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TOPOLOGICAL LATTICES $C_k(X)$ AND $C_p(X)$: EMBEDDINGS AND ISOMORPHISMS

By

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Abstract. For a Tychonoff space X, the topological lattices $C_k(X)$ and $C_p(X)$ of all real-valued continuous functions on X endowed respectively with the compact-open topology and the topology of pointwise convergence are studied. It is proved that $C_k(X)$ and $C_k(Y)$ are isomorphic if and only if $C_p(X)$ and $C_p(X)$ are isomorphic if and only if X and Y are homeomorphic. It is also shown that $C_p(Y)$ is embedded in $C_p(X)$ as a topological sublattice if and only if Y is a continuous image of a cozero-set of X.

1. Introduction

All spaces considered here are Tychonoff topological spaces. For a space X, the set of all real-valued continuous functions on X is denoted by $C(X)$. The subset of $C(X)$ consisting of bounded functions is denoted by $C^*(X)$. These sets can be regarded as lattices with respect to the order: $f \leq g$ if and only if $f(x) \leq g(x)$ at every point $x \in X$. Ring structures on $C(X)$ and $C^*(X)$ are also defined as usual and have been studied extensively. In case topological spaces are assumed to be compact, the following are famous.

KAPLANSKY THEOREM [4]. For compact spaces X and Y, if there is a lattice isomorphism between $C(X)$ and $C(Y)$, then X and Y are homeomorphic.

GELFAND-KOLMOGOROFF THEOREM [2]. For compact spaces X and Y, if there is a ring isomorphism between $C(X)$ and $C(Y)$, then X and Y are homeomorphic.

The Gelfand-Kolmogoroff theorem is considered as a corollary of the Kaplansky theorem since every ring isomorphism between function spaces is a

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lattice isomorphism. It is well known that $C(X)$ and $C(vX)$ are ring isomorphic for every space X, where vX is the Hewitt realcompactification of X [3]. So the Kaplansky theorem and the Gelfand-Kolmogoroff theorem can not be unconditionally extended to the class of Tychonoff spaces. However if some topological structure is added to $C(X)$, then the topological ring $C(X)$ happens to characterize the topology of X. The space $C(X)$ with the topology of pointwise convergence is denoted by $C_p(X)$. The space $C_k(X)$ is the space $C(X)$ with the compact-open topology. The following are known.

NAGATA THEOREM [6]. If $C_p(X)$ and $C_p(Y)$ are isomorphic as topological rings, then Tychonoff spaces X and Y are homeomorphic.

MORRIS-WULBERT THEOREM [5]. If $C_k(X)$ and $C_k(Y)$ are isomorphic as topological algebras, then Tychonoff spaces X and Y are homeomorphic.

It is also well-known that there are non-homeomorphic spaces X and Y such that $C_p(X)$ and $C_p(Y)$ (or $C_k(X)$ and $C_k(Y)$) are linearly homeomorphic (see [1]). Two topological lattices are called isomorphic if there exists a lattice isomorphism which is also a homeomorphism between these topological lattices. As mentioned above, every ring isomorphism between function spaces is a lattice isomorphism. And $C_p(X)$ and $C_k(X)$ are topological lattices in the sense that the operations $\vee$ and $\wedge$ are continuous. Hence the following question arises naturally:

Are X and Y homeomorphic if $C_k(X)$ and $C_k(Y)$ are isomorphic as topological lattices?

The same question is considered for function spaces with the topology of pointwise convergence. Notice that every order isomorphism between function spaces must be a lattice isomorphism. Hence, in order to see that $C_k(X)$ and $C_k(Y)$ are isomorphic as topological lattices, it suffices to show that there is an order-isomorphic homeomorphism between $C_k(X)$ and $C_k(Y)$. For Tychonoff spaces X and Y, we can show the following.

