TSUKUBA J. MATH.
Vol. 24 No. 1 (2000), 59–67

A NOTE ON NORMALLY GENERATED LINE BUNDLES ON COMPACT RIEMANN SURFACES II

By

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1. Introduction

Let X denote a compact Riemann surface of genus $g(X)$ and L an ample line bundle on X. Then two C-algebras $S(L)$ and $R(L)$ are defined as follows:

$$S(L) := \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n H^0(X, L), \quad R(L) := \bigoplus_{n=0}^{\infty} H^0(X, L^n).$$

Definition 1. (i) *L is said to be normally generated if the natural map $S(L) \to R(L)$ is surjective.*

(ii) *A normally generated line bundle L is said to be normally presented if the kernel I of the map in* (i) *is generated by its degree two part I_2 as an ideal of $S(L)$.*

Note that, by definition, a normally presented line bundle is normally generated and moreover, a normally generated line bundle is always very ample.

There are the following two sufficient conditions for line bundles on X to be normally generated or to be normally presented:

Theorem 1 (cf. [8], [12]). (i) *The canonical bundle K_X on X is normally generated if and only if X is nonhyperelliptic.*

(ii) *K_X is normally presented if and only if X is neither hyperelliptic, trigonal nor smooth plane quintic.*

Theorem 2 (cf. [9], [11]). (i) *If $\deg L \geq 2g(X) + 1$, then L is normally generated.*

(ii) *If $\deg L \geq 2g(X) + 2$, then L is normally presented.*

Received October 28, 1998
Revised June 25, 1999

On the other hand, Homma [6] classified all the normally generated line bundles on X when the genus of X is three.

THEOREM 3 (cf. [6]). *Suppose* $g(X)=3$. (i) *If* X *is hyperelliptic, then* L *is normally generated if and only if* $\deg L \geq 7$.

(ii) *If* X *is nonhyperelliptic, then* L *is normally generated if and only if* L *satisfies one of the following conditions*:

(a) $\deg L \geq 7$.

(b) $\deg L = 6$ *and* $H^0(X, L \otimes K_X^{-1}) = 0$.

(c) $L \cong K_X$.

Now let $\pi : X \to Y$ be a (possibly ramified) double covering of compact Riemann surfaces.

In [2], we study the following problem in the case of $g(X)=3$ and in the cases of $g(X) \geq 4$ and $g(Y) \leq 1$:

PROBLEM 1. Classify ample line bundles on Y such that the pull back on X are normally generated.

Now we set:

PROBLEM 2. Classify pairs of an ample line bundle M on Y and a point P on X such that the line bundles given as $\pi^* M \otimes \mathcal{O}_X(P)$ are normally generated on X.

The degree of $\pi^* M \otimes \mathcal{O}_X(P)$ is $2 \deg M + 1$. Hence the following are obvious by Theorems 2 and 3:

PROPOSITION 1. *If* $\deg M \geq g$, *then* $\pi^* M \otimes \mathcal{O}_X(P)$ *is normally generated.*

PROPOSITION 2. *If* $g(X)=3$, $\pi^* M \otimes \mathcal{O}_X(P)$ *is normally generated if and only if* $\deg M \geq 3$

In the following sections, we study Problem 2 in the cases of $g(X) \geq 4$ and $g(Y) \leq 1$. In §2, we will see some fundamantal properties of double coverings and in §3, we will summarize some basic facts on *Clifford index* of compact Riemann surfaces. In §4, Problem 2 will be studied in our cases.

The author would like to express his gratitude to Professors Sampei Usui and

Kazuhiro Konno for their helpful advices and encouragements. He also thanks to Professor Akira Ohbuchi, Dr. Masanori Asakura and Mr. Koichiro Yoshioka for useful discussions. Finally he thanks to the referee for many suggestions. In particular, Remark 1 and the first proof of Proposition 3 below are based on the referee's comments.

