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NON-EXISTENCE OF GENERIC ELEMENTARY EMBEDDINGS INTO THE GROUND MODEL

By

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Abstract. Kenneth Kunen showed that there is no non-trivial elementary embedding of the universe V into itself. We generalize this theorem by proving that there is no non-trivial generic elementary embedding of V into itself.

1. Introduction

The existence of certain elementary embeddings characterize various large cardinal axioms. And various authors have investigated elementary embeddings. One example is the following theorem of Kunen: *there is no non-trivial elementary embedding of the universe V into itself* [3]. One way to formulate the elementary embeddings of the theorem above is to restrict them to definable ones. So we treat definable elementary embeddings throughout this paper. We discuss whether or not, in a generic extension, there is a non-trivial elementary embedding of the ground model V into itself. Following the proof of Woodin of Kunen's theorem [2, p. 320], we will show that there is no such elementary embedding in a generic extension by a partially ordered set.

We reserve Greek letters for ordinal numbers, and κ and λ will be used for cardinal numbers. By NS_{λ} we denote the non-stationary ideal on λ. For a partially ordered set $\boldsymbol{P}$, we denote the class of $\boldsymbol{P}$-names in V by $V^{\boldsymbol{P}}$.

If $j: M \rightarrow N$ is an elementary embedding where M and N are transitive models for ZFC, then the *critical point* of j (crit(j)) is the least ordinal α such that $\alpha < j(\alpha)$, if such an α exists. An elementary embedding $j: M \rightarrow N$ is *non-trivial* if j is not the identity map on M. We would like to express our gratitude to Yo Matsubara for many useful discussions.

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2. Generic elementary embeddings of V

Let $j : V \to M$ be a $\boldsymbol{P}$-generic elementary embedding over V where $\boldsymbol{P}$ is a partially ordered set. Let G be a $\boldsymbol{P}$-generic filter over V.

DEFINITION. Assume that κ is a regular cardinal and λ is an ordinal such that $\kappa < \mathrm{cf}(\lambda)$. A subset C of λ is *κ-closed* on λ if for any strictly increasing sequence $\langle \alpha_\xi \mid \xi < \kappa \rangle$ of elements of C whose length is κ, $\sup\{\alpha_\xi : \xi < \kappa\} \in C$.

In particular we call a κ-closed unbounded subset of λ a *κ-club* subset of λ. Of course, any club subset of λ is κ-club on λ. Then we have the following well-known lemmas, whose proofs we omit:

LEMMA 1. *Assume that κ is a regular cardinal and λ is an ordinal such that $\kappa < \mathrm{cf}(\lambda)$. If S is a stationary subset of λ which is a subset of $\{\xi < \lambda : \mathrm{cf}(\xi) = \kappa\}$, and if C is κ-club on λ, then $S \cap C \neq \emptyset$.*

LEMMA 2. *If $j : V \to M$ is a $\boldsymbol{P}$-generic elementary embedding where $\boldsymbol{P}$ is a partially ordered set, then there exist unboundedly many ordinals λ such that*

$$1 \Vdash j(\check{\lambda}) = \check{\lambda} \wedge \mathrm{crit}(j) < \check{\lambda}.$$

LEMMA 3. *Assume that in V λ is an uncountable regular cardinal, and that $\boldsymbol{P}$ has the λ-c.c. Then, for any club subset C of λ in $V[G]$, there exists a club subset C' of λ in V such that $C' \subseteq C$.*

We can now prove our main theorem.

THEOREM 1. *Let $\boldsymbol{P}$ be a partially ordered set and $j : V \to M$ be a $\boldsymbol{P}$-generic elementary embedding, then $V \not\subseteq M$.*

PROOF. Fix an arbitrary $\boldsymbol{P}$-generic filter G over V. We argue in $V[G]$. Set $\kappa = \mathrm{crit}(j)$. By Lemma 2, we may pick a λ satisfying ($\boldsymbol{P}$ satisfies λ^+-c.c)^V, $k < \lambda$, and $j(\lambda) = \lambda$. Assume that $\mathscr{P}^V(\lambda^{++}) \subseteq M$; we will derive a contradiction.