THEOREM 1. *If topological lattices $C_k(X)$* and *$C_k(Y)$ are isomorphic, then X and Y are homeomorphic.*

THEOREM 2. *If topological lattices $C_k^*(X)$ and $C_k^*(Y)$ are isomorphic, then X and Y are homeomorphic.*

THEOREM 3. *If topological lattices $C_p(X)$ and $C_p(Y)$ are isomorphic, then X and Y are homeomorphic.*

THEOREM 4. *If topological lattices $C_p^*(X)$ and $C_p^*(Y)$ are isomorphic, then X and Y are homeomorphic.*

These theorems are generalizations of the Nagata theorem or the Morris-Wulbert theorem, but not generalizations of the Kaplansky theorem. In order to generalize the Kaplansky theorem to the class of Tychonoff spaces, a new topology on $C(X)$ is introduced. This idea is based on the following: In the Kaplansky theorem, the lattices $C(X)$ and $C(Y)$ can be thought of as topological lattices with discrete topologies. Further the following more general question is considered:

What is the space Y whose $C_k(Y)$ can be embedded in $C_k(X)$ as a topological sublattice?

For this question, it does not seem that such a space Y can be simply characterized. However if we consider the topology of pointwise convergence instead of the compact-open topology, then we have a simple characterization.

2. Topological-lattice Embeddings and Proofs

The essential parts of the proofs of the above theorems can be concentrated in the proof of the following theorem.

THEOREM 5. *There is a topological-lattice embedding Φ from $C_k(Y)$ into $C_k(X)$ such that $\{\Phi(f)(x) : f \in C(Y)\}$ is open in $\boldsymbol{R}$ for any $x \in X$ if and only if there is a continuous map ϕ from X onto Y such that for any compact subset K of Y there exists a compact subset K' of X with $\phi(K') \supset K$.*

Here, the topological-lattice embedding $\Phi : C_k(Y) \to C_k(X)$ is a homeomorphic embedding which satisfies $\Phi(f \vee g) = \Phi(f) \vee \Phi(g)$ and $\Phi(f \wedge g) = \Phi(f) \wedge \Phi(g)$ for any $f, g \in C(Y)$.

A subset I of the lattice $C(Y)$ is said to be a prime ideal (see [7], [8]) if the following conditions are satisfied:

1) if $f \in I$ and $g \leq f$, then $g \in I$,
2) If $f, g \in I$, then $f \vee g \in I$,
3) if $f \wedge g \in I$, then $f \in I$ or $g \in I$,
4) $I \neq \varnothing$, $I \neq C(Y)$.

Let y be an arbitrary point of Y. For any fixed real number r, the set

$$I_y^{<r} = \{f \in C(Y) : f(y) < r\}$$

is a prime ideal. In general, when a prime ideal I is given, I is said to be associated with a point y_0 in Y if $f \in I$, $g \in C(Y)$ and $g(y_0) < f(y_0)$ imply $g \in I$.

PROOF OF THEOREM 5. If there is a continuous map ϕ from X onto Y such that for any compact subset K of Y there exists a compact subset K' of X with $\phi(K') \supset K$, then the canonical map $\Phi : C_k(Y) \to C_k(X)$ defined by $\Phi(f) = f \circ \phi$ is a topological-lattice embedding with $\{\Phi(f)(x) : f \in C(Y)\} = \boldsymbol{R}$ for any $x \in X$.

We assume that there is a topological-lattice embeding $\Phi : C_k(Y) \to C_k(X)$ such that $\{\Phi(f)(x) : f \in C(Y)\}$ is open in $\boldsymbol{R}$ for any $x \in X$. Since $C_k(Y)$ is connected, $\{\Phi(f)(x) : f \in C(Y)\}$ must be a non-empty open interval (a_x, b_x) for any $x \in X$.

For any point y in Y and any real number r, the prime ideal $I_y^{<r}$ defined as above is an open subset of $C_k(Y)$. Conversely,

(1) For any open prime ideal I in $C_k(Y)$, there exists a unique point y_0 of Y such that I is associated with y_0.

