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THE INVERSE SURFACE AND THE OSSERMAN INEQUALITY*

By

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0. Introduction

In this paper, we shall work with surfaces of constant mean curvature one in hyper-bolic 3-space. We abbreviate constant mean curvature one by CMC-1. These surfaces share many properties with minimal surfaces in Euclidean 3-space. A striking result is that these surfaces have a hyperbolic analogue of Weierstrass representation formula [2]. Another important property is that the total curvature of CMC-1 surfaces is not necessarily an integral multiple of 4π, and does not generally satisfy Osserman inequality [4].

Let $f: M^2 \to \boldsymbol{H}^3(-1)$ be a CMC-1 immersion. Then there exist a null holomorphic immersion $F: \tilde{M}^2 \to SL(2, C)$, such that $f = F \cdot F^*$, where $\tilde{M}^2$ is the universal cover of M^2. By taking the inverse of the matrix F, we can construct a new CMC-1 surface $f_{-1}: \tilde{M}^2 \to \boldsymbol{H}^3(-1)$, call it the inverse surface (or dual surface [5]). Although the inverse surface is defined on the universal cover $\tilde{M}^2$, its metric ds_{-1}^2 is well defined on M^2. So we have two metrics on M^2, and they have the same completeness [6]. Umehara and Yamada have shown that if the surface $f: M^2 \to \boldsymbol{H}^3(-1)$ is complete and of finite total curvature, then the following inequality holds

$$\frac{1}{2\pi}\int_{M^2} k_{-1}\, dA_{-1} \leq \chi(M^2) - n, \tag{0.1}$$

where n is the number of ends of the original CMC-1 surface, the equality holds if and only if all the ends are regular and embedded [5].

By carefully observing, we may find that the condition of finite total curvature is not necessary. Indeed we have the following theorem

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THEOREM. *Let $f : M^2 \to \boldsymbol{H}^3(-1)$ be a complete CMC-1 immersion, then the Osserman inequality* (0.1) *holds.*

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1. The inverse surface

Let $f : M^2 \to \boldsymbol{H}^3(-1)$ be a complete CMC-1 immersion, $\tilde{M}^2$ the universal cover of M^2, which possess a holomorphic lift $F : \tilde{M}^2 \to SL(2, C)$, such that $f = F \cdot F^* : M^2 \to \boldsymbol{H}^3(-1)$ [2]. F satisfies the following equation

$$F^{-1}\,dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\omega, \tag{1.1}$$

where g and ω are meromophic function and holomorphic 1-form defined on $\tilde{M}^2$, respectively. The pair (g, ω) is called the Weierstrass data of the surface f, and $Q = \omega\, dg$ is the Hopf differential. By using the Weierstrass data, the first and second fundamental form ds^2 and Φ can be expressed as

$$ds^2 = (1 + |g|^2)^2 \omega\bar{\omega}, \tag{1.2}$$

$$\Phi = -\omega\, dg - \overline{\omega\, dg} + ds^2. \tag{1.3}$$

From (1.2) and (1.3), we easily know that the holomorphic quadratic differential Q is well defined on M^2. Moreover, the hyperbolic Gauss map can be written as

$$G : M^2 \to CP^1, \quad G(z) = [dF_1, dF_3], \tag{1.4}$$

here we have used the notation

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}, \quad \det F = 1.$$

The pseudometric $d\sigma^2 = -k\, ds^2$ can be expressed as

$$d\sigma^2 = \frac{4dg \cdot \overline{dg}}{(1 + |g|^2)^2}. \tag{1.5}$$

By (1.2), (1.5) and the definition of Q, we also have

$$d\sigma^2 \cdot ds^2 = 4Q \cdot \bar{Q}. \tag{1.6}$$

In what following, we give the definition of the inverse surface (see [5] and [6]).

DEFINITION 1.1. The inverse surface $f_{-1} : \tilde{M}^2 \to \boldsymbol{H}^3(-1)$ of the CMC-1 surface $f : M^2 \to \boldsymbol{H}^3(-1)$ is defined by

$$f_{-1} = (F^{-1}) \cdot (F^{-1})^*,$$

where F is the holomorphic lift of f, and F^{-1} is its inverse matrix.

Note that the inverse surface is defined on the universal cover $\tilde{M}^2$, generally, which can not be defined on M^2. About this problem Umehara and Yamada showed that it can be defined on M^2 if and only if the second Gauss map g is single-value on M^2 [5].

Now we demonstrate some important relations between the inverse surface and the original surface, their proofs can be found in related papers, so we omit them here.