Then we have $\lambda^{+V} \leqslant j(\lambda^{+V}) = \lambda^{+M}$ and we also have $\lambda^{+M} \leqslant \lambda^{+V[G]}$ because $M \subseteq V[G]$. But recalling that ($\boldsymbol{P}$ satisfies λ^+-c.c.)^V, we get $\lambda^{+V[G]} = \lambda^{+V}$. Thus we have $\lambda^{+V} = \lambda^{+M} = \lambda^{+V[G]}$. In the same way, we have $\lambda^{++V} = \lambda^{++M} = \lambda^{++V[G]}$.

For each $\xi < \lambda^{++}$, since $M \subseteq V[G]$, we have $\mathrm{cf}^{V[G]}(\xi) \leqslant \mathrm{cf}^M(\xi)$. Moreover we claim the following:

CLAIM. $\mathrm{cf}^M(\xi) \leqslant \mathrm{cf}^V(\xi)$.

Proof of Claim. We set $\alpha = \mathrm{cf}^V(\xi)$ and fix a cofinal function $f : \alpha \to \xi$ in V. Then because $f \in \mathscr{P}^V(\lambda^{++} \times \lambda^{++}) \subseteq M$ and $M \models$ "$f : \alpha \to \xi$ is a cofinal function," we have $\mathrm{cf}^M(\xi) \leqslant \alpha = \mathrm{cf}^V(\xi)$. □ (Claim)

On the other hand, since $\boldsymbol{P}$ preserves cofinalities $\geqslant \lambda^+$, we have

$$\mathrm{cf}^V(\xi) = \lambda^+ \Leftrightarrow \mathrm{cf}^M(\xi) = \lambda^+ \Leftrightarrow \mathrm{cf}^{V[G]}(\xi) = \lambda^+ \quad \text{for any } \xi < \lambda^{++}.$$

Therefore the stationary subset of λ^{++} given by $\{\xi < \lambda^{++} : \mathrm{cf}(\xi) = \lambda^+\}$ is absolute among V, M, and $V[G]$.

In V, we choose an injection $F : \kappa \to \mathscr{P}(\lambda^{++})$ such that $\mathrm{Ran}(F)$ is a partition of $\{\xi < \lambda^{++} : \mathrm{cf}(\xi) = \lambda^+\}$, and $F(\alpha)$ is a stationary subset of λ^{++} for each $\alpha < \kappa$. This is possible by a result of Solovay: a stationary subset of a regular cardinal ν can split into a set of ν many stationary subsets of ν [1, p. 433, Theorem 85].

Now, by the elementality of j, $j(F) : j(\kappa) \to \mathscr{P}^M(\lambda^{++})$ and $\mathrm{Ran}(j(F))$ is a partition of $\{\xi < \lambda^{++} : \mathrm{cf}(\xi) = \lambda^+\}$. This fact and $\kappa \in j(\kappa) = \mathrm{Dom}(j(F))$ imply that $j(F)(\kappa)$ is a stationary subset of λ^{++} in M. Furthermore in view of $\mathscr{P}^V(\lambda^{++}) \subseteq M$ and Lemma 3, we get

$$V[G] \models j(F)(\kappa) \text{ is stationary on } \lambda^{++}$$

and

$$V[G] \models \mathrm{Ran}(j(F)) \text{ is a partition of } \{\xi < \lambda^{++} : \mathrm{cf}(\xi) = \lambda^+\}.$$

Therefore by the λ^{++}-completeness of $NS_{\lambda^{++}}$, we can choose an $\alpha_0 < \kappa$ such that

$$V[G] \models F(\alpha_0) \cap j(F)(\kappa) \text{ is stationary on } \lambda^{++}.$$

Now set $C = \{\xi < \lambda^{++} : j(\xi) = \xi \wedge \mathrm{cf}(\xi) = \lambda^+\}$.

Claim. *C is λ^+-club on λ^{++}.*

Proof of Claim. First we show the unboundedness of C. Fix an arbitrary $\alpha < \lambda^{++}$; we find an $\bar{\alpha} \in C$ such that $\alpha < \bar{\alpha} < \lambda^{++}$. We define an increasing sequence $\langle \alpha_\eta \mid \eta < \lambda^+ \rangle$ as follows:

$$\alpha_0 = \alpha, \quad \alpha_{\eta+1} = j(\alpha_\eta) + 1, \quad \alpha_\zeta = \bigcup_{\eta<\zeta} \alpha_\eta \quad \text{if } \zeta \text{ is a limit ordinal}$$

