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REALIZATIONS OF SUBGROUPS OF TYPE D_8 OF CONNECTED EXCEPTIONAL SIMPLE LIE GROUPS OF TYPE E_8

By

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Introduction

In [4], we realized subgroups of type A_8, $A_4 \times A_4$ and $A_2 \times E_6$ of the compact simple Lie group of type E_8. In this paper, we shall realize subgroups of type D_8 of the compact and non-compact simple Lie groups of type E_8.

In [5], [6], [10] and [11], Yokota and some members of his school found all involutive automorphisms σ and realized subgroups G^σ of fixed points of connected exceptional simple Lie groups G explicitly, which correspond to Berger's result of simple Lie algebras [2]. But in their results concerning subgroups of type D_8 of Lie groups of type E_8, the definition of subgroups are not clear and proof of isomorphism is very difficult in comparison with their other results.

We improve those results in this paper. Our improvement make results that are more simple and intelligible. Hence they are of widely applicable to symmetric spaces. Our results are as follows.

type	G	G^σ	
E_8^C	E_8^C	$Ss(16, C)$	Theorem 6.1
E_8 (compact)	$(E_8^C)^\tau$	$Ss(16)$	Theorem 6.6(1)
$E_{8(8)}$	$(E_8^C)^{\tau\varepsilon}$	$Ss(16)$	Theorem 6.6(2)
	$(E_8^C)^{\tau\varepsilon_1}$	$Ss(8, 8) \times 2$	Theorem 6.8
	$(E_8^C)^{\tau\varepsilon J}$	$Ss^*(16) \times 2$	Theorem 6.11(2)
$E_{8(-24)}$	$(E_8^C)^{\tau\varepsilon_1\gamma_1}$	$Ss(4, 12)$	Theorem 6.9
	$(E_8^C)^{\tau J}$	$Ss^*(16) \times 2$	Theorem 6.11(1)

In §2, §3 and §4, we make a new realization of exceptional Lie algebras

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of type E_8. Since this new realization starts from the definition of spinor groups directly, the final results will be made clearly understandable.

In §5, we define connected exceptional simple Lie groups G of type E_8.

In §6, we find involutive automorphisms σ and realize subgroups G^σ which are isomorphic to semi-spinor groups.

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§1. Notations and Preliminaries

$V^C := V \oplus iV$ the complexification of a real vector space V, $C := \boldsymbol{R}^C$.

τ : the complex conjugation of V^C (resp. C^m, $M(n, C)$) with respect to V (resp. $\boldsymbol{R}^m$, $M(n, \boldsymbol{R})$).

I_m : the $m \times m$ unit matrix.

$I_{p,q} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$, $J_p := \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$.

$$SO(n, C) := \{A \in M(n, C) \mid {}^tAA = I_n, \det A = 1\},$$

$$SO(n) := \{A \in SO(n, C) \mid \tau A = A\},$$

$$SO(p, q) := \{A \in SO(p+q, C) \mid \tau(I_{p,q}AI_{p,q}) = A\},$$

$$SO^*(2p) := \{A \in SO(2p, C) \mid \tau(J_pAJ_p^{-1}) = A\},$$

$$\mathfrak{so}(n, C) := \{X \in M(n, C) \mid {}^tX + X = 0\},$$

$$\mathfrak{so}(n) := \{X \in \mathfrak{so}(n, C) \mid \tau X = X\},$$

$$\mathfrak{so}(p, q) := \{X \in \mathfrak{so}(p+q, C) \mid \tau(I_{p,q}XI_{p,q}) = X\},$$

$$\mathfrak{so}^*(2p) := \{X \in \mathfrak{so}(2p, C) \mid \tau(J_pXJ_p^{-1}) = X\}.$$

§1.1. Spinor Groups and Semi-spinor Groups ([1])

Let $\boldsymbol{K} = \boldsymbol{R}$ or C and $\{e_1, e_2, \dots, e_n\}$ be the canonical basis of $\boldsymbol{K}^n$. Let T be the tensor algebra of $\boldsymbol{K}^n$ and U the two-sided ideal of T generated as

$$x \otimes x + (x, x)1 \quad (x \in \boldsymbol{K}^n)$$

where $(\,,)$ is the symmetric bilinear form of $\boldsymbol{K}^n$ satisfying $(e_i, e_j) = \delta_{ij}$ (Kronecker's delta). Define the Clifford algebra $Cl(\boldsymbol{K}^n)$ by

$$Cl(\boldsymbol{K}^n) := T/U.$$

We denote the multiplication of $\alpha, \beta \in Cl(\boldsymbol{K}^n)$ by $\alpha \cdot \beta$. It is clear that $\boldsymbol{K}$ and $\boldsymbol{K}^n$ can be naturally considered as subspaces of $Cl(\boldsymbol{K}^n)$ and for $x, y \in \boldsymbol{K}^n$ we have

$$x \cdot y + y \cdot x = -2(x, y).$$

It is known that spinor groups $Spin(n) = Spin(n, \boldsymbol{R})$ and $Spin(n, \boldsymbol{C})$ are defined by

$$Spin(n, \boldsymbol{K}) := \left\{ \alpha = a_1 \cdot a_2 \cdots a_{2q} \in Cl(\boldsymbol{K}^n) \;\middle|\; \begin{array}{l} q = 1, 2, 3 \ldots, \\ a_i \in \boldsymbol{K}^n, \\ \prod_{i=1}^{2q} (a_i, a_i) = 1 \end{array} \right\}.$$

The unit element of $Spin(n, \boldsymbol{K})$ is $1 = -a \cdot a$ $(a \in \boldsymbol{K}^n, (a, a) = \pm 1)$ and the inverse element of

$$\alpha = a_1 \cdot a_2 \cdots a_{2q-1} \cdot a_{2q} \in Spin(n, \boldsymbol{K})$$

is

$$\alpha^{-1} = a_{2q} \cdot a_{2q-1} \cdots a_2 \cdot a_1 \in Spin(n, \boldsymbol{K}).$$

The vector representation $p : Spin(n, \boldsymbol{K}) \to SO(n, \boldsymbol{K})$ is given by

$$p(\alpha)x = \alpha \cdot x \cdot \alpha^{-1} \quad (x \in \boldsymbol{K}^n).$$

It is known that $Spin(n, \boldsymbol{K})$ is a covering group of $SO(n, \boldsymbol{K})$ (double covering), and $Spin(n, \boldsymbol{K})$ $(n \geq 3)$ is simply connected. Let

$$\omega = e_1 \cdot e_2 \cdot e_3 \cdots e_{2n-1} \cdot e_{2n} \in Spin(2n, \boldsymbol{K}).$$

It is known that the centers of spinor groups are

$$\begin{aligned} z(Spin(2n+1, \boldsymbol{K})) &= \{1, -1\} \cong \boldsymbol{Z}_2, \\ z(Spin(2, \boldsymbol{K})) &= Spin(2, \boldsymbol{K}), \\ z(Spin(4n, \boldsymbol{K})) &= \{1, -1\} \times \{1, \omega\} \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \quad (n \geq 1), \\ z(Spin(4n+2, \boldsymbol{K})) &= \{1, \omega, -1, -\omega\} \cong \boldsymbol{Z}_4 \quad (n \geq 1) \end{aligned}$$

and

$$Spin(n, \boldsymbol{K})/\{1, -1\} \cong SO(n, \boldsymbol{K}) \quad (n \geq 1).$$

Define semi-spinor groups $Ss(4n) = Ss(4n, \boldsymbol{R})$ and $Ss(4n, C)$ by

$$Ss(4n) := Spin(4n)/\{1, \omega\}, \quad Ss(4n, C) := Spin(4n, C)/\{1, \omega\}.$$

It is known that

$$Ss(4n, \boldsymbol{K}) \not\cong SO(4n, \boldsymbol{K}) \quad (n \geq 3),$$

and

$$Ss(4n, \boldsymbol{K}) \cong Spin(4n, \boldsymbol{K})/\{1, -\omega\}.$$

Let l, m be non-negative integers and $l + m = n$. Define the symmetric bilinear form $(\,,\,)_{l,m}$ of $\boldsymbol{R}^n$ satisfying

$$(\boldsymbol{e}_i, \boldsymbol{e}_j)_{l,m} := \begin{cases} -1 & (1 \leq i = j \leq l), \\ 1 & (l + 1 \leq i = j \leq l + m = n), \\ 0 & (i \neq j). \end{cases}$$

Let $U_{l,m}$ be the two-sided ideal of the tensor algebra T of $\boldsymbol{R}^n$ generated by

$$x \otimes x + (x, x)_{l,m} 1 \quad (x \in \boldsymbol{R}^n).$$

Define the Clifford algebra $Cl(\boldsymbol{R}^n)_{l,m}$ as

$$Cl(\boldsymbol{R}^n)_{l,m} := T/U_{l,m}.$$

We denote the multiplication of $\alpha, \beta \in Cl(\boldsymbol{R}^n)_{l,m}$ by $\alpha \cdot \beta$. The spinor group $Spin(l, m)$ defined as

$$Spin(l, m) := \left\{ \alpha = a_1 \cdots a_{2q} \in Cl(\boldsymbol{R}^n)_{l,m} \;\middle|\; \begin{array}{l} q = 1, 2, 3 \dots, \\ a_i \in \boldsymbol{R}^n, \\ \prod_{i=1}^{2q} (a_i, a_i)_{l,m} = 1 \end{array} \right\}.$$

Clearly $Cl(\boldsymbol{R}^n)_{0,n} = Cl(\boldsymbol{R}^n)$ and $Spin(0, n) = Spin(n)$.

For any C-linear transformation $K : C^n \to C^n$ and any element $\alpha = a_1 \cdot a_2 \cdots a_{2m} \in Spin(n, C)$, we define $K(\alpha) \in Spin(n, C)$ as

$$K(\alpha) = K(a_1) \cdot K(a_2) \cdots K(a_{2m}) \in Spin(n, C).$$

Now, we identify an element

$$\alpha = \begin{pmatrix} b_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} b_2 \\ c_2 \end{pmatrix} \in Spin(l, m), \quad (b_i \in \boldsymbol{R}^l, c_i \in \boldsymbol{R}^m)$$

with

$$\alpha' = \begin{pmatrix} ib_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} ib_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ -ic_1 \end{pmatrix} \cdot \begin{pmatrix} -b_2 \\ ic_2 \end{pmatrix} \in Spin(l+m, C).$$

Using this identification, we can consider the following

$$Spin(l,m) = \{\alpha \in Spin(l+m, C) \mid \tau I_{l,m}(\alpha) = \alpha\}.$$

Then $Spin(l,m)$ is a real subgroup of $Spin(l+m, C)$.

Let $l+m \geq 3$ and $lm \neq 0$. It is known that $SO(l,m)$ is not connected (it has two connected components) and $Spin(l,m)$ is connected. As same as in the case of $Spin(n)$, $Spin(l,m)$ is a double covering group of $SO(l,m)_0$ (the connected component of the identity of $SO(l,m)$). Furthermore $Spin(1,2)$ and $Spin(l,m)$ $(l,m \geq 2)$ are not simply connected, while $Spin(1,m)$ $(m \geq 3)$ is simply connected.