In fact, let f be an arbitrary element of I. Then there is a compact subset K of Y and an $\varepsilon > 0$ such that the canonical open set

$$\langle f, K, \varepsilon \rangle = \{g \in C(Y) : |g(y) - f(y)| < \varepsilon \ \forall y \in K\}$$

is a subset of I.

a) There is a point y_K in K which satisfies: if $g \in C(Y)$ and $g(y_K) < f(y_K)$, then $g \in I$.

Suppose that, for every point y in K, there exists $g_y \notin I$ such that $g_y(y) < f(y)$. Let $G_y = \{u \in Y : g_y(u) < f(u)\}$. then G_y is an open subset of Y containing y. Since K is compact, there are points $y_1, \dots, y_n \in K$ such that $K \subset G_{y_1} \cup \cdots \cup G_{y_n}$. Let

$$h = g_{y_1} \wedge \cdots \wedge g_{y_n}.$$

Then $h \notin I$ and $h|K < f|K$. However, since $(h \vee f)|K = f|K$, the supremum $h \vee f$ must be a member of I. Hence it follows that h is a member of I from the condition 1) of the prime ideal, which is a contradiction.

b) Such a point y_K is uniquely determined.

Assume that y_1 and y_2 be distinct points in K which satisfy the condition of a). Then for any $k \in C(Y)$ we can take $k_1, k_2 \in C(Y)$ with the following properties: $k = k_1 \vee k_2$, $k_1(y_1) < f(y_1)$ and $k_2(y_2) < f(y_2)$. This means that k_1,

$k_2 \in I$ and hence $k \in I$, which implies that $C(Y) = I$. This is a contradiction. By the same argument, we obtain the following

c) The point y_K does not depend on the choices of f and $\langle f, K, \varepsilon\rangle$.

Let y_0 be the point uniquely determined above. Then it is easy to see that I is associated with y_0.

Now, we can define a map ϕ from X to Y as follows: Take an arbitrary point x of X. For any real number a which satisfies $a_x < a < b_x$, let

$$J_x^{\prec a} = \{g \in C(Y) : \Phi(g)(x) < a\}.$$

Then this set is an open prime ideal in $C_p(Y)$ since Φ is a topological-lattice embedding. Hence a unique point y in Y, with which this open prime ideal is associated, is determined.

Since two open prime ideals I_1 and I_2 are associated with the same point if and only if $I_1 \cap I_2$ is a prime ideal,

(2) the point y does not depend on the choice of the value a.

This show that $\phi(x) = y$ is well-defined.

(3) ϕ is onto.

Let y be an arbitrary point of Y. Take a real number r and consider the open prime ideal $I_y^{<r} = \{f \in C(Y) : f(y) < r\}$ in $C_p(Y)$. Then, since $\Phi(I_y^{<r})$ is open in $\Phi(C_k(Y))$, if we take a function f in $I_y^{<r}$, then there is a compact subset K' of X and an $\varepsilon > 0$ such that

$$\langle \Phi(f), K', \varepsilon\rangle \cap \Phi(C(Y)) \subset \Phi(I_y^{<r}).$$

By the same argument as that in a) of (1), it is shown that there is a point x in K' with the following property: if $g \in C(Y)$ and $\Phi(g)(x) < \Phi(f)(x)$, then $g \in I_y^{<r}$. In fact, for any point $x \in K'$, assume that there exists $g_x \in C(Y)$ such that $\Phi(g_x)(x) < \Phi(f)(x)$ and $g_x \notin I_y^{<}$. Let $G_x = \{v \in X : \Phi(g_x)(v) < \Phi(f)(v)\}$ for each $x \in K'$. Since K' is compact, there exist $x_1, \ldots, x_n \in K'$ such that $K' \subset G_{x_1} \cup \cdots \cup G_{x_n}$. Let

$$g = g_{x_1} \wedge \cdots \wedge g_{x_n}.$$

Then $g \notin I_y^{<r}$ and $\Phi(g)|K' < \Phi(f)|K'$. Since $\Phi(g \vee f)|K' = \Phi(f)|K'$, $g \vee f$ must be in $I_y^{<r}$ and hence $g \in I_y^{<r}$. This is a contradiction. Let $a = \Phi(f)(x)$ and take the open prime ideal $J_x^{\prec a}$ in $C_p(Y)$ defined as that one in c) of (1). Then, since $J_x^{\prec a} \subset I_y^{<r}$, the open prime ideal $J_x^{\prec a}$ must be associated with y, which shows that $\phi(x) = y$.