PROPOSITION 1.2 [6]. *f_{-1} is complete if and only if f is complete.*

In [5] the completeness of the inverse surface is also shown under the hypothesis that all ends are regular. Another relation is

PROPOSITION 1.3 [5]. *The hyperbolic Gauss map, Weierstrass data and Hopf differential of the inverse surface can be represented as*

$$G_{-1} = g, \quad g_{-1} = G, \quad \omega_{-1} = -\frac{Q}{dG}, \quad Q_{-1} = -Q. \tag{1.7}$$

By (1.7) one can give the inverse metric

$$ds_{-1}^2 = (1 + |g_{-1}|^2)^2 \omega_{-1} \cdot \bar{\omega}_{-1} = (1 + |G|^2)^2 \frac{Q}{dG} \cdot \frac{\bar{Q}}{\overline{dG}}. \tag{1.8}$$

Because G, Q are both defined on M^2, ds_{-1}^2 is also well defined on it. Hence we may compute total curvature of the inverse metric on M^2. Set $d\sigma_{-1}^2 = -k_{-1}\, ds_{-1}^2$, which is the pseudometric of ds_{-1}^2 induced via $G : M^2 \to CP^1$

$$d\sigma_{-1}^2 = \frac{4dG \cdot \overline{dG}}{(1 + |G|^2)^2}. \tag{1.9}$$

Note that $Q_{-1} = \omega_{-1}\, dg_{-1} = -Q$. Combining (1.8) and (1.9) we get

$$d\sigma_{-1}^2 \cdot ds_{-1}^2 = 4Q_{-1} \cdot \bar{Q}_{-1} = 4Q \cdot \bar{Q}. \tag{1.10}$$

2. Monodromy conditions

Let $f: M^2 \to H^3(-1)$ be a complete CMC-1 immersion. We have known that the inverse metric ds_{-1}^2 is well defined on M^2. So one can compute total curvature $\int_{M^2} k_{-1}\,dA_{-1}$, where k_{-1} is the Gauss curvature of f_{-1}, and dA_{-1} is the volume element of f_{-1}. If $\int_{M^2} k_{-1}\,dA_{-1}$ is finite, then M^2 is conformal equivalent to a compact surface $\overline{M}^2$ with finite points $\{p_1,\dots,p_n\}$ removed, i.e. $M^2 = \overline{M}^2 \backslash \{p_1,\dots,p_n\}$. The point p_j $(j=1,\dots,n)$ corresponds to an end of f. At this time we immediately see that the hyperbolic Gauss map is meromorphically extended across all the ends $\{p_1,\dots,p_n\}$. Consequently, the total curvature is an integral multiple of 4π.

Notice that the Hopf differential $Q_{-1} = -Q$ is also defined on M^2. Like proposition 5 in [2], we have the following result

Proposition 2.1. *If the inverse metric ds_{-1}^2 is of finite total curvature on M^2, then the Hopf differential Q_{-1} can be meromorphically extended to $\overline{M}^2$.*

Proof. We first note a fact which is contained in the proof of Theorem 9.3 in [3].

Fact 1. Let $\Delta^* = \Delta\backslash\{0\}$ be a punctured unit disk on C and f, g holomorphic functions on Δ^* such that

$$ds^2 := (1+|g|^2)^2|f|^2|dz|^2$$

is positive definite on Δ^* and complete at the origin $z=0$. If g is meromorphic at $z=0$, so is f.

Since $d\sigma_{-1}^2 = G^*\,d\sigma_0^2$ ($d\sigma_0^2$ is the Fubini-Study metric on $CP^1 = C\cup\{\infty\}$) is of finite area, the hyperbolic Gauss map G must have at most pole, by the Great Picard Theorem. Since ds_{-1}^2 is complete by Proposition 1.2, the above fact yields that ω_{-1} has at most pole at the end. So the Hopf differential $Q_{-1} = \omega_{-1}dG$ has the same property. $\square$

In order to prove the main result, we should well understand the holomorphic representation around the end. Take a coordinate neighborhood of the end p_j, $\Delta_\varepsilon^* = \{z \in C \mid 0 < |z| < \varepsilon,\ z(p_j) = 0\}$ such that

$$g_{-1} = G = z^n, \quad n \geq 1,$$
$$\omega_{-1} = z^\nu \omega_0(z)\,dz, \quad \omega_0(0) \neq 0,$$

where n and v are integers, $\omega_0(z)$ is a holomorphic function on $\Delta_\varepsilon = \Delta_\varepsilon^* \cup \{0\}$. The holomorphic representation $F : \tilde{\Delta}_\varepsilon^* \to SL(2, C)$ satisfies

$$F \cdot dF^{-1} = \begin{pmatrix} g_{-1} & g_{-1}^2 \\ 1 & -g_{-1} \end{pmatrix} \omega_{-1}. \tag{2.1}$$

By a direct calculation, one easily get the following result, for details one can refer to [4].