Let $l \equiv m \equiv 0 \bmod 2$. Since $\tau I_{l,m}\omega = (-1)^l\omega = \omega$, we see $\omega \in Spin(l,m)$. If $l+m \equiv 0 \bmod 4$, we define a real subgroup $Ss(l,m)$ of $Ss(l+m, C)$ by

$$Ss(l,m) := Spin(l,m)/\{1,\omega\}.$$

Furthermore, consider a double covering group of $SO^*(2m)$. It is known that $SO^*(2m)$ is connected and not simply connected. Define

$$\begin{aligned} \tilde{G} := p^{-1}(SO^*(2m)) &= \{\alpha \in Spin(2m, C) \mid p(\alpha) \in SO^*(2m)\} \\ &= \{\alpha \in Spin(2m, C) \mid \tau(J_m p(\alpha) J_m^{-1}) = p(\alpha)\}. \end{aligned}$$

Since $\tau(J_m p(\alpha) J_m^{-1}) = p(J_m \tau\alpha)$ and $\operatorname{Ker} p = \{\pm 1\}$, we see

$$\tilde{G} = \{\alpha \in Spin(2m, C) \mid \tau J_m(\alpha) = \pm\alpha\} = \tilde{G}_+ \cup \tilde{G}_-,$$

where $\tilde{G}_\pm = \{\alpha \in \tilde{G} \mid J_m\tau\alpha = \pm\alpha\}$. It is clear that $\tilde{G}_+ \cap \tilde{G}_- = \phi$. Since τJ_m is an involutive automorphism of $Spin(2m, C)$, $\tilde{G}_+$ is connected ([7]). It is clear that $\tilde{G}_+$ is closed, because τJ_m is continuous.

Now, we assume $\tilde{G}_- \neq \phi$. For any $\alpha \in \tilde{G}_+$ and $\beta, \beta' \in \tilde{G}_-$, we have $\alpha \cdot \beta \in \tilde{G}_-$ and $\beta \cdot \beta' \in \tilde{G}_+$. This shows $\tilde{G}_- = \tilde{G}_+ \cdot \beta$ $(\beta \in \tilde{G}_-)$ and $\tilde{G}_-$ is closed and connected. Moreover, we can prove that $\tilde{G}$ is connected as follows. Let $\alpha \in \tilde{G}_+$, $\beta \in \tilde{G}_-$ and $A = p(\alpha)$, $B = p(\beta) \in SO^*(2m)$. There exists a continuous curve $\gamma : I \to SO^*(2m) \subset SO(2m, C)$ $(I = [0,1])$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Hence, we can choose a continuous curve $\tilde{\gamma} : I \to Spin(2m, C)$ such that $p(\tilde{\gamma}(t)) = \gamma(t)$ and $\tilde{\gamma}(0) = \alpha$. From the definition of $\tilde{G}$, $\tilde{\gamma} \subset \tilde{G}$ is clear. Since $p(\tilde{\gamma}(1)) = \gamma(1) = B$, we see $\tilde{\gamma}(1) = \beta$ or $-\beta$. This shows $\tilde{G} = \tilde{G}_+ \cup \tilde{G}_-$ is connected. This contradicts $\tilde{G}_+ \cap \tilde{G}_- = \phi$.

Hence we see $\tilde{G}_- = \phi$ and $\tilde{G} = \tilde{G}_+$ is a connected simple Lie group. Thus we can define a double covering group of $SO^*(2m)$ as follows:

$$Spin^*(2m) := \{\alpha \in Spin(2m, C) \mid \tau J_m(\alpha) = \alpha\}.$$

Clearly, $Spin^*(2m)$ is a real subgroup of $Spin(2m, C)$. From the fact above, $Spin^*(2m)$ is a connected simple Lie group. On the other hand, since $\pi_1(SO^*(2m)) = \boldsymbol{Z}$ (m : odd) or $\boldsymbol{Z} \times \boldsymbol{Z}_2$ (m : even) ([9]), $Spin^*(2m)$ is not simply connected.

Since $\tau J_m \omega = (-1)^m e_{m+1} \cdots e_{2m} \cdot e_1 \cdots e_m = (-1)^m (-1)^{m^2} \omega = \omega$, we see $\omega \in Spin^*(2m)$. Hence we define a real subgroup $Ss^*(4n)$ of $Ss(4n, C)$ as

$$Ss^*(4n) := Spin^*(4n)/\{1, \omega\}.$$

§1.2. Vector Representation of $Spin(n, \boldsymbol{K})$

For $1 \leq i \neq j \leq n$, define an element $\alpha_{ij}(t)$ of $Spin(n, \boldsymbol{K})$ as

$$\alpha_{ij}(t) := -e_i \cdot \left(\cos \frac{t}{2} e_i + \sin \frac{t}{2} e_j\right).$$

It is clear $\alpha_{ij}(0) = 1$. Since $\alpha_{ij}(t_1) \cdot \alpha_{ij}(t_2) = \alpha_{ij}(t_1 + t_2)$, we see

$$\{\alpha_{ij}(t) \in Spin(n, \boldsymbol{K}) \mid t \in \boldsymbol{R}\}$$

is a 1-parameter subgroup of $Spin(n, \boldsymbol{K})$. Since

$$x \cdot y \cdot x = -2(x, y)x + (x, x)y, \quad (x, y \in \boldsymbol{K}^n \subset Cl(\boldsymbol{K}^n)),$$

we see

$$p(\alpha_{ij}(t))e_i = \cos te_i - \sin te_j, \quad p(\alpha_{ij}(t))e_j = \sin te_i + \cos te_j,$$
$$p(\alpha_{ij}(t))e_k = e_k \quad (k \neq i, j).$$

§1.3. Cayley Algebra and Half-spinor Representations of $\mathfrak{so}(8, C)$ ([3], [8])

Let $\mathfrak{C}$ be the division Cayley algebra over $\boldsymbol{R}$ and $\mathfrak{C}^C$ its complexification. We denote the multiplication and the canonical conjugation of $\mathfrak{C}$ (resp. $\mathfrak{C}^C$) by xy and $\bar{x}$ ($x, y \in \mathfrak{C}$ (resp. $\mathfrak{C}^C$)). The symmetric bilinear form of $\mathfrak{C}$ (resp. $\mathfrak{C}^C$) is defined by

$$(x, y) := \frac{1}{2}(x\bar{y} + y\bar{x}).$$

Let $\{e_0, e_1, \ldots, e_7\}$ be the $\boldsymbol{R}$-basis (resp. C-basis) of $\mathfrak{C}$ (resp. $\mathfrak{C}^C$) with the following relation:

$$e_0e_k = e_ke_0 = e_k \ \ (0 \le k \le 7), \quad (e_0 = 1, \text{ the unit element}),$$
$$e_k^2 = -e_0 \ \ (1 \le k \le 7), \quad e_ke_l = -e_le_k \ \ (1 \le k \ne l \le 7),$$
$$e_1e_2 = e_3, \quad e_1e_4 = e_5, \quad e_2e_5 = e_7, \quad e_3e_4 = e_7,$$
$$e_3e_5 = e_6, \quad e_6e_4 = e_2, \quad e_6e_7 = e_1,$$
$$\overline{e_0} = e_0, \quad \overline{e_k} = -e_k \quad (k \ne 0),$$
$$(e_i, e_j) = \delta_{ij} \quad \text{(Kronecker's delta)}.$$

Let γ be the automorphism of $\mathfrak{C}^C$ satisfying

$$\gamma(e_i) = \begin{cases} e_i, & i = 0, 1, 2, 3, \\ -e_i, & i = 4, 5, 6, 7. \end{cases}$$

Define the split Cayley algebra $\mathfrak{C}'$ by

$$\begin{aligned} \mathfrak{C}' &:= \{x \in \mathfrak{C}^C \mid \tau\gamma x = x\} \\ &= \{e_0, e_1, e_2, e_3, \boldsymbol{i}e_4, \boldsymbol{i}e_5, \boldsymbol{i}e_6, \boldsymbol{i}e_7\}_{\boldsymbol{R}\text{-span}}. \end{aligned}$$

We identify $\mathfrak{C}^C$ with C^8 by

$$\sum_{i=0}^{7} x_ie_i = {}^t(x_0, x_1, \dots, x_7).$$

Then we see

$$\mathfrak{so}(8, C) = \mathfrak{so}(\mathfrak{C}^C) = \{X \in \mathfrak{gl}(\mathfrak{C}^C) \mid (Xx, y) + (x, Xy) = 0 \ \text{ for } \ x, y \in \mathfrak{C}^C\},$$
$$\mathfrak{so}(8) = \mathfrak{so}(\mathfrak{C}) = \{X \in \mathfrak{gl}(\mathfrak{C}) \mid (Xx, y) + (x, Xy) = 0 \ \text{ for } \ x, y \in \mathfrak{C}\},$$
$$\mathfrak{so}(4,4) = \mathfrak{so}(\mathfrak{C}') = \{X \in \mathfrak{gl}(\mathfrak{C}') \mid (Xx, y) + (x, Xy) = 0 \ \text{ for } \ x, y \in \mathfrak{C}'\}.$$

Define element G_{ij} $(0 \le i \ne j \le 7)$ of $\mathfrak{so}(8, C)$ and $\mathfrak{so}(8)$ by

$$G_{ij}e_k = \delta_{jk}e_i - \delta_{ik}e_j.$$

For an element $x \in \mathfrak{C}$ or $\mathfrak{C}^C$, we denote the left (resp. right) multiplication by L_x (resp. R_x). Define element F_{ij} $(0 \le i \ne j \le 7)$ of $\mathfrak{so}(8, C)$ or $\mathfrak{so}(8)$ as

$$F_{ij} = \frac{1}{2}L_{e_i}L_{\bar{e}_j}.$$

It is known that both

$$\{G_{ij} | 0 \leq i < j \leq 7\} \quad \text{and} \quad \{F_{ij} | 0 \leq i < j \leq 7\}$$

are C- (resp. $\boldsymbol{R}$-) bases of $\mathfrak{so}(8, C)$ (resp. $\mathfrak{so}(8)$). Furthermore, both

$$\{G_{ij}, G_{4+i4+j}, \boldsymbol{i}G_{k4+l} \mid 0 \leq i < j \leq 3, 0 \leq k, l \leq 3\}$$

and

$$\{F_{ij}, F_{4+i4+j}, \boldsymbol{i}F_{k4+l} \mid 0 \leq i < j \leq 3, 0 \leq k, l \leq 3\}$$

are $\boldsymbol{R}$-bases of $\mathfrak{so}(4, 4)$.

In the following statements of this subsection, we can replace $\mathfrak{so}(8, C)$ with $\mathfrak{so}(8)$ (resp. $\mathfrak{so}(4,4)$) and $\mathfrak{C}^C$ with $\mathfrak{C}$ (resp. $\mathfrak{C}'$).

