f'_\sigma(t)|\leq C\sqrt{f_\sigma(t)}\leq C(f_\sigma(t)|\xi|^{3/2}+|\xi|^{-3/2})$$

for $|\xi|\neq 0$. Therefore we have

$$\begin{aligned}&f_\sigma(x_1)(\log\langle\xi\rangle)^2+|f'_\sigma(x_1)|\log\langle\xi\rangle\\&\quad\leq\begin{cases}Cx_1^4(1+\xi_1^2)^{1/2}\leq C'p(x,\xi)\langle\xi\rangle^{-1} & \text{if } |\xi_1|\geq|\xi_2|,\\ Cp(x,\xi)\langle\xi\rangle^{-1}+Cp(x,\xi)\langle\xi\rangle^{-1/4}+C\langle\xi\rangle^{-1} & \text{if } |\xi_1|\leq|\xi_2|.\end{cases}\end{aligned}$$

This implies that $p(x,\xi)$ satisfies (A-4).

EXAMPLE 4.5. Let $n=2$ and $0<\sigma<2$. Put

$$p(x,\xi)=f_\sigma(x_1)\xi_1^2+x_1^4\xi_2^2+1.$$

Let us prove that $p(x,\xi)$ satisfies (A-0)–(A-4). It is obvious that $p(x,\xi)$ satisfies (A-0) with $m=2$, (A-2) and (A-3) with $r=1$. Fix $v>0$. Assume that $x_1^4(\log\langle\xi\rangle)^2\geq v$. Then we have

$$\begin{aligned}vf_\sigma(x_1)\langle\xi\rangle^2&\geq v\exp(-v^{-\sigma/4}(\log\langle\xi\rangle)^{\sigma/2})\langle\xi\rangle^2\\&\geq v\langle\xi\rangle^{2-v^{-\sigma/4}(\log\langle\xi\rangle)^{\sigma/2-1}}\\&\geq v\langle\xi\rangle\end{aligned}$$

if $\langle\xi\rangle \geq \exp(\nu^{-\sigma/(2(2-\sigma))})$. This gives

$$x_1^4(\log\langle\xi\rangle)^2 \leq \nu p(x,\xi) + C_\nu\langle\xi\rangle^{-1} \quad \text{if } |x_1| \leq 1,$$

where C_ν is a constant. Similarly, we have

$$\nu f_\sigma(x_1)\langle\xi\rangle^2 \geq \nu\langle\xi\rangle^{2-\nu^{-\sigma/3}(\log\langle\xi\rangle)^{\sigma/3-1}} \geq \nu\langle\xi\rangle$$

if $|x_1|^3\log\langle\xi\rangle \geq \nu$ and $\langle\xi\rangle \geq \exp(\nu^{-\sigma/(3-\sigma)})$. This gives, with some constant C_ν,

$$|x_1|^3\log\langle\xi\rangle \leq \nu p(x,\xi) + C_\nu\langle\xi\rangle^{-1} \quad \text{if } |x_1| \leq 1.$$

Therefore $p(x,\xi)$ satisfies (A-4), and $p(x,D)$ is hypoelliptic.

EXAMPLE 4.6. Let $n = 1$ and $C \in \boldsymbol{R}\backslash\{0\}$. Then

$$p(x,D) = -x^4\partial_x^2 - C^2.$$

does not satisfy (A-2). If we choose

$$u(x) = \begin{cases} x\exp(iCx^{-1}) & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

then for $\varphi(x) \in C_0^\infty(\boldsymbol{R})$

$$\begin{aligned}\langle p(x,D)u,\varphi\rangle &= \langle(-x^4\partial_x^2 - C^2)u,\varphi\rangle \\ &= -\langle\partial_x^2(x^4u) - 8\partial_x(x^3u) + (12x^2 + C^2)u,\varphi\rangle \\ &= \langle 0,\varphi\rangle.\end{aligned}$$

Therefore

$$p(x,D)u = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}).$$

However u is not differentiable at $x = 0$, that is,

$$0 \in \operatorname{sing\,supp} u.$$

Hence, $p(x,D)$ is not hypoelliptic at $x = 0$.

EXAMPLE 4.7. Let $C \in \boldsymbol{C}$. Then

$$p(x,D) = -x_1^2\Delta + C.$$

does not satisfy (A-4). Put

$$u(x) = (x_1)_+^\lambda = \begin{cases} x_1^\lambda & (x_1 > 0), \\ 0 & (x_1 \leq 0), \end{cases}$$

where $\lambda = (1 + \sqrt{1+4C})/2$ and we take a branch of $\sqrt{1+4C}$ satisfying $\mathrm{Re}\sqrt{1+4C} \geq 0$. Since $\mathrm{Re}\,\lambda \geq 1/2 > -1$, we have

$$x_1^2 \frac{d^2}{dx_1^2}((x_1)_+^\lambda) = \lambda(\lambda - 1)(x_1)_+^\lambda \quad \text{in } \mathscr{D}'(\boldsymbol{R}).$$

Therefore

$$x_1^2 \partial_{x_1}^2 u(x) = \lambda(\lambda - 1)u(x) \quad \text{in } \mathscr{D}'(\boldsymbol{R}^n).$$

Obviously,

$$\partial_{x_j}^2 u(x) = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}), \quad (2 \leq j \leq n).$$

Since $\lambda(\lambda - 1) - C = 0$, we obtain

$$P(x, D)u = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^n).$$

On the other hand, we have

$$0 \in \operatorname{sing\,supp} u.$$

Hence, $p(x, D)$ is not hypoelliptic at $x = 0$.

References

[1] L. Hörmander. Hypoelliptic differential operators. Ann. Inst. Fourier Grenoble, **11**: 477–492, XVI, 1961.

[2] L. Hörmander. The analysis of linear partial differential operators. III. Springer, Berlin-Heidelberg-New York-Tokyo, 1985.

[3] K. Kajitani and S. Wakabayashi. Propagation of singularities for several classes of pseudo-differential operators. Bull. Sci. Math. (2), **115**(4): 397–449, 1991.

[4] K. Katsuta. On the locally solvability of $-a(x)\Delta + b(x)$ (in Japanese). Master Thesis, University of Tsukuba, 1997.

[5] H. Kumano-go. Pseudodifferential operators. MIT Press, Cambridge, Mass., 1981. Translated from the Japanese by the author, Rémi Vaillancourt and Michihiro Nagase.

[6] N. Nakazawa. On hypoellipticity of $-a(x)\Delta + 1$ (in Japanese). Master Thesis, University of Tsukuba, 1998.

[7] M. Shubin. Pseudodifferential operators and spectral theory. Springer-Verlag, Berlin, 1987. Translated from the Russian by Stig I. Andersson.

[8] S. Wakabayashi and M. Suzuki. Microhypoellipticity for a class of pseudodifferential operators with double characteristics. Funkcial. Ekvac., **36**(3): 519–556, 1993.