F_3, F_4 satisfy the equation

$$X'' - \frac{\omega_{-1}'}{\omega_{-1}} X' - g_{-1}'\omega_{-1} X = 0; \tag{E.1}$$

F_1, F_2 satisfy the equation

$$Y'' - \frac{(g_{-1}^2\omega_{-1})'}{g_{-1}^2\omega_{-1}} Y' - g_{-1}'\omega_{-1} Y = 0, \tag{E.2}$$

where $' = d/dz$. Notice that

$$\frac{(\omega_{-1})'}{\omega_{-1}} = \frac{v}{z} + \frac{\omega_0'}{\omega_0}, \quad \frac{(g_{-1}^2\omega_1)'}{g_{-1}^2\omega_{-1}} = \frac{n+v}{z} + \frac{\omega_0'}{\omega_0}, \quad g_{-1}'\omega_{-1} = nz^{n+v-1}\omega_0.$$

Hence, the coefficients of (E.1) and (E.2) are all meromorphic functions on Δ_ε. Since we already assume that ds_{-1}^2 has finite total curvature on M^2, by proposition 2.1, then the Hopf differential Q_{-1} is meromorphic on Δ_ε. Now assume that the Order of Q_{-1} satisfies $Ord_0 Q_{-1} \geq -2$. Thus equations (E.1) and (E.2) have regular singularity at the point $z = 0$. If write

$$Q_{-1} = q\, dz^2 = \left(\sum_{j=-2}^{\infty} q_j z^j \right) dz^2,$$

by local theory of the ordinary differential equation [1], we obtain the indicial equations of (E.1) and (E.2) as follows

$$t^2 - (v+1)t - q_{-2} = 0, \tag{e.1}$$

$$t^2 - (2n+v+1)t - q_{-2} = 0. \tag{e.2}$$

Let λ_j and $\lambda_j - m_j$ are solutions of the indicial equations (e_j, $j = 1, 2$). Then the fundamental system of the solutions $\{X_1, X_2\}$ of (E.1) and $\{Y_1, Y_2\}$ of (E.2) can be written as

$$X_1 = z^{\lambda_1}\xi_1(z), \quad X_2 = z^{\lambda_1 - m_1}\xi_2(z) + k_1 X_1 \log z, \tag{2.2}$$

$$Y_1 = z^{\lambda_2}\eta_1(z), \quad Y_2 = z^{\lambda_2 - m_2}\eta_2(z) + k_2 Y_1 \log z, \tag{2.3}$$

where $\xi_i(0) = 1$, $\eta_i(0) = 1$, $(i = 1,2)$, k_1 and k_2 are constant.

LEMMA 2.2. *Let $f : \Delta_\varepsilon^* \to \boldsymbol{H}^3(-1)$ be a CMC-1 immersion, which is complete at $z = 0$, and the total curvature of ds_{-1}^2 on Δ_ε^* is finite. Then $k_1 = k_2 = 0$.*

PROOF. If m_1 is not an integral number, then the fundamental system of (E.1) must be in terms of

$$X_1 = z^{\lambda_1}\xi_1(z), \quad X_2 = z^{\lambda_1 - m_1}\xi_2(z).$$

So $k_1 = 0$ [1]. The same result will be hold for equation (E.2).

If m_1 is an integral number, without loss generality, assume $m_1 \geq 0$, and set

$$F_3 = b_{11}X_1 + b_{12}X_2, \quad F_4 = b_{21}X_1 + b_{22}X_2.$$

We calculate

$$\begin{aligned}
&|F_3|^2 + |F_4|^2 \\
&\quad = |b_{11}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{12}(z^{(\nu+1-m_1)/2}\xi_2(z) + k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z)|^2 \\
&\qquad + |b_{21}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{22}(z^{(\nu+1-m_1)/2}\xi_2(z) + k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z)|^2, \\
&\quad = |b_{11}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{12}(z^{(\nu+1-m_1)/2}\xi_2(z))|^2 + \underbrace{|b_{12}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z|^2}_{I} \\
&\qquad + \underbrace{[b_{11}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{12}(z^{(\nu+1-m_1)/2}\xi_2(z))]\overline{b_{12}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z}}_{II} \\
&\qquad + \underbrace{\overline{[b_{11}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{12}(z^{(\nu+1-m_1)/2}\xi_2(z))]}b_{12}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z}_{III} \\
&\qquad + |b_{21}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{22}(z^{(\nu+1-m_1)/2}\xi_2(z))|^2 + \underbrace{|b_{22}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z|^2}_{IV} \\
&\qquad + \underbrace{[b_{21}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{22}(z^{(\nu+1-m_1)/2}\xi_2(z))]\overline{b_{22}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z}}_{V} \\
&\qquad + \underbrace{\overline{b_{21}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{22}(z^{(\nu+1-m_1)/2}\xi_2(z))]}b_{22}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)\ln z}_{VI}.
\end{aligned}$$