(4) ϕ is continuous.

It suffices to show that, for any closed subset F of Y and any point $x \in X - \phi^{-1}(F)$, there are $g, h \in C(Y)$ which satisfy the following: $\Phi(h)(x) > \Phi(g)(x)$

and $\Phi(h)|\phi^{-1}(F) \leq \Phi(g)|\phi^{-1}(F)$. For $x \in X - \phi^{-1}(F)$ let $\phi(x) = y$. Since $\{\Phi(f)(x): f \in C(Y)\} = (a_x, b_x)$ is open, we can take a function $g \in C(Y)$ and $a \in (a_x, b_x)$ such that $\Phi(g)(x) < a$. Then the open prime ideal $J_x^{<a}$ defined as above is associated with $y \notin F$. Hence there is a function h in $C(Y)$ which satisfies; $h(u) < g(u)$ for any $u \in F$, and h is not a member of $J_x^{<a}$. It follows that $\Phi(h)(x) > \Phi(g)(x)$. We will show that $\Phi(h)|\phi^{-1}(F) \leq \Phi(g)|\phi^{-1}(F)$. Assume that there is a point x_0 in $\phi^{-1}(F)$ which satisfies $\Phi(g)(x_0) < \Phi(h)(x_0)$. Take a number r such that $\Phi(g)(x_0) < r < \Phi(h)(x_0)$. Then the open prime ideal $J_{x_0}^{<r}$ contains g but does not contain h. However, since this open prime ideal is associated with a point $\phi(x_0)$ in F and $h(\phi(x_0)) < g(\phi(x_0))$, a contradiction is obtained.

(5) For any compact subset K of Y there is a compact subset K' of X such that $\phi(K') \supset K$.

We can assume that K is nonempty. Take an $f \in C(Y)$ and $\varepsilon > 0$. Then there are a compact subset K' of X and a $\delta > 0$ such that

$$\langle \Phi(f), K', \delta \rangle \cap \Phi(C(Y)) \subset \Phi(\langle f, K, \varepsilon \rangle).$$

It is proved that $\phi(K') \supset K$. Assume that K is not a subset of $\phi(K')$. Then there is a point x_0 in $\phi^{-1}(K) - \phi^{-1}(\phi(K'))$. For any point $x' \in K'$, let $a_{x'} = \Phi(f)(x')$. Since the open prime ideal $J_{x'}^{<a_{x'}} = \{g \in C(Y) : \Phi(g)(x') < a_{x'}\}$ is associated with $\phi(x')$ and $\phi(x') \neq \phi(x_0)$, there exists $g_{x'} \in J_{x'}^{<a_{x'}}$ such that $g_{x'}(\phi(x_0)) \geq f(\phi(x_0)) + \varepsilon$. Using the same argument as that in (3), it can be shown that there is a function $g_0 \in C(Y)$ such that $g_0(\phi(x_0)) \geq f(\phi(x_0)) + \varepsilon$ and $\Phi(g_0)|K' < \Phi(f)|K'$. Since $\phi(x_0) \in K$, it follows that $g_0 \notin \langle f, K, \varepsilon \rangle$. Let $h = g_0 \vee f$. Then $\Phi(h)|K' = \Phi(f)|K'$ and $h(\phi(x_0)) \geq f(\phi(x_0)) + \varepsilon$. It follows that $\Phi(h) \in \Phi(\langle f, K, \varepsilon \rangle)$ and $h \notin \langle f, K, \varepsilon \rangle$ are satisfied. This is a contradiction.