2. Double coverings of compact Riemann surfaces

In this section, we will recall some fundamental facts on double coverings.

LEMMA 1 (cf. [10]). *Let B denote the branch locus of the double covering $\pi : X \to Y$ on Y. Then there exists a line bundle F on Y with $2F \cong B$ such that the following conditions hold*:

(i) *X is embedded into F and the projection of F to Y restricted on X coincides with π.*

(ii) *The canonical bundle K_X on X is linearly equivalent to $\pi^*(K_Y \otimes F)$ where K_Y is the canonical bundle on Y.*

(iii) *For any line bundle L on Y, we have*:

$$\pi_* \mathcal{O}_X(\pi^* L) \cong \mathcal{O}_Y(L) \oplus \mathcal{O}_Y(L \otimes F^{-1}).$$

COROLLARY 1. *For a double covering $\pi : X \to Y$ which is not unramified, the induced homomorphism $\pi^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ is injective.*

PROOF OF COROLLARY 1. Let M be a line bundle on Y such that the pull back $\pi^* M$ is trivial on X. Then we have $\deg M = 0$ and $h^0(X, \pi^* M) = 1$. Moreover, by the assumption that π is not unramified, we have $\deg F > 0$ and hence $h^0(Y, M \otimes F^{-1}) = 0$. Therefore, by Lemma 1 (iii), we conclude $h^0(Y, M) = h^0(X, \pi^* M) - h^0(Y, M \otimes F^{-1}) = 1$, that is, M is also trivial on Y. $\square$

REMARK 1. Corollary 1 was already given in [2, Corollary to Lemma 1] (as well as Lemma 1 in [2, Lemma 1]). However its statement was incomplete there. As in above, we need the assumption that the double covering π is *not* unramified.

The following may be well known:

PROPOSITION 3. *Let $\pi : X \to Y$ be a double covering with $g(Y) = 1$. Then we have*:

(i) *If* $g(X) \geq 4$, *then* X *is nonhyperelliptic.*
(ii) *If* $g(X) \geq 5$, *then* X *is neither trigonal nor smooth plane quintic.*

Here we will give two proofs of Proposition 3. One is based on the following theorem of Castelnuovo-Severi and the other is on Theorems 1, 2 and Lemma 1 above.

THEOREM 4 (cf. [4]. See also [1, Theorem 3.5] and [3, C.1 on p. 366].) *Let* X, Y *and* Z *be three compact Riemann surfaces of genus* $g(X)$, $g(Y)$ *and* $g(Z)$. *Let* $\varphi : X \to Y$ *and* $\psi : X \to Z$ *be two surjective holomorphic maps of degree* m *and* n *respectively. Suppose further that there does not exist a compact Riemann surface* W *of genus* $g(W) < g(X)$ *with three coverings* $\varphi' : W \to Y$, $\psi' : W \to Z$ *and* $\phi : X \to W$ *such that* $\varphi = \varphi'\phi$ *and* $\psi = \psi'\phi$.

Then we have:

$$g(X) \leq mg(Y) + ng(Z) + (m-1)(n-1).$$

COROLLARY 2. *Let* $\pi : X \to Y$ *be a double covering and* $\psi : X \to Z$ *be a surjective holomorphic map of degree* n. *If* $g(X) \geq 2g(Y) + n(g(Z)+1)$, *then* ψ *factors through* π, *that is, there exists a covering* $\psi' : Y \to Z$ *such that* $\psi = \psi'\pi$.

PROOF OF COROLLARY 2. Since the inequality in Theorem 4 does not hold, we have a compact Riemann surface W of genus $g(W) < g(X)$ with coverings $\phi : X \to W$ and $\pi' : W \to Y$ such that $\pi = \pi'\phi$. But, since the degree of π is two and that of ϕ is more than one, we conclude that $\deg \phi = 2$ and $\deg \pi' = 1$, that is, $\pi' : W \to Y$ is an isomorphism and hence we get the assertion. $\square$

THE FIRST PROOF OF PROPOSITION 3. Let $\psi : X \to \boldsymbol{P}^1$ be a surjective holomorphic map which does not factor through the double covering $\pi : X \to Y$. Then, by Corollary 2 above, we have $\deg \psi \geq g(X) - 1$.