Define C-linear transformations π, κ and ν of $\mathfrak{so}(8, C)$ as

$$\pi G_{ij} = F_{ij}, \quad \kappa X x = \overline{X\bar{x}} \quad (x \in \mathfrak{C}^C), \quad \nu = \pi\kappa.$$

It is known that π, κ and ν are outer automorphisms of the Lie algebra $\mathfrak{so}(8, C)$ and

$$\pi^2 = \kappa^2 = \nu^3 = (\nu\pi)^2 = \mathrm{id}, \quad \pi = \nu\kappa = \kappa\nu^2, \quad \kappa = \pi\nu = \nu^2\pi,$$

$$\nu^2 = \kappa\pi, \quad \nu\pi = \pi\nu^2 = \kappa\nu = \nu^2\kappa = \kappa\pi\kappa = \pi\kappa\pi,$$

where id denotes the identity map. Hence we have

LEMMA 1.1.

$$\nu G_{ij} = \frac{1}{2} L_{\bar{e}_i} L_{e_j}, \quad \nu^2 G_{ij} = \frac{1}{2} R_{\bar{e}_i} R_{e_j}, \quad \nu\pi G_{ij} = \frac{1}{2} R_{e_i} R_{\bar{e}_j}.$$

The following lemma is well known.

LEMMA 1.2. *For $X \in \mathfrak{so}(8, C)$ and $x, y \in \mathfrak{C}^C$, we have*

$$X(x\bar{y}) = (\pi X x)\bar{y} + x\overline{(\nu X y)}.$$

For $x, y \in \mathfrak{C}^C$, we define a C-linear transformation $x \times y$ of $\mathfrak{C}^C$ as

$$(x \times y)z = (y, z)x - (x, z)y \quad (z \in \mathfrak{C}^C).$$

Clearly, we see $x \times y = -y \times x$, $x \times y \in \mathfrak{so}(8, C)$ for any $x, y \in \mathfrak{C}^C$ and $e_i \times e_i = 0$, $e_i \times e_j = G_{ij}$ $(i \neq j)$. Hence we have the following

LEMMA 1.3. *For $x, y, z \in \mathfrak{C}^C$, the following relations are valid.*

$$\kappa(x \times y) = \bar{x} \times \bar{y}, \tag{1}$$

$$\pi(x \times y) = \frac{1}{2}L_x L_{\bar{y}} - \frac{1}{2}(x, y)\mathrm{id} = \frac{1}{4}(L_x L_{\bar{y}} - L_y L_{\bar{x}}), \tag{2}$$

$$\nu(x \times y) = \frac{1}{2}L_{\bar{x}} L_y - \frac{1}{2}(x, y)\mathrm{id} = \frac{1}{4}(L_{\bar{x}} L_y - L_{\bar{y}} L_x), \tag{3}$$

$$\nu^2(x \times y) = \frac{1}{2}R_{\bar{x}} R_y - \frac{1}{2}(x, y)\mathrm{id} = \frac{1}{4}(R_{\bar{x}} R_y - R_{\bar{y}} R_x), \tag{4}$$

$$\nu\pi(x \times y) = \frac{1}{2}R_x R_{\bar{y}} - \frac{1}{2}(x, y)\mathrm{id} = \frac{1}{4}(R_x R_{\bar{y}} - R_y R_{\bar{x}}), \tag{5}$$

$$x\bar{y} \times z = \pi(x \times zy) - \nu^2(\bar{z}x \times y). \tag{6}$$

PROOF. (1) It is easily obtained. (2) Since $\pi G_{ij} = F_{ij}$ and $L_{e_i} L_{\bar{e}_i} = \mathrm{id}$, the first equality is clear. Using this, we have

$$\begin{aligned} 4\pi(x \times y) &= 2\pi(x \times y) - 2\pi(y \times x) \\ &= \{L_x L_{\bar{y}} - (x, y)\mathrm{id}\} - \{L_y L_{\bar{x}} - (y, x)\mathrm{id}\} = L_x L_{\bar{y}} - L_y L_{\bar{x}}. \end{aligned}$$

The relations (3), (4) and (5) are obtained in a way similar to (2). (6) Let $p \in \mathfrak{C}^C$. Using (2) and (4), we have

$$\begin{aligned} &2\{\pi(x \times zy) - \nu^2(\bar{z}x \times y)\}p \\ &= x\{(\overline{zy})p\} - (py)(\overline{\bar{z}x}) = x\{(\bar{y}\bar{z})p\} - (py)(\bar{x}z) \\ &= x\{2(z, p)\bar{y} - (\bar{y}\bar{p})z\} - (py)(\bar{x}z) \\ &= 2(z, p)x\bar{y} - x\{(\overline{py})z\} - (py)(\bar{x}z) = 2(z, p)x\bar{y} - 2(x, py)z \\ &= 2(z, p)x\bar{y} - 2(x\bar{y}, p)z = 2(x\bar{y} \times z)p. \end{aligned}$$

□

The following is known.

LEMMA 1.4. *The representation π is an even half-spinor representation of $\mathfrak{so}(8, C)$ and the representation ν is an odd half-spinor representation of $\mathfrak{so}(8, C)$.*

§ 2. Spinor Group $Spin(16, \boldsymbol{K})$

We identify $(\mathfrak{C}^C)^2$ (resp. $\mathfrak{C}^2$) with C^{16} (resp. $\boldsymbol{R}^{16}$) as

$${}^t(x, y) = {}^t(x_0, x_1, \ldots, x_7, y_0, y_1, \ldots, y_7)$$

where $x = \sum_{i=0}^{7} x_i e_i$, $y = \sum_{i=0}^{7} y_i e_i$. Using this identification, we can consider that groups $Spin(16, C)$ and $Spin(16)$ are

$$Spin(16, C) = \left\{ \alpha = \tilde{a}_1 \cdots \tilde{a}_{2q} \in Cl((\mathfrak{C}^C)^2) \;\middle|\; \begin{array}{l} q = 1, 2, 3 \ldots, \\ \tilde{a}_i \in (\mathfrak{C}^C)^2 \\ \prod_{i=1}^{2q} (\tilde{a}_i, \tilde{a}_i) = 1 \end{array} \right\},$$

$$Spin(16) = \left\{ \alpha = \tilde{a}_1 \cdots \tilde{a}_{2q} \in Cl(\mathfrak{C}^2) \;\middle|\; \begin{array}{l} q = 1, 2, 3 \ldots \\ \tilde{a}_i \in \mathfrak{C}^2, (\tilde{a}_i, \tilde{a}_i) = 1 \end{array} \right\},$$

where $(\tilde{a}_1, \tilde{a}_2) = (a_1, a_2) + (b_1, b_2)$, $\tilde{a}_i = {}^t(a_i, b_i) \in \mathfrak{C}^2$.

Let

$$V = \mathfrak{C} \otimes \mathfrak{C} \oplus \mathfrak{C} \otimes \mathfrak{C}.$$

V is a 128 dimensional real vector space.

In the following statements of this section, the replacement $Spin(16, C)$ (resp. $\mathfrak{so}(16, C)$, $\mathfrak{C}^C$, $V^C, \ldots$ etc.) with $Spin(16)$ (resp. $\mathfrak{so}(16)$, $\mathfrak{C}$, $V, \ldots$ etc.) is possible.

§2.1. Even Half-spinor Representations of $Spin(16, C)$ and $\mathfrak{so}(16, C)$ ([8])

DEFINITION 2.1. *We define a representation* ρ *of* $Spin(16, C)$ *on* V^C *as*

$$\rho\left(\begin{pmatrix} a_1 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)(x \otimes y, 0)$$

$$= (-a_1(\bar{a}_2 x) \otimes y - x \otimes b_1(\bar{b}_2 y), \bar{a}_1 x \otimes \bar{b}_2 y - \bar{a}_2 x \otimes \bar{b}_1 y),$$

$$\rho\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right)(0, z \otimes w)$$

$$= (-a_1 z \otimes b_2 w + a_2 z \otimes b_1 w, -\bar{a}_1(a_2 z) \otimes w - z \otimes \bar{b}_1(b_2 w)),$$

$$\rho(\tilde{a}_1 \cdot \tilde{a}_2 \cdots \tilde{a}_{2m-1} \cdot \tilde{a}_{2m}) = \rho(\tilde{a}_1 \cdot \tilde{a}_2) \cdots \rho(\tilde{a}_{2m-1} \cdot \tilde{a}_{2m}).$$

Since $\rho(1) = \rho((-\tilde{a}) \cdot \tilde{a}) = \rho\left(\begin{pmatrix} -a \\ -b \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}\right) = 1$, ρ is well-defined.

For $i, j = 0, 1, \ldots, 7$, define elements $\alpha_{ij}^k(t)$ $(k = 1, 2, 3)$ of $Spin(16, C)$ as

$$\alpha_{ij}^1(t) = \begin{pmatrix} -e_i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos \frac{t}{2} e_i + \sin \frac{t}{2} e_j \\ 0 \end{pmatrix}, \quad (i \neq j)$$

$$\alpha_{ij}^2(t) = \begin{pmatrix} 0 \\ -e_i \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \cos \frac{t}{2} e_i + \sin \frac{t}{2} e_j \end{pmatrix}, \quad (i \neq j)$$

$$\alpha_{ij}^3(t) = \begin{pmatrix} -e_i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos \frac{t}{2} e_i \\ \sin \frac{t}{2} e_j \end{pmatrix}.$$

An element of $\mathfrak{so}(16, C)$ can be written by the sum of the following elements

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \begin{pmatrix} 0 & u^t v \\ -v^t u & 0 \end{pmatrix}$$

where $A, B \in \mathfrak{so}(8, C)$ and $u, v \in \mathfrak{C}^C = C^8$. From § 1.2, we have

$$\frac{d}{dt} p(\alpha_{ij}^1(t))\Big|_{t=0} = \begin{pmatrix} G_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{d}{dt} p(\alpha_{ij}^2(t))\Big|_{t=0} = \begin{pmatrix} 0 & 0 \\ 0 & G_{ij} \end{pmatrix},$$

$$\frac{d}{dt} p(\alpha_{ij}^3(t))\Big|_{t=0} = \begin{pmatrix} 0 & e_i{}^t e_j \\ -e_j{}^t e_i & 0 \end{pmatrix}.$$

PROPOSITION 2.2. *The representation $d\rho$ of $\mathfrak{so}(16, C)$ on V^C is given as follows.*

$$d\rho\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)(x \otimes y, z \otimes w)$$

$$= (\pi A x \otimes y + x \otimes \pi B y, \nu A z \otimes w + z \otimes \nu B w),$$

$$d\rho\left(\begin{pmatrix} 0 & u^t v \\ -v^t u & 0 \end{pmatrix}\right)(x \otimes y, z \otimes w) = \left(\frac{1}{2} uz \otimes vw, -\frac{1}{2} \bar{u} x \otimes \bar{v} y\right).$$

PROOF. From Lemma 1.1, we have

$$d\rho\left(\begin{pmatrix} G_{ij} & 0 \\ 0 & 0 \end{pmatrix}\right)(x \otimes y, z \otimes w)$$

$$= \frac{d}{dt} \rho(\alpha_{ij}^1(t))(x \otimes y, z \otimes w)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(e_i\left(\left(\cos\frac{t}{2}\bar{e}_i + \sin\frac{t}{2}\bar{e}_j\right)x\right) \otimes y, \bar{e}_i\left(\left(\cos\frac{t}{2}e_i + \sin\frac{t}{2}e_j\right)z\right) \otimes w\right)\Big|_{t=0}$$

$$= \left(\frac{1}{2}e_i(\bar{e}_j x) \otimes y, \frac{1}{2}\bar{e}_i(e_j z) \otimes w\right) = (\pi G_{ij}x \otimes y, \nu G_{ij}z \otimes w).$$