Fix $z = re^{i(\theta+2k\pi)}$, here $k = \pm 1, \pm 2, \ldots$. For convenience, assume $\theta = 0$. The part, which is relative with the number k, of the sum is

$$I + II + III + IV + V + VI = (|b_{12}|^2 + |b_{22}|^2)|k_1|^2|z^{(\nu+1+m_1)/2}\xi_1(z)|^2|\ln z|^2$$
$$+ (a+b)\,\overline{\ln z} + \overline{(a+b)}\ln z,$$

where

$$a = (b_{11}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{12}z^{(\nu+1-m_1)/2})\overline{b_{12}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)}, \tag{2.4}$$

$$b = (b_{21}z^{(\nu+1+m_1)/2}\xi_1(z) + b_{22}z^{(\nu+1-m_1)/2})\overline{b_{22}k_1 z^{(\nu+1+m_1)/2}\xi_1(z)}. \tag{2.5}$$

Since $|F_3|^2 + |F_4|^2$ is single-valued on Δ_ε^*, then it is constant when k varies. we get

$$(|b_{12}|^2+|b_{22}|^2)|k_1|^2|z^{(\nu+1+m_1)/2}\xi_1(z)|^2(2k\pi)^2+(a+b)(-2k\pi i)+\overline{(a+b)}(2k\pi i) = 0.$$

Thus

$$(|b_{12}|^2 + |b_{22}|^2)|k_1|^2|z^{(\nu+1+m_1)/2}\xi_1(z)|^2 = 0, \quad \overline{(a+b)} + (a+b) = 0. \tag{2.6}$$

If $k_1 = 0$, then the first equality of (2.6) holds, and (2.4) (2.5) yield the second equality of (2.6). If $k_1 \neq 0$, fix $z = re^{2k\pi i}$, and r is much small. Since

$$|z^{(\nu+1+m_1)/2}\xi_1(z)|^2 \neq 0,$$

then

$$|b_{12}|^2 + |b_{22}|^2 = 0, \quad \text{i.e. } b_{12} = b_{22} = 0.$$

It means that F_3 and F_4 are linear dependent. Therefore $g = -dF_4/dF_3$ is constant, and hence f is flat. So we have that G is a constant. This contradicts with $G = z^n$, $n \geq 1$, so $k_1 = 0$, similarly $k_2 = 0$. We complete the proof of the lemma 2.2. □

LEMMA 2.3. *Let $f : \Delta_\varepsilon^* \to \boldsymbol{H}^3(-1)$ be a CMC-1 immersion, complete at $z = 0$, ds_{-1}^2 of finite total curvature on Δ_ε^*. Then m_1, m_2 must be integers or non-integral real numbers, simultaneously.*

PROOF. We firstly show that if m_1 is an integer, then m_2 is also an integer and *vice versa*. By $G = dF_1/dF_3$, setting

$$F_1 = a_{11}Y_1 + a_{12}Y_2, \quad F_2 = a_{21}Y_1 + a_{22}Y_2,$$

we obtain

$$z^n = \frac{(a_{11}z^{(2n+\nu+m_2+1)/2}\eta_1(z) + a_{12}z^{(2n+\nu-m_2+1)/2}\eta_2(z))'}{(b_{11}z^{(\nu+m_1+1)/2}\xi_1(z) + b_{12}z^{(\nu-m_1+1)/2}\xi_2(z))'}. \tag{2.7}$$

Since n and ν are all integral numbers, from (2.7) we easily see that m_1 and m_2 must be integral numbers simultaneously.

Secondly, we prove that when m_1 and m_2 are not integral numbers, they should be real numbers. Using the representation

$$F_3 = b_{11}z^{\lambda_1}\xi_1(z) + b_{12}z^{\lambda_1 - m_1}\xi_2(z), \quad F_4 = b_{21}z^{\lambda_1}\xi_1(z) + b_{22}z^{\lambda_1 - m_1}\xi_2(z),$$

we obtain

$$|F_3|^2 + |F_4|^2 = |z^{\lambda_1}|^2|b_{11}\xi_1(z) + b_{12}z^{-m_1}\xi_2(z)|^2 + |z^{\lambda_1}|^2|b_{21}\xi_1(z) + b_{22}z^{-m_1}\xi_2(z)|^2.$$