Suppose X is hyperelliptic (resp. trigonal). Then there exists a map $\psi : X \to \boldsymbol{P}^1$ of degree two (resp. three), which does not factor through π. Hence we have $g(X) \leq 3$ (resp. $g(X) \leq 4$) by the above remark.

Now suppose X is a smooth plane quintic and hence $g(X) = 6$. Let P and Q be mutually distinct points on X with $\pi(P) = \pi(Q)$. Let R be another point on X such that P, Q and R are not collinear. Then the projection from R induces a covering $\varphi : X \to \boldsymbol{P}^1$ of degree four. But, since $\varphi(P) \neq \varphi(Q)$, φ does not factor through π and hence we have $6 = g(X) \leq \deg \varphi + 1 = 5$, which is a contradiction. $\square$

The Second Proof of Proposition 3. By Lemma 1 (ii), we have $K_X \cong \pi^* F$ and by Lemma 1 (iii), $H^0(X, K_X^n) \cong H^0(Y, F^n) \oplus H^0(Y, F^{n-1})$. Hence we get that K_X is normally generated (resp. normally presented) if F is normally generated (resp. normally presented) on Y. Since $\deg F = g - 1$, the assertions follow from Theorems 1 and 2. $\square$

3. Clifford index of compact Riemann surfaces

In this section, we will summarize some basic facts on *Clifford index* of compact Riemann surfaces. See for example, [8, §2] or [7, §2] for detail.

For a line bundle L on a compact Riemann surface X of genus $g(X) \geq 2$ and X itself, their *Clifford index* is defined as follows:

Definition 2. (i) *The Clifford index of L is the integer*

$$\mathrm{Cliff}(L) := \deg L - 2h^0(L) + 2.$$

(ii) *If $g(X) \geq 4$, the Clifford index of X is*

$$\mathrm{Cliff}(X) := \min\{\mathrm{Cliff}(L); h^0(L) \geq 2, h^1(L) \geq 2\}.$$

(iii) *If $g(X) = 2$ then* $\mathrm{Cliff}(X)$ *is always* 0 *and if $g(X) = 3$,*

$$\mathrm{Cliff}(X) := \begin{cases} 0 & \text{if } X \text{ is hyperelliptic} \\ 1 & \text{if } X \text{ is nonhyperelliptic.} \end{cases}$$

By Clifford's theorem, $\mathrm{Cliff}(X) \geq 0$ and the equality holds if and only if X is hyperelliptic. Moreover we have:

Lemma 2. $\mathrm{Cliff}(X) \leq (g-1)/2$.

Lemma 3. $\mathrm{Cliff}(X) = 1$ *if and only if X is either trigonal or smooth plane quintic.*

Lemma 4. *If there exists a surjective map $\varphi : X \to \boldsymbol{P}^1$ with $\deg \varphi \leq 4$, then we have* $\mathrm{Cliff}(X) \leq 2$.

Proof of Lemma 4. By Lemmas 2 and 3, we may assume $g(X) \geq 7$ and $\deg \varphi = 4$. Let D denote the divisor $\varphi^{-1}(\infty)$ on X where ∞ is the infinity on $\boldsymbol{P}^1$. Then we have $\deg D = 4$, $h^0(D) \geq 2$, $h^1(D) = h^0(K_X - D) \geq 3$ and $\mathrm{Cliff}(D) = \deg D - 2h^0(D) + 2 \leq 2$. $\square$

Hence we can rewrite Theorem 1 and Proposition 3 with the notion of Clifford index as follows:

THEOREM 1′. (i) K_X *is normally generated if and only if* $\mathrm{Cliff}(X) \geq 1$.
(ii) K_X *is normally presented if and only if* $\mathrm{Cliff}(X) \geq 2$.