Similarly we have

$$d\rho\left(\begin{pmatrix} 0 & 0 \\ 0 & G_{ij} \end{pmatrix}\right)(x \otimes y, z \otimes w) = (x \otimes \pi G_{ij}y, z \otimes \nu G_{ij}w).$$

Furthermore

$$d\rho\left(\begin{pmatrix} 0 & e_i{}^t e_j \\ -e_j{}^t e_i & 0 \end{pmatrix}\right)(x \otimes y, z \otimes w)$$

$$= \frac{d}{dt}\rho(\alpha_{ij}^3(t))(x \otimes y, z \otimes w)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(e_i\left(\cos\frac{t}{2}\bar{e}_i x\right) \otimes y + e_i z \otimes \sin\frac{t}{2}e_j w,\right.$$

$$\left. - \bar{e}_i x \otimes \sin\frac{t}{2}\bar{e}_j y + \bar{e}_i\left(\cos\frac{t}{2}e_i z\right) \otimes w\right)\Big|_{t=0}$$

$$= \left(\frac{1}{2}e_i z \otimes e_j w, -\frac{1}{2}\bar{e}_i x \otimes \bar{e}_j y\right). \qquad \square$$

It is clear that the representation $(Spin(16, C), \rho, V^C)$ is irreducible. Since $\dim V^C = 128$, we see the following

PROPOSITION 2.3. *The representation ρ (resp. $d\rho$) of the Lie group $Spin(16, C)$ (resp. the Lie algebra $\mathfrak{so}(16, C)$) is an even half-spinor representation.*

§ 2.2. Bilinear Map $\times$

DEFINITION 2.4. *Define an anti-symmetric bilinear map*

$$\times : V^C \times V^C \to \mathfrak{so}(16, C) \quad (V \times V \to \mathfrak{so}(16))$$

by

$$(x_1 \otimes y_1, 0) \times (x_2 \otimes y_2, 0) = \begin{pmatrix} (y_1, y_2)\pi(x_1 \times x_2) & 0 \\ 0 & (x_1, x_2)\pi(y_1 \times y_2) \end{pmatrix},$$

$$(0, z_1 \otimes w_1) \times (0, z_2 \otimes w_2) = \begin{pmatrix} (w_1, w_2)v^2(z_1 \times z_2) & 0 \\ 0 & (z_1, z_2)v^2(w_1 \times w_2) \end{pmatrix},$$

$$(x \otimes y, 0) \times (0, z \otimes w) = \begin{pmatrix} 0 & \frac{1}{2}(x\bar{z})\,{}^t(y\bar{w}) \\ -\frac{1}{2}(y\bar{w})\,{}^t(x\bar{z}) & 0 \end{pmatrix}.$$

LEMMA 2.5. *For $\alpha \in Spin(16, C)$ and $P, Q \in V^C$, we have*

$$\mathrm{Ad}(\alpha)(P \times Q) = p(\alpha)(P \times Q)p(\alpha)^{-1} = \rho(\alpha)P \times \rho(\alpha)Q.$$

PROOF. It is sufficient to prove for

$$\alpha = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in Spin(16, C), \quad (a_i, a_i) + (b_i, b_i) = 1$$

in the following 3 cases. Case 1. $P = (x_1 \otimes y_1, 0)$, $Q = (x_2 \otimes y_2, 0)$. Case 2. $P = (0, z_1 \otimes w_1)$, $Q = (0, z_2 \otimes w_2)$. Case 3. $P = (x \otimes y, 0)$, $Q = (0, z \otimes w)$.

CASE 1. Let

$$p(\alpha)(P \times Q)p(\alpha)^{-1} = \begin{pmatrix} A_1 & C_1 \\ -{}^tC_1 & B_1 \end{pmatrix}, \quad \rho(\alpha)P \times \rho(\alpha)Q = \begin{pmatrix} A_2 & C_2 \\ -{}^tC_2 & B_2 \end{pmatrix}.$$

Then we have

$$\begin{aligned} A_1 = {} & 4\{(y_1, y_2)(a_1, \pi(x_1 \times x_2)a_2) + (x_1, x_2)(b_1, \pi(y_1 \times y_2)b_2)\}a_1 \times a_2 \\ & + 4\{(a_1, a_2) + (b_1, b_2)\}(y_1, y_2)\pi(x_1 \times x_2)a_2 \times a_1 \\ & - 2(y_1, y_2)\pi(x_1 \times x_2)a_1 \times a_1 - 2(y_1, y_2)\pi(x_1 \times x_2)a_2 \times a_2 \\ & + (y_1, y_2)\pi(x_1 \times x_2) \end{aligned}$$

and

$$\begin{aligned} A_2 = {} & (y_1, y_2)\pi(a_1(\bar{a}_2x_1) \times a_1(\bar{a}_2x_2)) + (b_1(\bar{b}_2y_1), b_1(\bar{b}_2y_2))\pi(x_1 \times x_2) \\ & + (y_1, b_1(\bar{b}_2y_2))\pi(a_1(\bar{a}_2x_1) \times x_2) + (b_1(\bar{b}_2y_1), y_2)\pi(x_1 \times a_1(\bar{a}_2x_2)) \\ & + (\bar{b}_2y_1, \bar{b}_2y_2)v^2(\bar{a}_1x_1 \times \bar{a}_1x_2) + (\bar{b}_1y_1, \bar{b}_1y_2)v^2(\bar{a}_2x_1 \times \bar{a}_2x_2) \\ & - (\bar{b}_2y_1, \bar{b}_1y_2)v^2(\bar{a}_1x_1 \times \bar{a}_2x_2) - (\bar{b}_1y_1, \bar{b}_2y_2)v^2(\bar{a}_2x_1 \times \bar{a}_1x_2). \end{aligned}$$

From Lemma 1.3, we have

$$(y_1, b_1(\bar{b}_2 y_2)) = (\bar{b}_1 y_1, \bar{b}_1 y_2) = 2(b_1, \pi(y_1 \times y_2) b_2) + (y_1, y_2)(b_1, b_2),$$

$$\begin{aligned}\pi(a_1(\bar{a}_2 x_1) \times x_2) - v^2(\bar{a}_2 x_1 \times \bar{a}_1 x_2) &= a_1 \times x_2(\overline{\bar{a}_2 x_1}) \\ &= (x_1, x_2) a_1 \times a_2 + 2\pi(x_1 \times x_2) a_2 \times a_1,\end{aligned}$$

$$\begin{aligned}v^2(\bar{a} x_1 \times \bar{a} x_2) &= \pi(x_1 \times a(\bar{a} x_2)) - x_1(\overline{\bar{a} x_2}) \times a \\ &= (a, a)\pi(x_1 \times x_2) - 2\pi(x_1 \times x_2) a \times a,\end{aligned}$$

$$\begin{aligned}\pi(a_1(\bar{a}_2 x_1) \times a_1(\bar{a}_2 x_2)) = {} & (a_1, a_1)(a_2, a_2)\pi(x_1 \times x_2) - 2(a_2, a_2)\pi(x_1 \times x_2) a_1 \times a_1 \\ & - 2(a_1, a_1)\pi(x_1 \times x_2) a_2 \times a_2 + 4(a_1, a_2)\pi(x_1 \times x_2) a_2 \times a_1 \\ & + 4(a_1, \pi(x_1 \times x_2) a_2) a_1 \times a_2.\end{aligned}$$

Using these, we see $A_2 = A_1$. We can obtain $B_1 = B_2$ and $C_1 = C_2$ similarly. We can prove Cases 2 and 3 in a way similar to Case 1. □

This lemma implies the following

Lemma 2.6. *For $X \in \mathfrak{so}(16, C)$ and $P, Q \in V^C$, we have*

$$[X, P \times Q] = d\rho(X)P \times Q + P \times d\rho(X)Q.$$

Lemma 2.7. *For $P_i \in V^C$ $(i = 1, 2, 3)$, we have*

$$d\rho(P_1 \times P_2)P_3 + d\rho(P_2 \times P_3)P_1 + d\rho(P_3 \times P_1)P_2 = 0.$$

Proof. It is sufficient to prove the following 4 cases. Case 1. $P_i = (x_i \otimes y_i, 0)$, $(i = 1, 2, 3)$. Case 2. $P_i = (x_i \otimes y_i, 0)$, $(i = 1, 2)$, $P_3 = (0, z \otimes w)$. Case 3. $P_i = (0, z_i \otimes w_i)$, $(i = 1, 2)$, $P_3 = (x \otimes y, 0)$. Case 4. $P_i = (0, z_i \otimes w_i)$, $(i = 1, 2, 3)$.

Case 1.

$$\begin{aligned}d\rho(P_1 \times P_2)P_3 &= ((y_1, y_2)\{(x_1 \times x_2)x_3\} \otimes y_3 + (x_1, x_2)x_3 \otimes \{(y_1 \times y_2)y_3\}, 0) \\ &= ((x_2, x_3)x_1 \otimes (y_1, y_2)y_3 - (x_3, x_1)x_2 \otimes (y_1, y_2)y_3 \\ &\quad + (x_1, x_2)x_3 \otimes (y_2, y_3)y_1 - (x_1, x_2)x_3 \otimes (y_3, y_1)y_2, 0)\end{aligned}$$

$$
\begin{aligned}
&= -((x_3, x_1)x_2 \otimes (y_2, y_3)y_1 - (x_1, x_2)x_3 \otimes (y_2, y_3)y_1 \\
&\qquad + (x_2, x_3)x_1 \otimes (y_3, y_1)y_2 - (x_2, x_3)x_1 \otimes (y_1, y_2)y_3, 0) \\
&\quad - ((x_1, x_2)x_3 \otimes (y_3, y_1)y_2 - (x_2, x_3)x_1 \otimes (y_3, y_1)y_2 \\
&\qquad + (x_3, x_1)x_2 \otimes (y_1, y_2)y_3 - (x_3, x_1)x_2 \otimes (y_2, y_3)y_1, 0) \\
&= -d\rho(P_2 \times P_3)P_1 - d\rho(P_3 \times P_1)P_2
\end{aligned}
$$

Hence the formula can be proved in this case. The other cases can be probed in a way similar to Case 1. □

§ 2.3. Symmetric Bilinear Form in V^C

Define a symmetric bilinear form $(\,,)$ in V^C as

$$((x_1 \otimes y_1, z_1 \otimes w_1), (x_2 \otimes y_2, z_2 \otimes w_2)) = (x_1, x_2)(y_1, y_2) + (z_1, z_2)(w_1, w_2).$$

Then we have the following

LEMMA 2.8. *For $\alpha \in Spin(16, C)$, $X \in \mathfrak{so}(16, C)$ and $P, Q \in V^C$, we have*

(1) $$(\rho(\alpha)P, \rho(\alpha)Q) = (P, Q),$$

(2) $$(d\rho(X)P, Q) + (P, d\rho(X)Q) = 0,$$

(3) $$\operatorname{tr} X(P \times Q) = 2(d\rho(X)P, Q).$$

PROOF. (1) and (2) are clear. In order to prove (3), we consider the following lemma.

