

# TOTALLY REAL SECTIONAL CURVATURE OF A REAL HYPERSURFACE IN A QUATERNIONIC SPACE FORM

By

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**Abstract.** We characterize the real hypersurfaces of a non-flat quaternionic space form with constant quaternionic sectional curvature as the ones which have constant totally real sectional curvature.

## 1. Introduction

The sectional curvature is a basic tool when studying the geometry of a Riemannian manifold. Probably, the most studied manifolds are the real, complex and quaternionic space forms. These manifolds are characterized by the constancy of the sectional curvature, the holomorphic sectional curvature and the quaternionic sectional curvature respectively. Besides, in quaternionic kaehlerian manifolds, it is well-known that the constancy of quaternionic sectional curvature is equivalent to the constancy of totally real sectional curvature. In this way, we have considered the problem of studying whether this property is inherited to real hypersurfaces of a non-flat quaternionic space form  $QM^m(q)$ ,  $m \geq 3$ ,  $q \neq 0$ . Real hypersurfaces with constant quaternionic sectional curvature of  $QM^m(q)$ ,  $q \neq 0$ , have been classified by A. Martinez in [3] when  $q > 0$ , i.e., in the quaternionic projective space  $QP^m(q)$ , and by the authors in [6] when  $c < 0$ , i.e., in the quaternionic hyperbolic space  $QH^m(q)$ .

As  $QM^m(q)$ ,  $q \neq 0$ ,  $m \geq 3$ , is a quaternionic Kaehlerian manifold, there exists a 3-dimensional vector bundle  $\hat{V}$  of tensors of type (1, 1) and a local basis of almost Hermitian structures  $\{J_1, J_2, J_3\}$  on  $QM^m(q)$ . Let  $M$  be a connected real hypersurface of  $QM^m(q)$ ,  $q \neq 0$ ,  $m \geq 3$ , and  $N$  a local normal unit vector field on  $M$ . We shall denote  $U_k = -J_k N$   $k = 1, 2, 3$ . If  $X$  is a tangent vector field to  $M$ , we shall write  $J_k X = \phi_k X + f_k(X)N$ ,  $k = 1, 2, 3$ , where  $\phi_k X$  is the tangential component of  $J_k X$ . The maximal quaternionic distribution will be denoted by  $D$ .

Given a vector  $X$  tangent to  $M$ , we shall denote  $Q(X) = \text{Span}\{X, \phi_1 X, \phi_2 X, \phi_3 X\}$ . If  $\pi$  is a 2-plane tangent to  $M$ , we shall say that  $\pi$  is totally real if for any orthonormal basis  $\{X, Y\}$  of  $\pi$ ,  $Q(X) \perp Q(Y)$  and  $\pi$  is included in  $D$ . Let denote by  $T(\pi) = T(X, Y)$  the sectional curvature of any totally real 2-plane  $\pi = \text{Span}\{X, Y\}$  tangent to  $M$ . We shall call it the totally real sectional curvature of  $M$ .

We recall that  $M$  is ruled if  $D$  is integrable. We shall denote by  $D^\perp$  the orthogonal complement of  $D$  in  $TM$ . We need to write  $q = 4\varepsilon/k^2$ , where  $\varepsilon = \pm 1$  is the sign of  $q$  and  $k \neq 0$  is a real constant. Our results are

**THEOREM.** *Let  $M$  be a real hypersurface of  $QM^m(q)$ ,  $q \neq 0$ ,  $m \geq 3$ , on which  $T$  is constant. Then  $M$  is of one of the following:*

- a) Ruled,  $q = 4T$ .
- b) If  $\varepsilon = 1$ , an open subset of a tube of radius  $r > 0$  over a totally geodesic  $QP^{m-1}(q)$ ,  $T = (q/4) + (1/k^2) \cot^2 r$ .
- d) If  $\varepsilon = -1$  then  $M$  is an open subset of either
  - d.1) A tube of radius  $r > 0$  over a totally geodesic  $QH^{m-1}(q)$ ,  $(q/4) < T = (q/4) + (1/k^2) \tan^2 r < q/4 + 1/k^2$ .
  - d.2) A horosphere,  $T = (q/4) + (1/k^2)$ .
  - d.3) A geodesic hypersphere, i.e., a tube of radius  $r > 0$  over a point,  $(q/4) + (1/k^2) < T = (q/4) + (1/k^2) \cot^2 r$ .

