TSUKUBA J. MATH.
Vol. 22 No. 1 (1998), 165–178

A CHARACTERIZATION OF EINSTEIN REAL HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACE

By

Soo Hyo LEE, Juan de Dios PÉREZ and Young Jin SUH

Abstract. On a real hypersurface of quaternionic projective space QP^m we study the following condition: $\mathfrak{S}(R(X, Y)SZ) = 0$ where $\mathfrak{S}$ denotes the cyclic sum, R, respectively S, the curvature tensor, respectively the Ricci tensor, of the real hypersurface and $X, Y \in \mathscr{D}$, $Z \in \mathscr{D}^{\perp}$, $\mathscr{D}$ and $\mathscr{D}^{\perp}$ being certain distributions on the real hypersurface. We prove that such a real hypersurface must be Einstein.

1. Introduction

Let M be a connected real hypersurface of the quaternionic projective space QP^m, $m \geq 3$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_i = -J_iN$, $i = 1,2,3$, where $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic structure of QP^m, [2]. Several examples of such real hypersurfaces are well known, see for instance ([1], [3], [4]).

Let S be the Ricci tensor of M. In [4] it is proved that the unique real hypersurfaces of QP^m that are Einstein are geodesic hyperspheres of radius r, $0 < r < (\pi/2)$ and $cot^2(r) = (1/2m)$.

Recently, in [4] the second author has studied real hypersurfaces of QP^m, $m \geq 2$, such that

$$\mathfrak{S}(R(X, Y)SZ) = 0 \tag{1.1}$$

for any X, Y and Z tangent to M, where R denotes the curvature tensor of M and $\mathfrak{S}$ is the cyclic sum on X, Y, and Z, obtaining

1991 *Mathematics Subject Classification*. Primary 53C40; Secondary 53C15.
Partially supported by BSRI-97-1404 and TGRC-KOSEF
Received August 19, 1996.

THEOREM A. *A real hypersurface M of QP^m. $m \geq 2$ satisfies* (1.1) *if and only if it is Einstein.*

Now let us define a distribution $\mathscr{D}$ by $\mathscr{D}(x) = \{X \in T_xM : X \perp U_i(x), i = 1,2,3\}$, $x \in M$, of a real hypersurface M in QP^m, which is orthogonal to the structure vector fields $\{U_1, U_2, U_3\}$ and invariant with respect to structure tensors $\{\phi_1, \phi_2, \phi_3\}$, and by $\mathscr{D}^\perp = Span\{U_1, U_2, U_3\}$ its orthogonal complement in TM. In order to obtain a weaker condition than (1.1) it seems natural to propose to study real hypersurfaces of QP^m satisfying

$$\mathfrak{S}(R(X,Y)SZ) = 0 \tag{1.2}$$

for any $X, Y \in \mathscr{D}$, and $Z \in \mathscr{D}^\perp$

The purpose of the present paper is to study such a condition. Concretely we shall prove

THEOREM 1. *A real hypersurface M of QP^m, $m \geq 3$, satisfies* (1.2) *if and only if it is Einstein.*

2. Preliminaries

Let X be a tangent field to M. We write $J_iX = \phi_iX + f_i(X)N$, $i = 1,2,3$, where ϕ_iX is the tangent component of J_iX and $f_i(X) = g(X, U_i)$, $i = 1,2,3$. As $J_i^2 = -id$, $i = 1,2,3$, where id denotes the identity endomorphism on TQP^m, we get

$$\phi_i^2X = -X + f_i(X)U_i, \quad f_i(\phi_iX) = 0, \quad \phi_iU_i = 0, \quad i = 1,2,3 \tag{2.1}$$

for any X tangent to M. As $J_iJ_j = -J_jJ_i = J_k$, where (i,j,k) is a cyclic permutation of $(1,2,3)$ we obtain

$$\phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k \tag{2.2}$$

and

$$f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX) \tag{2.3}$$

for any vector field X tangent to M, where (i,j,k) is a cyclic permutation of $(1,2,3)$. It is also easy to see that for any X, Y tangent to M and $i = 1,2,3$

$$g(\phi_iX, Y) + g(X, \phi_iY) = 0, \quad g(\phi_iX, \phi_iY) = g(X,Y) - f_i(X)f_i(Y) \tag{2.4}$$

and

$$\phi_iU_j = -\phi_jU_i = U_k \tag{2.5}$$

(i, j, k) being a cyclic permutation of $(1,2,3)$. From the expression of the curvature tensor of QP^m, $m \geq 2$, we have the equations of Gauss and Codazzi are respectively given by

$$(2.6) \quad R(X, Y)Z = g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3} \{g(\phi_i Y, Z)\phi_i X - g(\phi_i X, Z)\phi_i Y + 2g(X, \phi_i Y)\phi_i Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

and

$$(2.7) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^{3} \{f_i(X)\phi_i Y - f_i(Y)\phi_i X + 2g(X, \phi_i Y)U_i\}$$

for any X, Y, Z tangent to M, where R denotes the curvature tensor of M, see [3]. Moreover, the Ricci tensor $S'(Z, Y) = g(SZ, Y) = Trace\{X \to R(X, Z)Y\}$ are defined by

$$(2.8) \qquad SZ = (4m+7)Z - 3\sum_k f_k(Z)U_k + hAZ - A^2 Z,$$

respectively.