Put $\lambda_1 = \dfrac{\nu+1+\sqrt{(\nu+1)^2+4q_{-2}}}{2}$, $\lambda_1 - m_1 = \dfrac{\nu+1-\sqrt{(\nu+1)^2+4q_{-2}}}{2}$, $m_1 = \sqrt{(\nu+1)^2+4q_{-2}}$ into the equation above, then

$$\begin{aligned}
&|F_3|^2 + |F_4|^2 \\
&\quad = (|b_{11}|^2 + |b_{21}|^2)|\xi_1(z)|^2|z^{(\nu+m_1+1)/2}|^2 \\
&\qquad + (|b_{12}|^2 + |b_{22}|^2)|\xi_1(z)|^2|z^{(\nu-m_1+1)/2}|^2 \\
&\qquad + b_{11}\bar{b}_{12}\xi_1\bar{\xi}_2 z^{(\nu+m_1=1)/2}\overline{z^{(\nu-m_1+1)/2}} + \bar{b}_{11}b_{12}\bar{\xi}_1\xi_2\overline{z^{(\nu+m_1=1)/2}}z^{(\nu-m_1+1)/2} \\
&\qquad + b_{21}b_{22}^{-}\xi_1\bar{\xi}_2 z^{(\nu+m_1+1)/2}\overline{z^{(\nu-m_1+1)/2}} + b_{21}^{-}b_{22}\bar{\xi}_1\xi_2\overline{z^{(\nu+m_1+1)/2}}z^{(\nu-m_1+1)/2}.
\end{aligned} \tag{2.8}$$

Fix $z = re^{i(\theta+2k\pi)}$, r is much small, and $k = \pm 1, \pm 2, \ldots$. For convenience, assume $\theta = 0$. Furthermore

$$\begin{aligned}
|z^{(\nu+m_1+1)/2}|^2 &= e^{(2(\nu+1)+m_1+\bar{m}_1)/2\ln r + (m_1-\bar{m}_1)k\pi i}, \\
|z^{(\nu-m_1+1)/2}|^2 &= e^{(2(\nu+1)-m_1-\bar{m}_1)/2\ln r + (-m_1+\bar{m}_1)k\pi i}, \\
z^{(\nu+m_1+1)/2}\overline{z^{(\nu-m_1+1)/2}} &= e^{(2(\nu+1)+m_1-\bar{m}_1)/2\ln r + (m_1+\bar{m}_1)k\pi i}, \\
\overline{z^{(\nu-m_1+1)/2}}z^{(\nu-m_1+1)/2} &= e^{(2(\nu+1)+m_1-\bar{m}_1)/2\ln r - (m_1+\bar{m}_1)k\pi i}.
\end{aligned}$$

Now, we set $m_1 = a + bi$, and

$$h_1 = (|b_{21}|^2 + |b_{11}|^2)|\xi_1|^2, \quad h_2 = (|b_{12}|^2 + |b_{22}|^2)|\xi_2|^2, \quad l = (b_{11}\overline{b_{12}} + b_{21}\overline{b_{22}})\xi_1\bar{\xi}_2.$$

Then

$$\begin{aligned}
|F_3|^2 + |F_4|^2 &= h_1 r^{\nu+a+1}e^{-2kb\pi} + h_2 r^{\nu-a+1}e^{2kb\pi} \\
&\quad + l r^{\nu+1}e^{(b\ln r + 2ka\pi)i} + \bar{l} r^{\nu+1}e^{-(b\ln r+2ka\pi)i}.
\end{aligned} \tag{2.9}$$

If $b \neq 0$, of course h_1 and h_2 do not all vanish, the last two terms in (2.9) are bounded, when k tends to ∞, right side of (2.9) will be infinite. However, $|F_3|^2 + |F_4|^2$ has to be constant when k varies. This is a contradiction. So $b = 0$, it means that $m_1 = a + bi = a$ is a real number. Similarly m_2 is also a real number. Lemma 2.3 is proved. □

Since $b = 0$, the terms containing k in (2.9) is the following

$$\begin{aligned}
&r^{v+1}(l(\cos 2ka\pi + i \sin 2ka\pi) + \bar{l}(\cos 2ka\pi - i \sin 2ka\pi)) \\
&= r^{v+1}((l + \bar{l}) \cos 2ka\pi + (li - \bar{l}i) \sin 2ka\pi) \\
&= r^{v+1}(2l_1 \cos 2ka\pi - 2l_2 \sin 2ka\pi) \\
&= 2r^{v+1}\sqrt{l_1^2 + l_2^2} \sin(\theta + 2ka\pi),
\end{aligned}$$

where

$$l = l_1 + il_2, \quad \sin\theta = \frac{l_1}{\sqrt{l_1^2 + l_2^2}}, \quad \cos\theta = \frac{l_2}{\sqrt{l_1^2 + l_2^2}}.$$

If $l \neq 0$, as $|F_3|^2 + |F_4|^2$ is not relevant with k, so a has to be an integral number, this contradicts the hypothesis, thus $l = 0$.