Given a 2-plane  $\pi$  tangent to  $M$ , we say that  $\pi$  is quaternionic if for any orthonormal basis  $\{X, Y\}$  of  $\pi$ ,  $Q(X) = Q(Y)$  and  $\pi$  is included in  $D$ . We recall that  $M$  has constant quaternionic sectional curvature if the sectional curvature of any tangent quaternionic 2-plane is constant. The next corollary relates this definition to the Theorem.

**COROLLARY.** *Let  $M$  be a real hypersurface of  $QM^m(q)$ ,  $q \neq 0$ ,  $m \geq 3$ . Then  $M$  has constant quaternionic sectional curvature if and only if  $M$  has constant totally real sectional curvature.*

**REMARK 1.** Let  $\pi$  be a totally real 2-plane tangent to  $M$ . The condition ' $\pi$  is included in  $D$ ' has the goal of preventing  $\phi_k \pi$  from being 1-dimensional for some  $k = 1, 2, 3$ .

If the quaternionic dimension of the quaternionic space form is  $m = 2$ , there are no totally real 2-planes tangent to  $M$ . Therefore, the totally real sectional curvature is meaningful when  $m \geq 3$ .

**2. Preliminaries**

Let  $\tilde{\nabla}$  be the Levi-Civita connection of  $QM^m(q)$ ,  $q \neq 0$ ,  $m \geq 3$ . As this is a quaternionic Kaehlerian manifold, there exists a 3-dimensional vector bundle  $\hat{V}$  of tensors of type  $(1, 1)$  and a local basis of almost Hermitian structures  $\{J_1, J_2, J_3\}$  on  $QM^m(q)$  which satisfy

$$(2.1) \quad J_1^2 = J_2^2 = J_3^2 = -Id, \quad J_1J_2 = -J_2J_1 = J_3$$

$$(2.2) \quad \tilde{\nabla}_X J_i = q_k(X)J_j - q_j(X)J_k \quad i = 1, 2, 3, \quad X \in TQM^m(q)$$

$$(2.3) \quad (dq_i + q_j \wedge q_k)(X, Y) = 4g(X, J_i Y) \quad i = 1, 2, 3, \quad X, Y \in TQM^m(q)$$

where  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$  and  $q_k$   $k = 1, 2, 3$  are local 1-forms on  $QM^m(q)$ .

Let  $M$  be a connected real hypersurface of  $QM^m(q)$ ,  $q \neq 0$ ,  $m \geq 3$ , and  $N$  a local normal unit vector field on  $M$ . We shall denote  $U_k = -J_k N$   $k = 1, 2, 3$ . If  $X$  is a tangent vector field to  $M$ , we shall write  $J_k X = \phi_k X + f_k(X)N$ ,  $k = 1, 2, 3$ , where  $\phi_k X$  is the tangential component of  $J_k X$  and  $f_k(X) = g(X, U_k)$ ,  $k = 1, 2, 3$ , where  $g$  is the induced metric on  $M$ . The distribution  $D^\perp$  is locally spanned by the set  $\{U_1, U_2, U_3\}$ . By (2.1),

$$(2.4) \quad \phi_k^2 X = -X + f_k(X)U_k \quad f_k(\phi_k X) = 0, \quad \phi_k U_k = 0, \quad k = 1, 2, 3.$$

for any  $X$  tangent to  $M$ .