If Φ is a topological-lattice isomorphism from $C_k(Y)$ onto $C_k(X)$, then the inverse of the continuous map ϕ in the above proof must correspond to the continuous map from Y onto X constracted similarly by using Φ^{-1} instead of Φ. Hence it is shown that Theorem 1 is true.

Quite similarly we can prove the following.

THEOREM 6. *There is a topological-lattice embedding Φ from $C_k^*(Y)$ into $C_k^*(X)$ such that $\{\Phi(f)(x) : f \in C^*(Y)\}$ is open in $\boldsymbol{R}$ for any $x \in X$ if and only if there is a continuous map ϕ from X onto Y such that for any compact subset K of Y there exists a compact subset K' of X with $\phi(K') \supset K$.*

In the proof of Theorem 5, if we replace compact sets with finite sets, then analogous results are obtained for the function spaces with the topology of pointwise convergence. Further, if $\{\Phi(f)(x) : f \in C(Y)\}$ contains at least 2 values, then the interior of this set is a non-empty open interval (a_x, b_x). Inquiring into the proof of Theorem 5, we have the following.

THEOREM 7. *There is a topological-lattice embedding Φ from $C_p(Y)$ into $C_p(X)$ such that $\{\Phi(f)(x) : f \in C(Y)\}$ contains at least 2 values for any $x \in X$ if and only if there is a continuous map ϕ from X onto Y.*

THEOREM 8. *There is a topological-lattice embedding Φ from $C_p^*(Y)$ into $C_p^*(X)$ such that $\{\Phi(f)(x) : f \in C^*(Y)\}$ contains at least 2 values for any $x \in X$ if and only if there is a continuous map ϕ from X onto Y.*

It has been already obvious that Theorem 2, 3 and 4 are true.

If we turn to look at above theorems, then the following problem arises: *For a space X, how can we characterize such a space Y whose $C_k(Y)$ (or $C_p(Y)$) is embedded in $C_k(X)$ (or $C_p(X)$) as a topological sublattice?* In case C_p we have a simple characterization of such a space Y. The following lemma is obvious.

LEMMA. *Let A be a topological sublattice of $C_p(X)$ and let $Z = \{x \in X : |A(x)| \geq 2\}$, where $A(x) = \{f(x) : f \in A\}$ and $||$ means the cardinality of a set. Let $r : C_p(X) \to C_p(Z)$ be the restriction $r(f) = f|Z$. Then $r|A : A \to r(A)$ is a topological-lattice isomorphism.*

THEOREM 9. *$C_p(Y)$ is embedded in $C_p(X)$ as a topological sublattice if and only if Y is a continuous image of a cozero-set of X.*

PROOF. For a cozero-set U of X, assume that there is a continuous map ϕ from U onto Y. Then there is a canonical embedding $\Phi : C_p(Y) \to C_p(U)$ defined by $\Phi(f) = f \circ \phi$ for any $f \in C_p(Y)$. Let t be an order-preserving homeomorphism from the real line $\boldsymbol{R}$ onto the open interval $(-1, 1)$ such as $(2/\pi)\, tan^{-1}$. Then the map $H : C_p(Y) \to C_p(U)$ defined by

$$H(f)(u) = t(\Phi(f)(u))$$

is a topological-lattice embedding, where $f \in C(Y)$ and $u \in U$. Further, we can take a continuous map $s : X \to [0, 1]$ such that $s^{-1}(0) = X - U$. Let $\Psi : C_p(Y) \to C_p(X)$ be the map defined as follows: $\Psi(f)(x) = 0$ if $x \in X - U$ and $\Psi(f)(x) = s(x)H(f)(x)$ if $x \in U$. Then it is not difficult to see that Ψ is a topological-lattice embedding.