Corollary 2.4. *If m_1, m_2 are not integral numbers, then coefficients of F_i $(i = 1, 2, 3, 4)$ satisfy*

$$b_{11}\bar{b}_{12} + b_{21}\bar{b}_{22} = 0, \quad a_{11}\bar{a}_{12} + a_{21}\bar{a}_{22} = 0.$$

Lemma 2.5. *If m_1, m_2 are not integral numbers, then following equations hold*

$$m_1 = m_2 = m, \quad n = -(v + 1),$$

$$a_{11}(m - v - 1) = b_{11}(m + v + 1), \quad a_{21}(m - v - 1) = b_{21}(m + v + 1),$$

$$a_{12}(m + v + 1) = b_{12}(m - v - 1), \quad a_{22}(m + v + 1) = b_{22}(m - v - 1).$$

Proof. By using $G = dF_1/dF_3$ we have

$$z^n = \frac{[a_{11}z^{(2n+v+1+m_2)/2}\eta_1(z) + a_{12}z^{(2n+v+1-m_2)/2}\eta_2(z)]'}{[b_{11}z^{(v+1+m_1)/2}\xi_1(z) + b_{12}z^{(v+1-m_1)/2}\xi_2(z)]'}.$$

A direct computation shows that

$$a_{11}\left(\frac{2n+\nu+1+m_2}{2}\eta_1+z\eta_1'\right)z^{m_2/2}+a_{12}\left(\frac{2n+\nu+1-m_2}{2}\eta_2+z\eta_2'\right)z^{-m_2/2}$$
$$=b_{11}\left(\frac{\nu+1+m_1}{2}\xi_1+z\xi_1'\right)z^{m_1/2}+b_{12}\left(\frac{\nu+1-m_1}{2}\xi_2+z\xi_2'\right)z^{-m_1/2}, \qquad (2.10)$$

where

$$m_1=\sqrt{(\nu+1)^2+4q_{-2}}>0, \quad m_2=\sqrt{(2n+\nu+1)^2+4q_{-2}}>0.$$

Since m_1 and m_2 are not integral numbers, then

$$\frac{\nu+1+m_1}{2},\ \frac{\nu+1-m_1}{2},\ \frac{2n+\nu+1+m_2}{2},\ \frac{2n+\nu+1-m_2}{2}$$

do not vanish.

1). If $a_{12}=0$, then $b_{12}=0$. Otherwise $b_{12}\neq 0$, when z tends to 0, the left hand side of the equation (2.10) converges to 0, and the right hand side is divergent. This is a contradiction. In this case $m_1=m_2$ must hold and hence $n=-(\nu+1)$. Moreover applying $\eta_2(0)=1$ and $\xi_1(0)=1$ we get

$$a_{11}\frac{2n+\nu+1+m_2}{2}=b_{11}\frac{\nu+1+m_1}{2}.$$

2). If $a_{12}\neq 0$, then $b_{12}\neq 0$. Assume $m_2>m_1$. We multiply the equation (2.10) by $z^{m_1/2}$. When z tends to 0, the right side of the equation (2.10) tends to a constant, and the left side divergent, we get a repugnance, similarly $m_1>m_2$ does not hold.

Thus $m_1=m_2$ and hence $n=-(\nu+1)$. Now put $m_1=m_2=m$ into the equation (2.10)

$$a_{11}\left(\frac{2n+\nu+1+m}{2}\eta_1+z\eta_1'\right)z^{m}+a_{12}\left(\frac{2n+\nu+1-m}{2}\eta_2+z\eta_2'\right)$$
$$=b_{11}\left(\frac{\nu+1+m}{2}\xi_1+z\xi_1'\right)z^{m}+b_{12}\left(\frac{\nu+1-m}{2}\xi_2+z\xi_2'\right).$$

Take $z\to 0$, we get

$$a_{12}\left(\frac{2n+\nu+1-m}{2}\right)=b_{12}\left(\frac{\nu+1-m}{2}\right).$$

On the other hand, the coefficients of z^m on two side should be equal to each other. If not, $z^m=h_0/h_1$, h_0 and h_1 are holomorphic functions. z^m is a multiplevalued holomorphic function. This is a contradiction. So

$$a_{11}\frac{2n+v+1+m}{2}=b_{11}\frac{v+1+m}{2}.$$

From $z^n = dF_2/dF_4$, the other equations can be verified. Lemma 2.5 is proved. □

Next we prove the main result in this section.