PROPOSITION 3′. *Let* $\pi : X \to Y$ *be a double covering with* $g(Y) = 1$. *Then we have*:
(i) $\mathrm{Cliff}(X) = 1$ *if* $g(X) = 4$.
(ii) $\mathrm{Cliff}(X) = 2$ *if* $g(X) \geq 5$.

PROOF OF PROPOSITION 3′. Since Y is a double covering of $\boldsymbol{P}^1$, there is a surjective map $X \to \boldsymbol{P}^1$ of degree four and hence we have $\mathrm{Cliff}(X) \leq 2$ by Lemma 4. Hence we get the assertion by Proposition 3. □

Now we can state a result of Green-Lazarsfeld [5]:

THEOREM 5. *Let* L *be a very ample line bundle on* X *with*

$$\deg L \geq 2g(X) + 1 - 2h^1(L) - \mathrm{Cliff}(X).$$

Then L *is normally generated.*

4. The case of $g(X) \geq 4$ and $g(Y) \leq 1$

In the following arguments, we will denote the line bundle $\pi^*M \otimes \mathcal{O}_X(P)$ on X by L. Moreover we set $\pi(P) = R \in Y$ and $\pi^{-1}(R) = \{P, Q\}$. Then we have:

$$H^0(X, \pi^*M) \subset H^0(X, L) \subset H^0(X, \pi^*(M \otimes \mathcal{O}_Y(R))). \tag{1}$$

4.1. The cases of $g(Y) = 0$

If $g(Y) = 0$, then $\deg F = g(X) + 1$ where F is the line bundle in Lemma 1. Hence, if $\deg M \leq g(X) - 1$, we have

$$H^0(X, L) \subset H^0(X, \pi^*(M \otimes \mathcal{O}_Y(R))) \cong H^0(Y, M \otimes \mathcal{O}_Y(R))$$

by Lemma 1 (iii) and (1). Hence the (rational) map defined by the linear system $|L|$ factors through the double covering $\pi : X \to Y$. We therefore conclude that L is not very ample and hence not normally generated in this case.

On the other hand, by Proposition 1, L is normally generated if $\deg M \geq g(X)$.

Consequently we have:

PROPOSITION 4. *Suppose* $g(Y) = 0$. *Then* $L = \pi^*M \otimes \mathcal{O}_X(P)$ *is normally generated if and only if* $\deg M \geq g(X)$.

4.2. The cases of $g(Y) = 1$

If $g(Y) = 1$, then we have $\deg F = g(X) - 1$ and $K_X \cong \pi^*F$.

Let M be a line bundle on Y. If $H^0(X, M \otimes \mathcal{O}_Y(R) \otimes F^{-1}) = 0$, then, by the same arguments as in §3.1, we can conclude that $L = \pi^*M \otimes \mathcal{O}(P)$ is not very ample and hence not normally generated. Hence we may assume either $\deg M \geq g(X) - 1$ or $\deg M = g(X) - 2$ and $M \otimes \mathcal{O}_Y(R) \cong F$.

On the other hand, by Proposition 1, L is normally generated if $\deg M \geq g(X)$ so that it suffices to consider only the cases of $\deg M = g(X) - 1$ or $M \otimes \mathcal{O}_Y(R) \cong F$.

Suppose $\deg M = g(X) - 1$. Then we have $\deg L = 2g(X) - 1$ and hence $h^0(L) = g(X)$, $h^0(L^2) = 3g(X) - 1$.

By Lemma 1 (iii) and (1), we have

$$H^0(X, L) \supset H^0(X, \pi^*M) \cong H^0(Y, M) \oplus H^0(Y, M \otimes F^{-1}).$$

If $M \cong F$, then we have $K_X \cong \pi^*M$ and hence $L \cong K_X \otimes \mathcal{O}_X(P)$. Then, since $h^0(X, L) = g(X) = h^0(X, K_X) = h^0(X, L \otimes \mathcal{O}_X(-P))$, we get that P is a base point of L and therefore L is not very ample.