LEMMA 2.9. *For $A \in \mathfrak{so}(8, C)$ and $x, y \in \mathfrak{C}^C$, we have*

(4) $$\operatorname{tr} A\pi(x \times y) = 2(\pi A x, y),$$

(5) $$\operatorname{tr} A v^2(x \times y) = 2(v^2 A x, y).$$

PROOF. Since $\mathfrak{so}(8, C)$ is spanned by $a \times b$ $(a, b \in \mathfrak{C}^C)$, it is sufficient to prove for $A = a \times b$. From Lemma 1.3, we have

$$
\begin{aligned}
\operatorname{tr} A\pi(x \times y) &= \sum_i (\pi(x \times y)Ae_i, e_i) \\
&= \sum_i \frac{1}{2}(x(\bar{y}Ae_i), e_i) - \sum_i \frac{1}{2}(x, y)(Ae_i, e_i)
\end{aligned}
$$

$$= \sum_i \frac{1}{2}\{(b, e_i)(x(\bar{y}a), e_i) - (a, e_i)(x(\bar{y}b), e_i)\}$$

$$= \frac{1}{2}\{(x(\bar{y}a), b) - (x(\bar{y}b), a)\} = \frac{1}{2}\{(a(\bar{b}x), y) - (b(\bar{a}x), y)\}$$

$$= 2\left(\frac{1}{4}(L_a L_{\bar{b}} - L_b L_{\bar{a}})x, y\right) = 2(\pi A x, y).$$

Thus (4) is proved. (5) can be proved similarly. □

PROOF OF LEMMA 2.8 (3). Let $P = (x_1 \otimes y_1, z_1 \otimes w_1)$ and $Q = (x_2 \otimes y_2, z_2 \otimes w_2)$. For $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ $(A, B \in \mathfrak{so}(8, G))$, we have

$$\begin{aligned} \operatorname{tr} X(P \times Q) &= (y_1, y_2)\operatorname{tr} A\pi(x_1 \times x_2) + (w_1, w_2)\operatorname{tr} A\nu^2(z_1 \times z_2) \\ &\quad + (x_1, x_2)\operatorname{tr} B\pi(y_1 \times y_2) + (z_1, z_2)\operatorname{tr} B\nu^2(w_1 \times w_2) \\ &= 2(y_1, y_2)(\pi A x_1, x_2) + 2(w_1, w_2)(\nu^2 A z_1, z_2) \\ &\quad + 2(x_1, x_2)(\pi B y_1, y_2) + 2(z_1, z_2)(\nu^2 B w_1, w_2) \\ &= 2(d\rho(X)P, Q). \end{aligned}$$

For $X = \begin{pmatrix} 0 & u\,{}^t v \\ -v\,{}^t u & 0 \end{pmatrix}$, we have

$$\begin{aligned} 2\operatorname{tr} X(P \times Q) &= -\operatorname{tr} u\,{}^t v\,{}^t\{(x_1\bar{z}_2)\,{}^t(y_1\bar{w}_2) - (x_2\bar{z}_1)\,{}^t(y_2\bar{w}_1)\} \\ &\quad - \operatorname{tr} v\,{}^t u\{(x_1\bar{z}_2)\,{}^t(y_1\bar{w}_2) - (x_2\bar{z}_1)\,{}^t(y_2\bar{w}_1)\} \\ &= -(v, y_1\bar{w}_2)(u, x_1\bar{z}_2) + (v, y_2\bar{w}_1)(u, x_2\bar{z}_1) \\ &\quad - (u, x_1\bar{z}_2)(v, y_1\bar{w}_2) + (u, x_2\bar{z}_1)(v, y_2\bar{w}_1) \\ &= 2(uz_1, x_2)(vw_1, y_2) - 2(\bar{u}x_1, z_2)(\bar{v}y_1, w_2) \\ &= 4(d\rho(X)P, Q). \end{aligned}$$

□

§3. Complex Exceptional Lie Algebra $\mathfrak{g}^C$ of Type E_8

§3.1. Lie Algebra $\mathfrak{g}^C$

Let

$$\mathfrak{g} = \mathfrak{so}(16) \oplus V.$$

$\mathfrak{g}$ is a 248 dimensional real vector space.

Definition 3.1. *We define an anti-symmetric bilinear multiplication* $[\,,]$ *in* $\mathfrak{g}$ (*resp.* $\mathfrak{g}^C$) *as*

$$[(X,P),(Y,Q)] = ([X,Y] - P\times Q, d\rho(X)Q - d\rho(Y)P)$$

where $X, Y \in \mathfrak{so}(16)$ (*resp.* $\mathfrak{so}(16,C)$) *and* $P, Q \in V$ (*resp.* V^C).

Lemma 3.2. $\mathfrak{g}$ *and* $\mathfrak{g}^C$ *are Lie algebras with the multiplication* $[\,,]$.

Proof. We can prove the Jacobi identity using Lemmas 2.6 and 2.7. □

§3.2. Simplicity of $\mathfrak{g}^C$ and Type of $\mathfrak{g}^C$

Lemma 3.3. $\mathfrak{g}^C$ *is a simple Lie algebra.*

Proof. Let $\mathfrak{a}$ be a non-zero ideal of $\mathfrak{g}^C$. There are three cases to be considered: Case 1. $\mathfrak{so}(16,C)\cap\mathfrak{a}\neq\{0\}$. Case 2. $V^C\cap\mathfrak{a}\neq\{0\}$. Case 3. $\mathfrak{so}(16,C)\cap\mathfrak{a}=\{0\}$ and $V^C\cap\mathfrak{a}=\{0\}$.

Case 1. Since $\mathfrak{so}(16,C)\cap\mathfrak{a}$ is a non-zero ideal of $\mathfrak{so}(16,C)$, we see $\mathfrak{so}(16,C)\subset\mathfrak{a}$. Moreover we have

$$V^C = d\rho(\mathfrak{so}(16,C))V^C = [\mathfrak{so}(16,C), V^C] \subset [\mathfrak{a}, V^C] \subset \mathfrak{a}.$$

Then we have $\mathfrak{a} = \mathfrak{g}^C$.

Case 2. For any non-zero element $P\in V^C\cap\mathfrak{a}$, we can choose an element $Q\in V^C$ such that $P\times Q\neq 0$. Hence we can reduce this case to the Case 1.

Case 3. Let $q:\mathfrak{g}^C\to\mathfrak{so}(16,C)$ denote the projection. Since $q(\mathfrak{a})$ is a non-zero ideal of $\mathfrak{so}(16,C)$, we see $q(\mathfrak{a})=\mathfrak{so}(16,C)$. Let $V^C=\sum_{\alpha_i}(V^C)_{\alpha_i}$ be a weight decomposition of the representation $(\mathfrak{so}(16,C), d\rho, V^C)$ with respect to a Cartan subalgebra $\mathfrak{h}_{\mathfrak{so}(16,C)}\subset\mathfrak{so}(16,C)$. Choose an element $H\in\mathfrak{h}_{\mathfrak{so}(16,C)}$ such that $\alpha_i(H)\neq 0$ for any weight α_i. Since $H\in\mathfrak{h}_{\mathfrak{so}(16,C)}\subset q(\mathfrak{a})$, there exist a non-zero element

$$P = \sum_{\alpha_i} b_{\alpha_i}P_{\alpha_i} \in V^C$$

where $P_{\alpha_i}\in(V^C)_{\alpha_i}$ is a weight vector and $b_{\alpha_i}\in C$, such that $(H,P)\in\mathfrak{a}$. Then we have

$$[(H,0),(H,P)] = (0, d\rho(H)P) = \left(0, \sum_{\alpha_i} \alpha_i(H) b_{\alpha_i} P_{\alpha_i}\right) \in V^C \cap \mathfrak{a} = \{0\}.$$

This shows $\alpha_i(H) b_{\alpha_i} P_{\alpha_i} = 0$. Since $\alpha_i(H) \neq 0$, we see $b_{\alpha_i} = 0$, i.e., $P = \sum_{\alpha_i} b_{\alpha_i} P_{\alpha_i} = 0$. This is a contradiction. □

Since $\dim_C \mathfrak{g}^C = 248$, we have the following

THEOREM 3.4. *$\mathfrak{g}^C$ is a complex exceptional Lie algebra of type E_8.*

§3.3. Killing Form of $\mathfrak{g}^C$

PROPOSITION 3.5. *The Killing form B of $\mathfrak{g}^C$ is given by*

$$B((X,P),(Y,Q)) = 30 \operatorname{tr} XY - 60(P,Q).$$

PROOF. Let us define a symmetric bilinear form as

$$B_1((X,P),(Y,Q)) = \operatorname{tr} XY - 2(P,Q).$$

Using Lemma 2.8, we see that B_1 is $\mathfrak{g}^C$-invariant. Since $\mathfrak{g}^C$ is simple, there exists some $\alpha \in C$ such that $B = \alpha B_1$. Since $B(R,R) = 60$ and $B_1(R,R) = 2$ for $R = \left(\begin{pmatrix} iG_{ij} & 0 \\ 0 & 0 \end{pmatrix}, 0\right) \in \mathfrak{g}^C$, we see $B = 30B_1$. □

§4. Real Forms of $\mathfrak{g}^C$

§4.1. Involutions of $\mathfrak{g}^C$

Let γ be the automorphism of $\mathfrak{C}^C$ defined in §**1.2**. Using this, we define a C-linear transformation γ_1 on $\mathfrak{g}^C$ as

$$\gamma_1\left(\begin{pmatrix} A & u\,{}^t v \\ -v\,{}^t u & B \end{pmatrix}, (x \otimes y, z \otimes w)\right)$$

$$= \left(\begin{pmatrix} \gamma A \gamma^{-1} & (\gamma u)\,{}^t u \\ -v\,{}^t(\gamma u) & B \end{pmatrix}, (\gamma x \otimes y, \gamma z \otimes w)\right).$$

It is clear $\gamma_1^2 = \mathrm{id}$. Furthermore, define C-linear transformations ε_1, ε_2 and ε of

$\mathfrak{g}^C$ as

$$\varepsilon_1(X, (x \otimes y, z \otimes w)) = (I_{8,8} X I_{8,8}, (-x \otimes y, z \otimes w)),$$
$$\varepsilon_2(X, (x \otimes y, z \otimes w)) = (I_{8,8} X I_{8,8}, (x \otimes y, -z \otimes w)),$$
$$\varepsilon(X, (x \otimes y, z \otimes w)) = (X, (-x \otimes y, -z \otimes w)).$$

It is clear that $\varepsilon = \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1$ and $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon^2 = \mathrm{id}$. From the definition, it is clear that γ_1, ε_1, ε_2 and ε are commutable with each other. Furthermore we have the following Lemma through straightforward calculations.

LEMMA 4.1. *γ_1, ε_1, ε_2 and ε are involutive automorphisms of $\mathfrak{g}^C$.*

Let J be a C-linear transformation of $\mathfrak{g}^C$ defined by

$$J(X, (x \otimes y, z \otimes w)) = (J_8 X J_8^{-1}, (y \otimes x, w \otimes z))$$

where $J_8 = \begin{pmatrix} & -I_8 \\ I_8 & \end{pmatrix}$. It is clear that $J^2 = \mathrm{id}$ and $\varepsilon J = J\varepsilon$. Furthermore we have the following Lemma through straightforward calculations.