Conversely, let $\Phi : C_p(Y) \to C_p(X)$ be a topological-lattice embedding. Let

$$F = \{x \in X : |\{\Phi(f)(x) : f \in C(Y)\}| = 1\}.$$

If $F = \emptyset$, then we have already shown that Y is a continuous image of X in Theorem 7. So we can assume that $F \neq \emptyset$. Further, we can assume that $\Phi(0_Y) = 0_X$, since $\Phi' : C_p(Y) \to C_p(X)$ defined by $\Phi'(f) = \Phi(f) - \Phi(0_Y)$ is also a topological-lattice embedding, where 0_X and 0_Y are real-valued constant functions on X and Y respectively with values 0. Hence it follows that $\Phi(f)(x) = 0$ is satisfied for any $x \in F$ and any $f \in C(Y)$.

(0) F is a zero-set.

For each integer i, let $i_Y \in C(Y)$ be the real-valued constant function on Y with the value i. It suffices to show that

$$F = \bigcap\{\Phi(i_Y)^{-1}(0) : i = 0, \pm 1, \pm 2, \dots\}.$$

Assume that there is a point x in $X - F$ such that $x \in \Phi(i_Y)^{-1}(0)$ for every integer i. Then there exists a function $g \in C(Y)$ such that $\Phi(g)(x) \neq 0$. Now, let $a = (1/2)\Phi(g)(x)$. Take

$$J_x^{<a} = \{f \in C(Y) : \Phi(f)(x) < a\}.$$

Then $J_x^{<a}$ is an open prime ideal in $C_p(Y)$. Hence there is a point $y \in Y$ such that $J_x^{<a}$ is associated with y. But this is a contradiction, since if $a > 0$ then $i_Y \in J_x^{<a}$ for all i, and if $a < 0$ then $i_Y \notin J_x^{<a}$ for all i.

Let $U = X - F$. Then there is a topological-lattice embedding of $C_p(Y)$ into $C_p(U)$ by Lemma, which satisfies the condition of Theorem 7. Hence Y is a continuous image of U.

It is obvious that the similar theorem is obtained for $C_p^*(X)$ and $C_p^*(Y)$.

COROLLARY 1. *Let X be a Lindelöf space. If $C_p(Y)$ is embedded in $C_p(X)$ as a topological sublattice, then Y is also Lindelöf.*

The following example show that topological-lattice embeddings can not be replaced with topological, order-isomorphic embeddings in Theorem 5, 6, 7, 8.

EXAMPLE. There exist spaces X and Y with the following properties:

1) There is an order-isomorphic, topological embedding Φ from $C_k(Y)$ $(C_p(Y))$ into $C_k(X)$ $(C_p(X))$ such that $\{\Phi(f)(x) : f \in C(Y)\} = \boldsymbol{R}$ for any $x \in X$.

2) Y is not a continuous image of X.

In fact, let X be the unit interval $[0,1]$ and Y the two-points space $\{0,1\}$. Then obviously there is no continuous map from X onto Y. The map $\Phi : C_k(Y) \to C_k(X)$, which satisfies the condition 1), is defined as follows: For $f \in C(Y)$ and $x \in [0,1]$,

$$\Phi(f)(x) = (1-x)f(0) + xf(1).$$

3. Generalizations of the Kaplansky Theorem

Let X be a space. For $f \in C(X)$ and a compact subset K of X, let

$$[f,K] = \{g \in C(X) : g|K = f|K\}.$$

Then we can define the topology on $C(X)$ generated by

$$\{[f,K] : f \in C(X), K \in \mathscr{K}\},$$

where $\mathscr{K}$ is the family of all compact subsets of X. This topology is called the compact-discrete topology. The space $C(X)$ with the compact-discrete topology is denoted by $C_d(X)$. The meaning of $C_d^*(X)$ is obvious.

The compact-discrete topology is related to a topology on the power set of a topological space. The power set $P(X)$ is the set of all subsets of a space X. We define the topology τ_K on $P(X)$ as follows: For each pair A, B of disjoint compact subsets of X, let

$$\langle A,B\rangle = \{Y \in P(X) : A \subset Y, B \cap Y = \varnothing\}.$$

Considering the family of all these subsets $\langle A,B\rangle$ as an open (sub-)base, we can introduce a topology on $P(X)$. This topology is called the compact-cocompact topology.