If $M \not\cong F$, then we have $h^0(X, \pi^*M) = h^0(Y, M) = g(X) - 1$. Hence there exists an element $\lambda \neq 0$ in $H^0(X, L)$ such that $H^0(X, L) \cong H^0(Y, M) \oplus \boldsymbol{C}\lambda$. Then the image of the natural map $\mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^2)$ is, as a $\boldsymbol{C}$-vector space, generated by $H^0(Y, M^2)$, $\lambda H^0(Y, M)$ and λ^2. But since $h^0(Y, M^2) = 2g(X) - 2$ and $h^0(Y, M) = g(X) - 1$, the dimension of that image is at most $3g(X) - 2$, which is less than $3g(X) - 1 = h^0(X, L^2)$. Consequently we get that, if $\deg M = g(X) - 1$, then L is not normally generated.

Now suppose $M \otimes \mathcal{O}_Y(R) \cong F$. Then we have:

$$L = \pi^*M \otimes \mathcal{O}_X(P) \cong K_X \otimes \mathcal{O}_X(-Q).$$

To apply Theorem 5 to the above L, we will see whether L is very ample or not.

Let P_1, P_2 be arbitrary two points on X. Then we have an exact sequence

$$0 \to L \otimes \mathcal{O}_X(-P_1 - P_2) \to L \to \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2} \to 0,$$

which implies that the map

$$H^0(X, L) \to \boldsymbol{C}_{P_1} \oplus \boldsymbol{C}_{P_2}$$

is surjective if and only if the surjective map

$$H^1(X, L \otimes \mathcal{O}_X(-P_1 - P_2)) \to H^1(X, L)$$

is injective. But since $h^1(L) = h^0(Q) = 1$, we conclude that $L = K_X \otimes \mathcal{O}_X(-Q)$ is very ample if and only if, for any two points P_1, P_2 on X (which are not necessarily mutually distinct), $h^1(X, L \otimes \mathcal{O}_X(-P_1 - P_2)) = h^0(X, \mathcal{O}_X(Q+P_1+P_2)) = 1$, that is X is not trigonal. (By Proposition 3′ (i) and the assumption of $g(X)$ being more than three, X is always nonhyperelliptic).

Now suppose $\mathrm{Cliff}(X) = 2$. Then, since $\deg L = 2\deg M + 1 = 2g(X) - 3$ and $h^1(L) = 1$, we have $2g(X) + 1 - 2h^1(L) - \mathrm{Cliff}(X) = 2g(X) - 3 = \deg L$. Moreover, since X is not trigonal, L is very ample by the above remark. Consequently, if $\mathrm{Cliff}(X) = 2$, the assumptions of Theorem 5 are satisfied and hence we can conclude that L is normally generated.

Therefore we have:

PROPOSITION 5. *Suppose* $g(Y) = 1$. (i) *If* $\mathrm{Cliff}(X) = 1$ *then we have* $g(X) = 4$ *and* $L = \pi^* M \otimes \mathcal{O}_X(P)$ *is normally generated if and only if* $\deg M \geq g$.

(ii) *If* $\mathrm{Cliff}(X) = 2$ *then* L *is normally generated if and only if one of the following is satisfied*:

(a) $\deg M \geq g$.

(b) $M \otimes \mathcal{O}_Y(R) \cong F$ *where* $R = \pi(P)$.

PROOF. It suffices to show (i). If $\mathrm{Cliff}(X) = 1$, then we have $g(X) = 4$ by Proposition 3′ (ii) and hence X is trigonal by Lemma 3. Therefore $K_X \otimes \mathcal{O}_X(-Q)$ is not very ample, so that $L = \pi^* M \otimes \mathcal{O}_X(P)$ is normally generated only if $\deg M \geq g$. □

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