THEOREM 2.6. *Let $f: M^2 \to \boldsymbol{H}^3(-1)$ be a complete CMC-1 immersion. In the following three conditions any two conditions imply the another,*

i) $\int_{M^2} k_{-1} dA_{-1}$ *is finite,*
ii) $\int_{M^2} k\, dA$ *is finite,*
iii) $Ord_{p_j} Q \geq -2$, $(j=1,2,\dots,n)$.

PROOF. In [2] Bryant has shown that i) is equivalent to iii) under the condition ii). So we only need to prove that i) and iii) imply ii). It is sufficient to prove that $\int_{\Delta_\varepsilon^*} k\, dA$ is finite, Δ_ε^* is a coordinate neighborhood near the end p_j $(j=1,2,\dots,n)$. By Lemma 2.3, m_1 m_2 are integral numbers or both are not simultaneously.

1). m_1 and m_2 are integral numbers. The second Gauss map g is

$$g=-\frac{b_{21}\lambda_1 z^{-1}\xi_1+b_{21}\xi_1'+b_{22}(\lambda_1-m_1)z^{-m_1-1}\xi_2+b_{22}z^{-m_1}\xi_2'}{b_{11}\lambda_1 z^{-1}\xi_1+b_{12}\xi_1'+b_{12}(\lambda_1-m_1)z^{-m_1-1}\xi_2+b_{12}z^{-m_1}\xi_2'}. \tag{2.11}$$

From (2.11) we know that g is a meromorphic function on Δ_ε. Moreover $-\int_{\Delta_\varepsilon^*} k\, dA$ is the area of the image of $g: \Delta_\varepsilon^* \to CP^1$, so $\int_{\Delta_\varepsilon^*} kd$ is finite.

2). m_1 and m_2 are not integral numbers. By lemma 2.5 $m_1 = m_2 = m > 0$ and $n = -(v+1)$, using corollary 2.4 we can prove that $\int_{\Delta_\varepsilon^*} kd$ is finite in three cases as follows.

CASE 1. If $b_{11} \neq 0$, $b_{12} = 0$. Then $b_{21} = 0$, $b_{22} \neq 0$. The second Gauss map is

$$g=-\frac{1}{z^m}\frac{b_{22}(\xi_2+z\xi_2')}{b_{11}(\xi_1+z\xi_1')}.$$

Take Δ_ε^* very small such that

$$\frac{b_{22}(\xi_2+z\xi_2')}{b_{11}(\xi_1+z\xi_1')}\neq 0$$

for all $z \in \Delta_\varepsilon^*$. Consider a conformal transformation $w : \Delta_\varepsilon \to \Delta_\varepsilon'$

$$w(z) = z\left(\frac{b_{22}(\xi_2 + z\xi_2')}{b_{11}(\xi_1 + z\xi_1')}\right)^{-1/m}.$$

It is obviously that

$$w' = \left(\frac{b_{22}(\xi_2 + z\xi_2')}{b_{11}(\xi_1 + z\xi_1')}\right)^{-1/m} + z\left[\left(\frac{b_{22}(\xi_2 + z\xi_2')}{b_{11}(\xi_1 + z\xi_1')}\right)^{-1/m}\right]'.$$

So $w'(0) \neq 0$. Thus on the new coordinate neighborhood Δ_ε' the second Gauss map is

$$g = -\frac{1}{w^m}.$$

It is clear that

$$\int_{\Delta_\varepsilon'} k\,dA = \int_{\Delta_\varepsilon'} \frac{4m^2|w|^{2(m-1)}\,dw \cdot d\bar{w}}{(1 + |w|^{2m})^2}$$

is finite.

CASE 2. If $b_{21} \neq 0$, $b_{11} = 0$. Then $b_{21} \neq 0$, $b_{22} = 0$. That is similar with case 1.

CASE 3. If $b_{11} \neq 0$, $b_{12} \neq 0$. Then $b_{21} \neq 0$, $b_{22} \neq 0$. We compute the second Gauss map

$$g = -\frac{\frac{\nu+1}{2}(b_{21}\xi_1 z^m + b_{22}\xi_2) + \frac{m}{2}b_{21}z^m + b_{21}z^{m+1}\xi_1' + b_{22}z\xi_2' - \frac{m}{2}b_{22}\xi_2}{\frac{\nu+1}{2}(b_{11}\xi_1 z^m + b_{12}\xi_2) + \frac{m}{2}b_{11}z^m + b_{11}z^{m+1}\xi_1' + b_{12}z\xi_2' - \frac{m}{2}b_{12}\xi_2}$$

and

$$g(0) = -\frac{\frac{\nu+1}{2}b_{22} - \frac{m}{2}b_{22}}{\frac{\nu+1}{2}b_{12} - \frac{m}{2}b_{12}} = -\frac{b_{22}}{b_{12}} \neq 0.$$