and put the desired $\bar{\alpha}$ as

$$\bar{\alpha} = \sup\{\alpha_\eta : \eta < \lambda^+\}.$$

It is sufficient to show that $j(\bar{\alpha}) = \bar{\alpha}$. We know that ($\boldsymbol{P}$ has λ^+-c.c.)^V and $\mathrm{cf}^{V[G]}(\bar{\alpha}) = \lambda^+$, hence we can see that $\mathrm{cf}^V(\bar{\alpha}) = \lambda^+$. Therefore in V there exists an increasing sequence $\langle \beta_\eta \mid \eta < \lambda^+ \rangle$ such that $\bar{\alpha} = \sup\{\beta_\eta : \eta < \lambda^+\}$. Now for any $\eta < \lambda^+$, there exists an $\bar{\eta} < \lambda^+$ such that $\beta_\eta \leqslant \alpha_{\bar{\eta}}$, and hence $j(\beta_\eta) \leqslant j(\alpha_{\bar{\eta}}) (< \alpha_{\bar{\eta}+1})$. Consequently we see that

$$\begin{aligned} j(\bar{\alpha}) &= \sup\{j(\beta)_\eta : \eta < j(\lambda^+)\} \\ &= \sup\{j(\beta_\eta) : \eta < \lambda^+\} \\ &\leqslant \sup\{\alpha_{\eta+1} : \eta < \lambda\} = \bar{\alpha}. \end{aligned}$$

Next we show the λ^+-closedness of C. In $V[G]$ choose an arbitrary strictly increasing sequence $\langle \alpha_\eta \mid \eta < \lambda^+ \rangle$ of elements of C; we show that $\alpha = \sup\{\alpha_\eta : \eta < \lambda^+\} \in C$. It is sufficient to show that $j(\alpha) = \alpha$. We can pick some increasing sequence in V $\langle \beta_\eta \mid \eta < \lambda^+ \rangle$ such that $\alpha = \sup\{\beta_\eta : \eta < \lambda^+\}$. Now for any $\eta < \lambda^+$, there exists an $\bar{\eta} < \lambda^+$ such that $\beta_\eta \leqslant \alpha_{\bar{\eta}}$ and hence $j(\beta_\eta) \leqslant j(\alpha_{\bar{\eta}}) = \alpha_{\bar{\eta}}$, since we picked $\alpha_{\bar{\eta}}$ as an element of C. Consequently $j(\alpha) = \alpha$ holds as before. $\square$ (Claim)

By Lemma 1, a stationary subset of $\{\xi < \lambda^{++} : \mathrm{cf}(\xi) = \lambda^+\}$ must meet any λ^+-club on λ^{++}. Thus there exists a $\xi_0 \in (F(\alpha_0) \cap j(F)(\kappa)) \cap C$. Then $j(\xi_0) \in j(F)(\alpha_0)$, because $\xi_0 \in F(\alpha_0)$ and $\alpha_0 < \mathrm{crit}(j)$. But recalling that $\xi_0 \in C$, we have $\xi_0 = j(\xi_0) \in j(F)(\alpha_0)$. So we get $\xi_0 \in j(F)(\alpha_0) \cap j(F)(\kappa)$. But this contradicts that $\mathrm{Ran}(j(F))$ is a partition. So we have $\mathscr{P}^V(\lambda^{++}) \not\subseteq M$. $\square$ (Theorem 1)

REMARK 1. We used the regularity of λ^+, but it is not necessary that λ^+ be a successor cardinal. When there is a limit regular cardinal λ satisfying ($\boldsymbol{P}$ has the λ-c.c.)^V and $j(\lambda) = \lambda$, it is known that $\mathscr{P}^V(\lambda^+) \not\subseteq M$.

REMARK 2. The theorem of Kunen holds for all elementary embeddings not just definable ones. Likewise our proof can be adapted for non-definable generic elementary embeddings.

REMARK 3. The theorem above does not hold for the case $\boldsymbol{P}$ is a proper class. Indeed, Woodin shows that if there are unboundedly many completely Jonsson cardinals, then we have a partially ordered class $\boldsymbol{P}$ and a $\boldsymbol{P}$-generic elementary embedding $j : V \to V[G]$ where $V[G]$ is a generic extension of V by $\boldsymbol{P}$ [4].

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