$$(2.5) \quad \phi_i X = \phi_j \phi_k X - f_k(X)U_j = -\phi_k \phi_j X + f_j(X)U_k \quad i = 1, 2, 3$$

$$f_i(X) = f_j(\phi_k X) = -f_k(\phi_j X)$$

for any  $X$  tangent to  $M$ , where  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$ . It is also easy to check

$$(2.6) \quad \begin{aligned} \phi_i U_j &= U_k = -\phi_j U_i \\ g(\phi_i X, Y) + g(X, \phi_i Y) &= 0, \quad g(\phi_i X, \phi_i Y) = g(X, Y) - f_i(X)f_i(Y) \end{aligned}$$

for any  $X, Y$  tangent to  $M$ ,  $i = 1, 2, 3$   $(i, j, k)$  being a cyclic permutation of  $(1, 2, 3)$ . If we denote by  $\nabla$  the induced connection of  $QM^m(q)$  on  $M$ , the Gauss and Weingarten formulae are given respectively by

$$(2.7) \quad \tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$$

$$(2.8) \quad \tilde{\nabla}_X N = -AX$$

for any  $X, Y$  tangent to  $M$ , where  $A$  is the Weingarten endomorphism of the immersion.

We shall denote  $UD(p) = \{X \in T_p M : f_k(X) = 0, k = 1, 2, 3, \|X\| = 1\}$ . The Gauss equation allows us to compute the following expression of the totally real sectional curvature of  $M$

$$(2.9) \quad T(X, Y) = \frac{q}{4} + g(AX, X)g(AY, Y) - g(AX, Y)^2$$

where  $X, Y \in UD$ ,  $\text{Span}\{X, Y\}$  is totally real and  $g(X, Y) = 0$ . A similar formula of the sectional curvature of any quaternionic 2-plane  $\pi$  tangent to  $M$  is given by

$$(2.10) \quad H(X, Y) = q + g(AX, X)g(AY, Y) - g(AX, Y)^2$$

where  $\pi = \text{Span}\{X, Y\}$ ,  $\{X, Y\}$  being an orthonormal basis of  $\pi$ .

In [5], ruled real hypersurfaces are characterized by the condition  $g(AX, Y) = 0$  for any  $X, Y \in D$ .

Finally, we need the following results to prove ours.

**THEOREM 1** [3]. *Let  $M$  be a real hypersurface of  $\mathbf{QP}^m(q)$ ,  $q > 0$ ,  $m \geq 3$ , which has constant quaternionic sectional curvature  $H$ . Then  $M$  is one of the following cases:*

- a) *An open subset of a geodesic hypersphere,  $4H > q$ .*
- b) *Ruled,  $4H = q$ .*

**THEOREM 2** [6]. *Let  $M$  be a real hypersurface of  $\mathbf{QH}^m(q)$ ,  $q < 0$ ,  $m \geq 3$ , which has constant quaternionic sectional curvature  $H$ . Then  $M$  is one of the following cases:*

- a) *An open subset of a geodesic hypersphere, i.e., a tube of radius  $r > 0$  over a point,  $(q/4) + (1/k^2) < H = (q/4) + (1/k^2) \cot h^2 r$ .*
- b) *An open subset of a horosphere,  $H = (q/4) + (1/k^2)$ .*
- c) *An open subset of a tube of radius  $r > 0$  over a hyperplane  $\mathbf{QH}^{m-1}$ ,  $(q/4) < H = (q/4) + (1/k^2) \tan h^2 r < (q/4) + (1/k^2)$ .*
- d) *Ruled,  $4H = q$ .*

**REMARK 2.** In the reference [3], Theorem 1 can be found with the hypothesis  $m > 4$ . Anyway, the proof of Theorem 2 can be repeated with slight differences for the cases  $m = 3$  and  $m = 4$ .