LEMMA 4.2. *J is an involutive automorphism of $\mathfrak{g}^C$.*

Since τ is the complex conjugation, $\tau\gamma_1$, $\tau\varepsilon_j$, $\tau\varepsilon$, $\tau\gamma_1\varepsilon_j$, $\tau\gamma_1\varepsilon$, τJ and $\tau\varepsilon J$ $(j = 1, 2)$ are complex conjugate linear involutions of the complex Lie algebra $\mathfrak{g}^C$.

§4.2. Real Forms of $\mathfrak{g}^C$

Since the Killing form B of $\mathfrak{g}^C$ is negative definite on $\mathfrak{g}$,

$$\mathfrak{g} = (\mathfrak{g}^C)^\tau = \{R \in \mathfrak{g}^C \mid \tau R = R\} = \mathfrak{so}(16) \oplus V$$

is a compact real form of $\mathfrak{g}^C$.

Let us consider the $\boldsymbol{R}$-subalgebra $(\mathfrak{g}^C)^{\tau\varepsilon_1}$ of $\mathfrak{g}^C$ defined as

$$(\mathfrak{g}^C)^{\tau\varepsilon_1} := \{R \in \mathfrak{g}^C \mid \tau\varepsilon_1 R = R\} = \mathfrak{so}(8, 8) \oplus (V^C)^{\tau\varepsilon_1},$$

where

$$(V^C)^{\tau\varepsilon_1} = \boldsymbol{i}\mathfrak{C} \otimes \mathfrak{C} \oplus \mathfrak{C} \otimes \mathfrak{C}.$$

It is clear that $(\mathfrak{g}^C)^{\tau\varepsilon_1}$ is a real form of $\mathfrak{g}^C$, i.e., $((\mathfrak{g}^C)^{\tau\varepsilon_1})^C = \mathfrak{g}^C$. Let

$$\mathfrak{so}(8, 8) = \mathfrak{k}_{\mathfrak{so}(8,8)} \oplus \mathfrak{p}_{\mathfrak{so}(8,8)}$$

be a Cartan decomposition of $\mathfrak{so}(8,8)$. The Cartan decomposition of $(\mathfrak{g}^C)^{\tau\varepsilon_1}$ is given by

$$(\mathfrak{g}^C)^{\tau\varepsilon_1} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{k}_{\mathfrak{so}(8,8)} \oplus \mathfrak{k}(V^C)^{\tau\varepsilon_1},$$

$$\mathfrak{p} = \mathfrak{p}_{\mathfrak{so}(8,8)} \oplus \mathfrak{p}(V^C)^{\tau\varepsilon_1},$$

where

$$\mathfrak{k}_{(V^C)^{\tau\varepsilon_1}} = \{(0, z \otimes w) \mid z, w \in \mathfrak{C}\},$$

$$\mathfrak{p}_{(V^C)^{\tau\varepsilon_1}} = \{(\boldsymbol{i}x \otimes y, 0) \mid x, y \in \mathfrak{C}\}.$$

The Cartan involution is ε_1 and the Cartan index of $(\mathfrak{g}^C)^{\tau\varepsilon_1}$ is $\dim \mathfrak{p} - \dim \mathfrak{k} = 128 - 120 = 8$. Hence $(\mathfrak{g}^C)^{\tau\varepsilon_1}$ is an exceptional non-compact real simple Lie algebra of type $E_{8(8)}$. Similarly, let us consider the other real forms of $\mathfrak{g}^C$ as follows:

$$(\mathfrak{g}^C)^{\tau\varepsilon} := \{R \in \mathfrak{g}^C \mid \tau\varepsilon R = R\} = \mathfrak{so}(16) \oplus \boldsymbol{i}V,$$

(Cartan involution ε, Cartan index 8),

$$(\mathfrak{g}^C)^{\tau\varepsilon_1\gamma_1} := \{R \in \mathfrak{g}^C \mid \tau\varepsilon_1\gamma_1 R = R\} = \mathfrak{so}(4,12) \oplus (V^C)^{\tau\varepsilon_1\gamma_1},$$

(Cartan involution $\varepsilon_1\gamma_1$, Cartan index -24),

$$(\mathfrak{g}^C)^{\tau J} := \{R \in \mathfrak{g}^C \mid \tau J R = R\} = \mathfrak{so}^*(16) \oplus (V^C)^{\tau J},$$

(Cartan involution J, Cartan index -24),

$$(\mathfrak{g}^C)^{\tau\varepsilon J} := \{R \in \mathfrak{g}^C \mid \tau\varepsilon J R = R\} = \mathfrak{so}^*(16) \oplus \boldsymbol{i}(V^C)^{\tau J},$$

(Cartan involution εJ, Cartan index 8),

where

$$(V^C)^{\tau\varepsilon_1\gamma_1} = \boldsymbol{i}\mathfrak{C}' \otimes \mathfrak{C} \oplus \mathfrak{C}' \otimes \mathfrak{C},$$

$$(V^C)^{\tau J} = (\mathfrak{C}^C \otimes \mathfrak{C}^C)^{\tau J} \oplus (\mathfrak{C}^C \otimes \mathfrak{C}^C)^{\tau J},$$

$$(\mathfrak{C}^C \otimes \mathfrak{C}^C)^{\tau J} = \left\{ \sum_k x_k \otimes y_k \,\middle|\, \sum_k x_k \otimes y_k = \sum_k \tau y_k \otimes \tau x_k \right\}.$$

Thus we have the following

THEOREM 4.3. (1) $\mathfrak{g}$ *is a compact exceptional real Lie algebra of type* E_8.

(2) $(\mathfrak{g}^C)^{\tau\varepsilon} \cong (\mathfrak{g}^C)^{\tau\varepsilon_1} \cong (\mathfrak{g}^C)^{\tau\varepsilon J}$ *is a non-compact exceptional real Lie algebra of type* $E_{8(8)}$.

(3) $(\mathfrak{g}^C)^{\tau\varepsilon_1\gamma_1} \cong (\mathfrak{g}^C)^{\tau J}$ *is a non-compact exceptional real Lie algebra of type* $E_{8(-24)}$.

§5. Exceptional Simple Lie Groups of Type E_8 ([5], [6], [10], [11])

Define a positive definite Hermitian inner product of $\mathfrak{g}^C$ as

$$\langle R_1, R_2 \rangle := -B(R_1, \tau R_2), \quad (R_i \in \mathfrak{g}^C).$$

We define a complex exceptional simple Lie group and a compact exceptional simple Lie group of type E_8 as

$$E_8^C := \mathrm{Aut}_C \mathfrak{g}^C,$$

$$E_8 := \{\alpha \in E_8^C \mid \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle\}.$$

For a non-compact exceptional simple Lie algebra $\mathfrak{e}_{8(k)}$ of type $E_{8(k)}$ ($k = 8$ or -24), a non-compact connected exceptional simple Lie group $E_{8(k)}$ can be defined as

$$E_{8(k)} := \mathrm{Aut}_{\boldsymbol{R}} \mathfrak{e}_{8(k)}$$

$$:= ((E_8)^C)^{\sigma_k} = \{\alpha \in E_8^C \mid \sigma_k \alpha = \alpha \sigma_k\},$$

where σ_k is the complex conjugation of $\mathfrak{e}_8^C = \mathfrak{g}^C$ with respect to $e_{8(k)}$. Hence from Theorem 4.3 we have the following

THEOREM 5.1.

(1) $$E_8 \cong (E_8^C)^\tau := \{\alpha \in E_8^C \mid \tau\alpha = \alpha\tau\},$$

(2) $$E_{8(8)} \cong (E_8^C)^{\tau\varepsilon} \cong (E_8^C)^{\tau\varepsilon_1} \cong (E_8^C)^{\tau\varepsilon J},$$

(3) $$E_{8(-24)} \cong (E_8^C)^{\tau\varepsilon_1\gamma_1} \cong (E_8^C)^{\tau J}.$$

From definitions of the transformation ε and the real forms of $\mathfrak{g}^C$, ε can be considered as an element of the groups E_8^C, E_8, $E_{8(8)}$ and $E_{8(-24)}$.

§6. Subgroups of Type D_8

Clearly, the complex exceptional Lie algebra $\mathfrak{g}^C$ of type E_8 has a classical subalgebra $\mathfrak{so}(16, C)$ of type D_8. For real exceptional Lie algebras of type E_8, from §4.2 we see the following (1), (2) and (3).

(1) The compact exceptional Lie algebra of type E_8 has a classical subalgebra $\mathfrak{so}(16)$ of type D_8.
(2) The non-compact exceptional Lie algebra of type $E_{8(8)}$ has classical subalgebras $\mathfrak{so}(16)$, $\mathfrak{so}(8,8)$ and $\mathfrak{so}^*(16)$ of type D_8.
(3) The non-compact exceptional Lie algebra of type $E_{8(-24)}$ has classical subalgebras $\mathfrak{so}(4,12)$ and $\mathfrak{so}^*(16)$ of type D_8.

In this section, we consider subgroups of type D_8 of complex and real exceptional Lie groups of type E_8.

§6.1. Subgroups of Type D_8 of E_8^C

In this subsection, consider subgroup of E_8^C;

$$(E_8^C)^\varepsilon = \{\alpha \in E_8^C \mid \varepsilon\alpha\varepsilon = \alpha\} \subset E_8^C.$$

We define a mapping $\varphi : Spin(16,C) \to (E_8^C)^\varepsilon$ as

$$\varphi(\alpha)(X,P) = (\mathrm{Ad}(\alpha)X, \rho(\alpha)P).$$

The Lie algebra of $(E_8^C)^\varepsilon$ is isomorphic to $(\mathfrak{e}_8^C)^\varepsilon = \mathfrak{so}(16,C) = \mathrm{Lie}Spin(16,C)$. Thus the differential of φ is surjective. Since $(E_8^C)^\varepsilon$ is connected ([7]), φ is surjective. Since

$$z(Spin(16,C)) = \{1,-1\} \times \{1,\omega\}$$

where

$$\omega = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} e_7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ e_7 \end{pmatrix}$$

and $\rho(\pm 1) = \rho(\pm\omega) = \pm\mathrm{id}$, we see $\mathrm{Ker}\varphi = \{1,\omega\}$. Hence we have the following

Theorem 6.1. *The complex exceptional Lie group E_8^C of type E_8 has the following subgroup of type D_8.*

$$(E_8^C)^\varepsilon \cong Ss(16,C) := Spin(16,C)/\{1,\omega\}.$$

§6.2. Preliminaries for Non-compact Case

Define elements ω_1, $\omega_2 \in Spin(16,C)$ as

$$\omega_1 = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} e_7 \\ 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 \\ e_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ e_7 \end{pmatrix}.$$

It is clear that $\omega_1^2 = \omega_2^2 = 1$ and $\omega = \omega_1 \cdot \omega_2 = \omega_2 \cdot \omega_1$. Since

$$p(\omega_1) = I_{8,8}, \quad p(\omega_2) = -I_{8,8} \in SO(16, C)$$

and

$$L_{e_0}L_{e_1}L_{e_2}L_{e_3}L_{e_4}L_{e_5}L_{e_6}L_{e_7} = \mathrm{id}_{\mathfrak{e}^C},$$

we see the following

LEMMA 6.2.