By $ds^2 = (1 + |g|^2)^2\omega\bar{\omega}$ we get $\mathrm{Ord}_0\, ds^2 = \mathrm{Ord}_0\omega$. On the other hand

$$\begin{aligned}\omega &= F_1\,dF_3 - F_3\,dF_1 \\ &= z^{-1-m}(\nu + 1)\{a_{12}b_{12}\xi_2\eta_2 + a_{11}b_{11}\xi_1\eta_1 z^{2m} + z^m(a_{12}b_{11}\xi_1\eta_2 + a_{11}b_{12}\xi_2\eta_1) \\ &\quad + \frac{z^{m+1}}{\nu + 1}(\cdots)\}\,dz.\end{aligned}$$

When $z \to 0$, the value of $(\cdots)$ is finite, so we get $\mathrm{Ord}_0\omega = -m - 1$, hence

$$\mathrm{Ord}_0\, ds^2 = -m - 1. \tag{2.12}$$

By hypothesis $\mathrm{Ord}_0 Q = \mathrm{Ord}_0 Q_{-1} \geq -2$, when $q_{-2} = 0$, m_1, m_2 are integral numbers, thus $\mathrm{Ord}_0 Q = -2$. Note that

$$\mathrm{Ord}_0\, ds^2 + \mathrm{Ord}_0\, d\sigma^2 = \mathrm{Ord}_0 Q = -2. \tag{2.13}$$

In conjunction with (2.12) and (2.13) we get $\mathrm{Ord}_0 d\sigma^2 = m - 1$, $m > 0$. Thus $-\int_{\Delta_\varepsilon^*} k\, dA = -\int_{\Delta_\varepsilon^*} d\sigma^2$ is finite.

Up to now we have proved that total curvature around all the ends is finite, so ii) holds. Theorem 2.6 is proved. □

3. Osserman inequality

In this section we prove the main result

THEOREM 3.1. *Let $f : M^2 \to H^3(-1)$ be a complete CMC-1 immersion, then the Osserman inequality*

$$\frac{1}{2\pi} \int_{M^2} k_{-1}\, dA_{-1} \leq \chi(M^2) - n, \tag{3.1}$$

holds, where n is the boundary number of the surface f.

In order to prove Theorem 3.1, we need to establish a lemma as follows.

LEMMA 3.2. *Let ds_{-1}^2 be of finite total curvature on M^2. Then the inequality*

$$Ord_{p_j}\, d\sigma_{-1}^2 > Ord_{p_j} Q_{-1} + 1 \tag{3.2}$$

holds, where p_j corresponding to an end of f.

PROOF. We apply the following fact to prove the lemma.

FACT 2 [5, Lemma 3]. Let ds_{-1}^2 is of finite total curvature on M^2. Then the following inequality holds

$$Ord_{p_j}\, d\sigma_{-1}^2 > Ord_{p_j} Q + 1.$$

Suppose that $\mathrm{Ord}_{pj}\, d\sigma_{-1}^2 \leq \mathrm{Ord}_{p_j} Q_{-1} + 1$. Since $\mathrm{Ord}_{p_j}\, d\sigma_{-1}^2 > -1$, we have

$$\mathrm{Ord}_{p_j} Q_{-1} > -2.$$

Since we assume that $d\sigma_{-1}^2$ is of finite total curvature at $z = p_j$, so is ds^2 by Theorem 2.6. Thus we get a contradiction by the above fact 2. Lemma 3.2 is proved. □

We have the following corollary

COROLLARY 3.3. *$Ord_{p_j}\, ds^2_{-1} \le -2$.*

PROOF OF THEOREM 3.1. If ds^2_{-1} has infinite total curvature, the result is obviously. If ds^2_{-1} is of finite total curvature, by Corollary 3.3 and using the method in [4], the Theorem 3.1 can be proved. □

Now let ds^2_{-1} be of finite total curvature, and the equality in (3.1) holds. This means $\mathrm{Ord}_{p_j} = -2$ at every end p_j, $(j = 1, 2, \ldots, n)$. Because $\mathrm{Ord}_{p_j}\, d\sigma^2_{-1} = n - 1 \ge 0$, and $\mathrm{Ord}_{p_j} Q_{-1} = \mathrm{Ord}_{p_j}\, d\sigma^2_{-1} + \mathrm{Ord}_{p_j}\, ds^2_{-1}$, then the inequality

$$\mathrm{Ord}_{p_j} Q_{-1} \ge -2, \quad j = 1, 2, \ldots, n$$

holds. By this fact and applying Theorem 2.6 we have that the total curvature of ds^2 is finite. Then we obtain

COROLLARY 3.5 [5]. *If the inverse metric ds^2_{-1} is of finite total curvature, then the equality in* (3.1) *holds if and only if all the ends of f are regular and embedded.*

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