### 3. Proof of the Theorem

Let  $p$  be a point of  $M$ . Take  $X \in UD(p)$ . Let  $Y(t)$ ,  $t \in (-\delta, \delta)$ , be a curve in  $D(p)$  such that  $\text{Span}\{X, Y(t)\}$  is totally real,  $Y(0) = Y$ ,  $Y'(0) = Z$  and

$g(Y, Z) = 0$ . By (2.9),

$$\frac{d}{dt} \Big|_{t=0} g(AX, X)g(AY(t), Y(t)) - g(AX, Y(t))^2 = 0$$

A straightforward computation shows

$$(3.1) \quad 0 = g(AX, X)g(AY, Z) - g(AX, Y)g(AX, Z)$$

for any  $X, Y, Z \in UD(p)$  such that  $\text{Span}\{X, Y\}$  and  $\text{Span}\{X, Z\}$  are totally real, and  $g(Y, Z) = 0$ . In the sequel, we shall denote by  $(*)_{\mathcal{D}}$  the component of  $(*)$  in  $\mathcal{D}$ . Take  $\{U_1, U_2, U_3, E_1, \dots, E_{4m-4}\}$  an orthonormal basis of  $T_pM$  such that

$$(3.2) \quad (AE_i)_{\mathcal{D}} = a_i E_i \quad i = 1, \dots, 4m - 4$$

where  $a_i$  are functions on  $M$ . Choose  $i \in \{1, \dots, 4m - 4\}$ . If we substitute  $X = E_i$  in (3.1),

$$(3.3) \quad 0 = a_i g(AY, Z)$$

where  $Y, Z \in UD$ ,  $\text{Span}\{Y, Z\} \perp Q(E_i)$  and  $g(Y, Z) = 0$ . From (3.3), we have to discuss two cases:

A)  $a_1 = \dots = a_{4m-4} = 0$ .

B) There exists  $i \in \{1, \dots, 4m - 4\}$  such that  $a_i \neq 0$ .

We begin by studying case A). In such a case, formulae (3.2) show  $g(AX, Y) = 0$  for any  $X, Y \in \mathcal{D}$ , and therefore  $M$  is ruled.

Next, we pay attention to case B). We can suppose without losing any generality  $i = 1$  and  $a_1 \neq 0$  in an dense open subset of  $M$ . From (3.3),

$$(3.4) \quad g(AY, Z) = 0$$

for any  $Y, Z \in \mathcal{D}$  such that  $Y, Z \in Q(E_1)^\perp$  and  $g(Y, Z) = 0$ . Given  $Y, Z \in UD$  in these latter conditions, we put  $X = \phi_k E_1$ ,  $k = 1, 2, 3$  in (3.1) and we obtain

$$(3.5) \quad 0 = g(A\phi_k E_1, Y)g(A\phi_k E_1, Z) \quad k = 1, 2, 3.$$

for any  $Y, Z \in UD$  such that  $Y, Z \in Q(E_1)^\perp$  and  $g(Y, Z) = 0$ . Besides, the vectors  $Y' = (1/\sqrt{2})(Y + Z)$ ,  $Z' = (1/\sqrt{2})(Y - Z)$  satisfy the conditions of (3.5), so we can introduce them in that equation to obtain  $g(A\phi_k E_1, Y)^2 = g(A\phi_k E_1, Z)^2$ . This and (3.5) yield

$$(3.6) \quad g(A\phi_k E_1, Y) = 0 \quad \text{for any } Y \in \mathcal{D} \cap Q(E_1)^\perp, \quad k = 1, 2, 3.$$

Moreover, given  $Y, Z \in UD$  which satisfy the conditions of (3.4), the vectors  $Y' = Y + Z$  and  $Z' = Y - Z$  also satisfy them, and if we introduce these vectors

in (3.4),  $0 = g(A(Y + Z), Y - Z)$  and then

$$(3.7) \quad g(AY, Y) = g(AZ, Z)$$

for any  $Y, Z \in \mathbf{D}$  such that  $Y, Z \in Q(E_1)^\perp$  and  $g(Y, Z) = 0$ . Now, from (3.4), (3.6) and (3.7) there exists a local orthonormal basis  $\{U_1, U_2, U_3, E_1, \phi_1 E_1, \phi_2 E_1, \phi_3 E_1, \dots, E_{m-1}, \phi_1 E_{m-1}, \phi_2 E_{m-1}, \phi_3 E_{m-1}\}$  of  $TM$  such that

$$(3.8) \quad \begin{aligned} (AE_1)_\mathbf{D} &= a_1 E_1 \quad a_1 \neq 0, \\ (A\phi_k E_1)_\mathbf{D} &\in \text{Span}\{\phi_1 E_1, \phi_2 E_1, \phi_3 E_1\} \quad k = 1, 2, 3. \\ (AX)_\mathbf{D} &= bX \quad \text{for any } X \in \mathbf{D} \cap Q(E_1)^\perp \end{aligned}$$

where  $b$  is a function on  $M$ .