$$\varphi(\omega_1) = \varphi(\omega_2) = \varepsilon_2, \quad \varphi(-\omega_1) = \varphi(-\omega_2) = \varepsilon_1,$$
$$\varphi(1) = \varphi(\omega) = 1, \quad \varphi(-1) = \varphi(-\omega) = \varepsilon.$$

Furthermore we define elements ω_3, $\omega_4 \in Spin(16, C)$ as

$$\omega_3 = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_3 \\ 0 \end{pmatrix},$$
$$\omega_4 = \begin{pmatrix} e_4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_6 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ e_7 \end{pmatrix}.$$

It is clear that $\omega_3^2 = \omega_4^2 = 1$ and $\omega = \omega_3 \cdot \omega_4 = \omega_4 \cdot \omega_3$. Since

$$p(\omega_3) = I_{4,12}, \quad p(\omega_4) = -I_{4,12} \in SO(16, C)$$

and

$$L_{e_0}L_{e_1}L_{e_2}L_{e_3} = L_{e_4}L_{e_5}L_{e_6}L_{e_7} = -\gamma,$$

we have the following

LEMMA 6.3.

$$\varphi(\omega_3) = \varphi(\omega_4) = \varepsilon_1\gamma_1, \quad \varphi(-\omega_3) = \varphi(-\omega_4) = \varepsilon_2\gamma_1.$$

Define an element $\boldsymbol{j} \in Spin(16, C)$ as

$$\boldsymbol{j} = \begin{pmatrix} \frac{1}{\sqrt{2}}e_0 \\ \frac{1}{\sqrt{2}}e_0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}}e_1 \\ \frac{1}{\sqrt{2}}e_1 \end{pmatrix} \cdots \begin{pmatrix} \frac{1}{\sqrt{2}}e_7 \\ \frac{1}{\sqrt{2}}e_7 \end{pmatrix}.$$

It is clear that $\boldsymbol{j}^2 = 1$.

Lemma 6.4.

$$\boldsymbol{j} = \omega_1 \cdot \boldsymbol{j} \cdot \omega_2 = \omega_2 \cdot \boldsymbol{j} \cdot \omega_1, \quad \omega_1 \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \omega_2, \quad \omega_2 \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \omega_1.$$

Proof. Since $\begin{pmatrix} e_k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e_k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_k \end{pmatrix} = -1$, we see

$$\begin{pmatrix} e_k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_k \\ e_k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_k \end{pmatrix} = \begin{pmatrix} e_k \\ 0 \end{pmatrix} \cdot \left\{ \begin{pmatrix} e_k \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e_k \end{pmatrix} \right\} \cdot \begin{pmatrix} 0 \\ e_k \end{pmatrix} = -\begin{pmatrix} e_k \\ e_k \end{pmatrix}.$$

Hence

$$\omega_1 \cdot \boldsymbol{j} \cdot \omega_2 = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} e_0 \\ \frac{1}{\sqrt{2}} e_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_0 \end{pmatrix} \cdots \begin{pmatrix} e_7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} e_7 \\ \frac{1}{\sqrt{2}} e_7 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_7 \end{pmatrix}$$

$$= (-1)^8 \begin{pmatrix} \frac{1}{\sqrt{2}} e_0 \\ \frac{1}{\sqrt{2}} e_0 \end{pmatrix} \cdots \begin{pmatrix} \frac{1}{\sqrt{2}} e_7 \\ \frac{1}{\sqrt{2}} e_7 \end{pmatrix} = \boldsymbol{j}.$$

Furthermore we see $\omega_1 \cdot \boldsymbol{j} = \omega_1 \cdot \boldsymbol{j} \cdot \omega_2^2 = \boldsymbol{j} \cdot \omega_2$. $\square$

Lemma 6.5.

$$\varphi(\boldsymbol{j} \cdot \omega_1) = \varphi(\boldsymbol{j} \cdot \omega_2) = J, \quad \varphi(-\boldsymbol{j} \cdot \omega_1) = \varphi(-\boldsymbol{j} \cdot \omega_2) = \varepsilon J.$$

Proof. From §**1.2**, we have $p(\boldsymbol{j} \cdot \omega_1) = J_8$. Hence

$$\varphi(\boldsymbol{j} \cdot \omega_1)(X, P) = (J_8 X J_8^{-1}, \rho(\boldsymbol{j} \cdot \omega_1) P).$$

Let $\boldsymbol{j}_k = \begin{pmatrix} \frac{1}{\sqrt{2}} e_k \\ \frac{1}{\sqrt{2}} e_k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_k \end{pmatrix}$. Then $\boldsymbol{j} \cdot \omega_1 = \boldsymbol{j}_0 \cdot \boldsymbol{j}_1 \cdots \boldsymbol{j}_7$, and we have

$$\rho(\boldsymbol{j}_k)(x \otimes y, z \otimes w) = \frac{1}{\sqrt{2}}(-x \otimes y + E_k(z \otimes w), -E_k(x \otimes y) - z \otimes w),$$

where $E_k(z \otimes w) = e_k z \otimes e_k w$. Using this, we have

$$\rho(\boldsymbol{j} \cdot \omega_1)(x \otimes y, z \otimes w) = \tfrac{1}{8}(A(x \otimes y), A(z \otimes w)),$$

where

$$A = 1 - \sum_{k<l} E_k E_l + \sum_{1 \le k<l<m} E_k E_l E_m.$$

Throught the straightforward calculations, we see $A(e_i \times e_j) = 8e_j \times e_i$ for any $0 \le i,\ j \le 7$. Then we have $\varphi(\boldsymbol{j} \cdot \omega_1) = J$. Since $\varphi(\omega) = 1$ and $\varphi(-\omega) = \varepsilon$, others are clear. $\square$

§ 6.3. Subgroup $Ss(16)$ of E_8 and $E_{8(8)}$

From § 6.1, we have the following

THEOREM 6.6. (1) *The compact exceptional Lie group $(E_8^C)^\tau$ of type E_8 has the following subgroup of type D_8.*

$$((E_8^C)^\tau)^\varepsilon \cong Ss(16) := Spin(16)/\{1, \omega\}.$$

(2) *The non-compact exceptional Lie group $(E_8^C)^{\tau\varepsilon}$ of type $E_{8(8)}$ has the following subgroup of type D_8.*

$$((E_8^C)^{\tau\varepsilon})^\varepsilon \cong Ss(16) := Spin(16)/\{1, \omega\}.$$

PROOF. (1)

$$\begin{aligned} ((E_8^C)^\tau)^\varepsilon = ((E_8^C)^\varepsilon)^\tau &= \{\varphi(\alpha) \mid \alpha \in Spin(16, C), \tau\varphi(\alpha)\tau = \varphi(\alpha)\} \\ &= \{\varphi(\alpha) \mid \alpha \in Spin(16)\} \\ &\cong Ss(16) := Spin(16)/\{1, \omega\}. \end{aligned}$$

(2) $((E_8^C)^\tau)^\varepsilon = ((E_8^C)^{\tau\varepsilon})^\varepsilon$ is clear. $\square$

§ 6.4. Subgroup $Ss(8, 8) \times 2$ of $E_{8(8)}$

In this subsection, consider subgroup of $(E_8^C)^{\tau\varepsilon_1}$;

$$((E_8^C)^{\tau\varepsilon_1})^\varepsilon = \{\alpha \in (E_8^C)^{\tau\varepsilon_1} \mid \varepsilon\alpha\varepsilon = \alpha\} \subset (E_8^C)^{\tau\varepsilon_1} \cong E_{8(8)}.$$

Since

$$\begin{aligned} ((E_8^C)^{\tau\varepsilon_1})^\varepsilon = ((E_8^C)^\varepsilon)^{\tau\varepsilon_1} &= \{\varphi(\alpha) \mid \alpha \in Spin(16, C), \varepsilon_1\tau\varphi(\alpha)\tau\varepsilon_1 = \varphi(\alpha)\} \\ &= \{\varphi(\alpha) \mid \alpha \in Spin(16, C), \varepsilon_1\varphi(\tau\alpha)\varepsilon_1 = \varphi(\alpha)\}, \end{aligned}$$

we consider

$$G = \{\alpha \in Spin(16, C) \mid \varepsilon_1 \varphi(\tau\alpha)\varepsilon_1 = \varphi(\alpha)\}.$$

Clearly, $((E_8^C)^{\tau\varepsilon_1})^{\varepsilon} = \varphi(G)$. Let $\alpha \in G$. From Lemma 6.2, we have

$$\varphi(\alpha) = \varepsilon_1 \varphi(\tau\varphi)\varepsilon_1 = \varphi(-\omega_i)\varphi(\tau\alpha)\varphi(-\omega_j) = \varphi(\omega_i \cdot \tau\alpha \cdot \omega_j), \quad (i, j = 1, 2).$$

Since $\mathrm{Ker}\varphi = \{1, \omega\}$ and $\omega \cdot \omega_1 = \omega_2$, we see

$$\alpha = \omega_1 \cdot \tau\alpha \cdot \omega_1 \quad \text{or} \quad \alpha = \omega_1 \cdot \tau\alpha \cdot \omega_2.$$

Hence we have

$$G = G_1 \cup G_2, \quad \text{where } G_1 = \{\alpha \in Spin(16, C) \mid \omega_1 \cdot \tau\alpha \cdot \omega_1 = \alpha\},$$
$$G_2 = \{\beta \in Spin(16, C) \mid \omega_1 \cdot \tau\beta \cdot \omega_2 = \beta\}.$$

LEMMA 6.7. *For $\alpha \in G_1$ and $\beta, \beta' \in G_2$, we have*

$$\alpha \cdot \beta \in G_2, \quad \beta \cdot \beta' \in G_1.$$

PROOF.

$$\omega_1 \cdot \tau(\alpha \cdot \beta) \cdot \omega_2 = \omega_1 \cdot \tau\alpha \cdot \tau\beta \cdot \omega_2 = \omega_1 \cdot \tau\alpha \cdot \omega_1 \cdot \omega_1 \cdot \tau\beta \cdot \omega_2 = \alpha \cdot \beta,$$
$$\omega_1 \cdot \tau(\beta \cdot \beta') \cdot \omega_1 = \omega_1 \cdot \tau\beta \cdot \tau\beta' \cdot \omega_1 = \omega_1 \cdot \tau\beta \cdot \omega_2 \cdot \omega_2 \cdot \tau\beta' \cdot \omega_1 = \beta \cdot \beta'. \quad \square$$

From Lemma 6.4, we have $\boldsymbol{j} \in G_2$. Clearly, $G_1 \cap G_2 = \phi$. Then we have

$$G_2 = G_1 \cdot \boldsymbol{j} \quad \text{and} \quad G = G_1 \cup G_2 = G_1 \times \{1, \boldsymbol{j}\} = G_1 \times 2.$$