THEOREM 10. *The topology τ_K is T_1 and zero-dimensional, and hence Tychonoff.*

PROOF. Let G be an arbitrary point and take another point H in $P(X)$. Then there exists a point x in X such that (1) $x \in H$ and $x \notin G$ or (2) $x \in G$ and $x \notin H$. In case (1), H is an element of the basic open set $\langle\{x\},\varnothing\rangle$, but G is not in $\langle\{x\},\varnothing\rangle$. If (2) is satisfied, then $\langle\varnothing,\{x\}\rangle$ is a neighborhood of H which does not contain G. In order to show the zero-dimensionality, it suffices to show that every basic open set $\langle A,B\rangle$ is closed. Let $C \notin \langle A,B\rangle$. Then there is a point x in X such that either $x \in A\backslash C$ or $x \in B \cap C$. Using the same argument above, it is shown that $\langle\varnothing,\{x\}\rangle$ or $\langle\{x\},\varnothing\rangle$ is a neighborhood of C which does not intersect with $\langle A,B\rangle$.

Considering graphs of functions, the set $C(X)$ can be thought of as a subset of the power set $P(X \times \boldsymbol{R})$ of the product space $X \times \boldsymbol{R}$.

THEOREM 11. *The compact-discrete topology on* $C(X)$ *coincides with the relative topology of the compact-cocompact topology on* $P(X \times \boldsymbol{R})$.

PROOF. Let f be an arbitrary point in $C(X)$. Any basic neighborhood $[f, K]$ of f with respect to the compact-discrete topology is equal to the neighborhood $\langle\{(x, f(x)) : x \in K\}, \varnothing\rangle \cap C(X)$ of f with respect to the relative compact-cocompact topology. Conversely, for any basic neighborhood $\langle A, B\rangle \cap C(X)$ of f with respect to the relative compact-cocompact topology, the set $[f, \pi_X(A) \cup \pi_X(B)]$ is a neighborhood of f with respect to the compact-discrete topology which is included in $\langle A, B\rangle$, where π_X is the natural projection from $X \times \boldsymbol{R}$ onto X.

The following is easy.

THEOREM 12. *The space* $C_d(X)$ *has the following properties*:
1) $C_d(X)$ *is a zero-dimensional Tychonoff space.*
2) $C_d(X)$ *is a topological ring.*
3) $C_d(X)$ *is a topological lattice.*
4) $C_d(X)$ *is discrete if and only if* X *is compact.*

As mentioned in Introduction, algebraic lattices are regarded as topological lattices with discrete topologies. So we can generalize the Kaplansky theorem as follows:

THEOREM 13. *If topological lattices* $C_d(X)$ *and* $C_d(Y)$ *are isomorphic, then* X *and* Y *are homeomorphic.*

THEOREM 14. *If topological lattices* $C_d^*(X)$ *and* $C_d^*(Y)$ *are isomorphic, then* X *and* Y *are homeomorphic.*

These theorems follow from the following, which can be proved similarly as Theorem 5.

THEOREM 15. *There is a topological-lattice embedding* Φ *from* $C_d(Y)$ *into* $C_d(X)$ *such that* $\{\Phi(f)(x) : f \in C(Y)\}$ *is open in* $\boldsymbol{R}$ *for any* $x \in X$ *if and only if* *there is a continuous map ϕ from X onto Y such that for any compact subset K of Y there exists a compact subset K' of X with $\phi(K') \supset K$.*

THEOREM 16. *There is a topological-lattice embedding Φ from $C_d^*(Y)$ into $C_d^*(X)$ such that $\{\Phi(f)(x) : f \in C^*(Y)\}$ is open in $\boldsymbol{R}$ for any $x \in X$ if and only if there is a continuous map ϕ from X onto Y such that for any compact subset K of Y there exists a compact subset K' of X with $\phi(K') \supset K$.*

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