Let us suppose that the function  $b$  vanishes in an open subset of  $M$ . If we take  $X = E_1$ ,  $Y = E_2$ , by (2.9) and (3.8) we see

$$(3.9) \quad 4T = q.$$

Next, we consider the vectors  $X = (1/\sqrt{2})(E_1 + E_2)$ ,  $Y = (1/\sqrt{10})(2E_1 + \phi_k E_1 - 2E_2 - \phi_k E_2)$ . It is easy to check  $\text{Span}\{X, Y\}$  is totally real. Introducing them in (2.9) and bearing in mind (3.8) and (3.9), easy computations yield

$$(3.10) \quad 0 = g(A\phi_k E_1, \phi_k E_1) \quad k = 1, 2, 3$$

Now, given  $i = 1, 2, 3$ , we choose  $X = (1/\sqrt{2})(\phi_i E_1 + \phi_i E_2)$  and  $Y = (1/\sqrt{10})(2\phi_i E_1 + \phi_k E_1 - 2\phi_i E_2 - \phi_k E_2)$  where  $k \in \{1, 2, 3\} \setminus \{i\}$ . By virtue of (2.9), (3.8), (3.9) and (3.10) we obtain

$$(3.11) \quad 0 = g(A\phi_i E_1, \phi_k E_1) \quad i \neq k.$$

The assumption  $b = 0$  and formulae (3.8), (3.10) and (3.11) yield there exists a tangent vector field  $E$  and a non-vanishing function  $a$  in an open subset of  $M$  such that

$$(3.12) \quad (AX)_\mathbf{D} = ag(X, E)E \quad \text{for any } X \in \mathbf{D}$$

If we introduce (3.12) in (2.10) we see that  $M$  has constant quaternionic sectional curvature  $q$ . By Theorem 1 and Theorem 2,  $M$  must be ruled, and therefore  $g(AX, Y) = 0$  for any  $X, Y \in \mathbf{D}$ . This implies that the function  $a$  must vanish, which is a contradiction.

As we have shown,  $b$  does not vanish in an dense open subset of  $M$ . Now we choose  $X = E_2$  and  $Y, Z \in Q(E_1) \cap U\mathbf{D}$  such that  $g(Y, Z) = 0$ . Introducing them in (3.1) we see  $0 = bg(AY, Z)$ , that is to say,  $g(AY, Z) = 0$  for any  $Y, Z \in Q(E_1)$

such that  $g(Y, Z) = 0$ . From this and (3.8) it is easy to check  $g(AX, Y) = 0$  for any  $X, Y \in \mathbf{D}$  such that  $g(X, Y) = 0$ . If  $X, Y$  are unitary, the vectors  $X + Y$  and  $X - Y$  are orthogonal, so  $g(A(X + Y), X - Y) = 0$ , which implies  $g(AX, X) = g(AY, Y)$ . From this and (3.8) we deduce

$$(3.13) \quad g(AX, Y) = bg(X, Y) \quad \text{for any } X, Y \in \mathbf{D}$$

where  $b$  is a non-vanishing function in an dense open subset of  $M$ . Introducing (3.13) in (2.9) we obtain  $4T = q + 4b^2$ , which implies that  $b$  must be constant. From this, (3.13) and (2.10),  $M$  has constant quaternionic sectional curvature. But all model spaces of Theorem 1 and Theorem 2 have constant totally real sectional curvature. This finishes the proof. ■

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