For

$$\alpha = \tilde{a}_1 \cdot \tilde{a}_2 \cdots \tilde{a}_{2m} \in Spin(16, C), \quad \tilde{a}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix},$$

we see

$$\omega_1 \cdot \tau\alpha \cdot \omega_1 = \omega_1 \cdot \tau\tilde{a}_1 \cdot \omega_1 \cdot \omega_1 \cdot \tau\tilde{a}_2 \cdot \omega_1 \cdots \omega_1 \cdot \tau\tilde{a}_{2m} \cdot \omega_1$$

and

$$\omega_1 \cdot \tau\tilde{a}_i \cdot \omega_1 = I_{8,8}\tau\tilde{a}_i = \begin{pmatrix} -\tau a_i \\ \tau b_i \end{pmatrix}.$$

Then, from §**1.1**, we see

$$G_1 = Spin(8,8) = \{\alpha \in Spin(16,C) \mid I_{8,8}\tau\alpha = \alpha\}.$$

From Lemmas 6.2 and 6.5, we see $\varphi(\boldsymbol{j}) = J\varepsilon_2$. Since $\omega \in Spin(8,8)$ and $\operatorname{Ker}\varphi|_{Spin(8,8)} = \{1,\omega\}$, we see

$$\varphi(G) \cong Spin(8,8)/\{1,\omega\} \times \{1,\varphi(\boldsymbol{j})\} = Ss(8,8) \times \{1, J\varepsilon_2\}.$$

Thus we have the following

THEOREM 6.8. *The non-compact exceptional Lie group* $(E_8^C)^{\tau\varepsilon_1}$ *of type* $E_{8(8)}$ *has the following subgroup of type* D_8.

$$((E_8^C)^{\tau\varepsilon_1})^{\varepsilon} \cong Ss(8,8) \times \{1, J\varepsilon_2\} = Ss(8,8) \times 2.$$

§6.5. Subgroup $Ss(4,12)$ of $E_{8(-24)}$

In this subsection, consider subgroup of $(E_8^C)^{\tau\varepsilon_1\gamma_1}$;

$$((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon} = \{\alpha \in (E_8^C)^{\tau\varepsilon_1\gamma_1} \mid \varepsilon\alpha\varepsilon = \alpha\} \subset (E_8^C)^{\tau\varepsilon_1\gamma_1} \cong E_{8(-24)}.$$

Since

$$\begin{aligned}((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon} &= ((E_8^C)^{\varepsilon})^{\tau\varepsilon_1\gamma_1} \\ &= \{\varphi(\alpha) \mid \alpha \in Spin(16,C), \gamma_1\varepsilon_1\tau\varphi(\alpha)\tau\varepsilon_1\gamma_1 = \varphi(\alpha)\} \\ &= \{\varphi(\alpha) \mid \alpha \in Spin(16,C), \gamma_1\varepsilon_1\varphi(\tau\alpha)\varepsilon_1\gamma_1 = \varphi(\alpha)\},\end{aligned}$$

we consider

$$G' = \{\alpha \in Spin(16,C) \mid \gamma_1\varepsilon_1\varphi(\tau\alpha)\varepsilon_1\gamma_1 = \varphi(\alpha)\}.$$

From Lemma 6.3, the same as §**6.4** we have the following

$$G' = G_1' \cup G_2', \text{ where } G_1' = \{\alpha \in Spin(16,C) \mid \omega_3 \cdot \tau\alpha \cdot \omega_3 = \alpha\},$$
$$G_2' = \{\beta \in Spin(16,C) \mid \omega_3 \cdot \tau\beta \cdot \omega_4 = \beta\}.$$

For

$$\alpha = \tilde{\alpha}_1 \cdot \tilde{\alpha}_2 \cdots \tilde{\alpha}_{2m} \in Spin(16,C), \quad \tilde{\alpha}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix},$$

we see

$$\omega_3 \cdot \tau\alpha \cdot \omega_3 = \omega_3 \cdot \tau\tilde{\alpha}_1 \cdot \omega_3 \cdot \omega_3 \cdot \tau\tilde{a}_2 \cdot \omega_3 \cdots \omega_3 \cdot \tau\tilde{a}_{2m} \cdot \omega_3$$

and

$$\omega_3 \cdot \tau\tilde{a}_i \cdot \omega_3 = I_{4,12}\tau\tilde{a}_i = \begin{pmatrix} -\gamma\tau a_i \\ \tau b_i \end{pmatrix}.$$

Then, from §**1.1**, we see

$$G_1' = Spin(4,12) = \{\alpha \in Spin(16, C) \mid I_{4,12}\tau\alpha = \alpha\}.$$

It is known that the symmetric space $(E_8^C)^{\tau\varepsilon_1\gamma_1}/((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon}$ is simply connected ([9]). Then we have the following exact sequence.

$$\begin{array}{ccccccc} \pi_1(E/E^{\varepsilon}) & \longrightarrow & \pi_0(E^{\varepsilon}) & \longrightarrow & \pi_0(E) & \longrightarrow & \pi_0(E/E^{\varepsilon}) \\ \| & & & & \| & & \| \\ 0 & & & & 0 & & 0 \end{array}$$

where $E = (E_8^C)^{\tau\varepsilon_1\gamma_1}$. Then $E^{\varepsilon} = ((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon}$ is connected. On the other side we see

$$((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon} = \varphi(G') = \varphi(G_1') \cup \varphi(G_2')$$

and $\varphi(G_1') \cap (G_2') = \phi$. Hence we have $G_2' = \phi$, $G' = G_1' = Spin(4,12)$ and

$$((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon} = \varphi(G_1').$$

Since $\omega \in Spin(4,12)$ and $\mathrm{Ker}\,\varphi|_{Spin(4,12)} = \{1, \omega\}$, we see

$$\varphi(G') = \varphi(G_1') \cong Spin(4,12)/\{1,\omega\} = Ss(4,12).$$

Thus we have the following

THEOREM 6.9. *The non-compact exceptional Lie group $(E_8^C)^{\tau\varepsilon_1\gamma_1}$ of type $E_{8(-24)}$ has the following subgroup of type D_8.*

$$((E_8^C)^{\tau\varepsilon_1\gamma_1})^{\varepsilon} \cong Ss(4,12).$$

§6.6. Subgroup $Ss^*(16) \times 2$ of $E_{8(8)}$ and $E_{8(-24)}$

In this subsection, consider subgroups of $(E_8^C)^{\tau J}$ and $(E_8^C)^{\tau\varepsilon J}$;

$$((E_8^C)^{\tau J})^{\varepsilon} = \{\alpha \in (E_8^C)^{\tau J} \mid \varepsilon\alpha\varepsilon = \alpha\} \subset (E_8^C)^{\tau J} \cong E_{8(-24)},$$

$$((E_8^C)^{\tau\varepsilon J})^{\varepsilon} = \{\alpha \in (E_8^C)^{\tau\varepsilon J} \mid \varepsilon\alpha\varepsilon = \alpha\} \subset (E_8^C)^{\tau\varepsilon J} \cong E_{8(8)}.$$

It is clear that

$$((E_8^C)^{\tau\varepsilon J})^{\varepsilon} = ((E_8^C)^{\tau J})^{\varepsilon}.$$

Since

$$((E_8^C)^{\tau J})^{\varepsilon} = ((E_8^C)^{\varepsilon})^{\tau J} = \{\varphi(\alpha) \mid \alpha \in Spin(16, C), J\tau\varphi(\alpha)\tau J = \varphi(\alpha)\}$$
$$= \{\varphi(\alpha) \mid \alpha \in Spin(16, C), J\varphi(\tau\alpha)J = \varphi(\alpha)\},$$

we consider

$$G'' = \{\alpha \in Spin(16, C) \mid J\varphi(\tau\alpha)J = \varphi(\alpha)\}.$$

From Lemmas 6.4 and 6.5, the same as §**6.4** we have the following

$$G'' = G_1'' \cup G_2'', \quad \text{where } G_1'' = \{\alpha \in Spin(16, C) \mid \boldsymbol{j} \cdot \omega_1 \cdot \tau\alpha \cdot \omega_1 \cdot \boldsymbol{j} = \alpha\},$$
$$G_2'' = \{\beta \in Spin(16, C) \mid \boldsymbol{j} \cdot \omega_1 \cdot \tau\beta \cdot \omega_2 \cdot \boldsymbol{j} = \beta\}.$$

The same as Lemma 6.7, we have the following

Lemma 6.10. *For $\alpha \in G_1''$ and β, $\beta' \in G_2''$, we have*

$$\alpha \cdot \beta \in G_2'', \quad \beta \cdot \beta' \in G_1''.$$

From Lemma 6.4, we see

$$\boldsymbol{j} \cdot \omega_1 \cdot \tau\omega_1 \cdot \omega_2 \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \omega_2 \cdot \boldsymbol{j} = \omega_1 \cdot \boldsymbol{j}^2 = \omega_1.$$

i.e., $\omega_1 \in G_2''$. Clearly, $G_1'' \cap G_2'' = \phi$. Then we have

$$G_2'' = G_1'' \cdot \omega_1 \text{ and } G'' = G_1'' \cup G_2'' = G_1'' \times \{1, \omega_1\} = G_1'' \times 2.$$

For

$$\alpha = \tilde{a}_1 \cdot \tilde{a}_2 \cdots \tilde{a}_{2m} \in Spin(16, C), \quad \tilde{a}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix},$$

we see

$$\boldsymbol{j} \cdot \omega_1 \cdot \tau\alpha \cdot \omega_1 \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \omega_1 \cdot \tau\tilde{a}_1 \cdot \omega_1 \cdot \boldsymbol{j} \cdot \boldsymbol{j} \cdot \omega_1 \cdot \tau\tilde{a}_2 \cdot \omega_1 \cdot \boldsymbol{j} \cdots \boldsymbol{j} \cdot \omega_1 \cdot \tau\tilde{a}_{2m} \cdot \omega_1 \cdot \boldsymbol{j}$$

and

$$\boldsymbol{j} \cdot \omega_1 \cdot \tau\tilde{a}_i \cdot \omega_1 \cdot \boldsymbol{j} = J_8\tau\tilde{a}_i = \begin{pmatrix} -\tau b_i \\ \tau a_i \end{pmatrix}.$$

Then, from §**1.1**, we see

$$G_1'' = Spin^*(16) = \{\alpha \in Spin(16, C) \mid J_8\tau\alpha = \alpha\}.$$

From Lemma 6.2, we have $\varphi(\omega_1) = \varepsilon_2$. Since $\omega \in Spin^*(16)$ and $\operatorname{Ker} \varphi|_{Spin^*(16)} = \{1, \omega\}$, we see

$$\varphi(G'') \cong Spin^*(16)/\{1,\omega\} \times \{1, \varphi(\omega_1)\} = Ss^*(16) \times \{1, \varepsilon_2\}.$$

Thus we have the following

THEOREM 6.11. (1) *The non-compact exceptional Lie group* $(E_8^C)^{\tau J}$ *of type* $E_{8(-24)}$ *has the following subgroup of type* D_8.

$$((E_8^C)^{\tau J})^{\varepsilon} \cong Ss^*(16) \times \{1, \varepsilon_2\} = Ss^*(16) \times 2.$$

(2) *The non-compact exceptional Lie group* $(E_8^C)^{\tau\varepsilon J}$ *of type* $E_{8(8)}$ *has the following subgroup of type* D_8.

$$((E_8^C)^{\tau\varepsilon J})^{\varepsilon} \cong Ss^*(16) \times \{1, \varepsilon_2\} = Ss^*(16) \times 2.$$

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