From the expressions of the covariant derivatives of J_i, $i = 1,2,3$, it is easy to see that

$$(2.9) \qquad \nabla_X U_i = -p_j(X)U_k + p_k(X)U_j + \phi_i AX$$

and

$$(2.10) \qquad (\nabla_X \phi_i)Y = -p_j(X)\phi_k Y + p_k(X)\phi_j Y + f_i(Y)AX - g(AX, Y)U_i$$

for any X, Y tangent to M, (i, j, k) being a cyclic permutation of $(1,2,3)$ and p_i, $i = 1,2,3$, local 1-forms on QP^m.

3. Key Lemma

Let M be a real hypersurface in a quaternionic projective space QP^m satisfying

$$(3.1) \qquad \mathfrak{S} R(X, Y)SZ = 0$$

for any $X, Y \in \mathscr{D}$, and $Z \in \mathscr{D}^\perp$. Now let us take an orthonormal basis

$$\{E_1, \ldots, E_{4m-4}, U_1, U_2, U_3\}$$

of the tangent space of $T_x(M)$ at any point $x \in M$. Then for a case where $X = E_i$, $Y = \phi_1 E_i$ and $Z = U_1$ the above formula (3.1) gives

$$R(E_i, \phi_1 E_i)SU_1 + R(\phi_1 E_i, U_1)SE_i + R(U_1, E_i)S\phi_1 E_i = 0. \tag{3.2}$$

Now let us denote by $H = hA - A^2$. Then the first term of the left side of (3.2) becomes

$$\begin{aligned} R(E_i, \phi_1 E_i)SU_1 = {} & g(\phi_1 E_i, HU_1)E_i - g(E_i, HU_1)\phi_1 E_i \\ & - g(E_i, HU_1)\phi_1 E_i + g(\phi_1 E_i, HU_1)E_i - 2g(\phi_3 E_i, HU_i)\phi_2 E_i \\ & + 2g(\phi_2 E_i, HU_1)\phi_3 E_i - 2\phi_1 HU_1 + g(A\phi_1 E_i, SU_1)AE_i \\ & - g(AE_i, SU_1)A\phi_1 E_i \end{aligned}$$

The second term gives

$$\begin{aligned} R(\phi_1 E_i, U_1)SE_i = {} & g(HU_1, E_i)\phi_1 E_i - (4m+7)g(E_i, \phi_1 E_i)U_1 \\ & - g(HE_i, \phi_1 E_i)U_1 + g(U_3, HE_i)\phi_3 E_i \\ & - g(\phi_3 E_i, HE_i)U_3 - g(\phi_2 E_i, HE_i)U_2 \\ & + g(U_2, HE_i)U_2 - g(AU_1, S\phi_1 E_i)AE_i + g(AE_i, S\phi_1 E_i)AU_1. \end{aligned}$$

Also the third term of (3.2) gives

$$\begin{aligned} R(U_1, E_i)S\phi_1 E_i = {} & -g(HU_1, \phi_1 E_i)E_i + (4m+7)g(E_i, \phi_1 E_i)U_1 \\ & + g(H\phi_1 E_i, E_i)U_1 + g(U_3, H\phi_1 E_i)\phi_2 E_i \\ & - g(U_2, H\phi_1 E_i)\phi_3 E_i - g(\phi_2 E_i, H\phi_1 E_i)U_3 \\ & + g(\phi_3 E_i, H\phi_1 E_i)U_2 - g(AU_1, S\phi_1 E_i)AE_i \\ & + g(AE_i, S\phi_1 E_i)AU_1. \end{aligned}$$

Thus summing up the above formulas, we have

$$\begin{aligned} \mathfrak{S}R(E_i, \phi_1 E_i)SU_1 = {} & \{g(\phi_3 E_i, H\phi_1 E_i) - g(H\phi_2 E_i, E_i)\}U_2 \\ & - \{g(H\phi_3 E_i, E_i) + g(H\phi_2 E_i, \phi_1 E_i)\}U_3 \\ & + g(\phi_1 E_i, HU_1)E_i - g(E_i, HU_1)\phi_1 E_i - 2\phi_1 HU_1 \\ & + \{g(U_2, HE_i) + g(U_3, H\phi_1 E_i) - 2g(\phi_3 E_i, HU_1)\}\phi_2 E_i \\ & + \{g(U_3, HE_i) - g(U_2, H\phi_1 E_i) + 2g(\phi_2 E_i, HU_1)\}\phi_3 E_i \\ & + 3g(AU_1, E_i)A\phi_1 E_i - 3g(AU_1, \phi_1 E_i)AE_i \\ = {} & 0. \end{aligned} \tag{3.3}$$

For a case where $j=2$ we can also calculate the following

$$
\begin{aligned}
(3.4)\quad \mathfrak{S}R(E_i,\phi_2E_i)SU_2 &= \{g(\phi_1E_i,H\phi_2E_i)-g(\phi_3E_i,HE_i)\}U_3\\
&\quad-\{g(H\phi_1E_i,E_i)+g(H\phi_3E_i,\phi_2E_i)\}U_1\\
&\quad+g(\phi_2E_i,HU_2)E_i-g(E_i,HU_2)\phi_2E_i-2\phi_2HU_2\\
&\quad+\{g(U_3,HE_i)+g(U_1,H\phi_2E_i)-2g(\phi_1E_i,HU_2)\}\phi_3E_i\\
&\quad+\{g(U_1,HE_i)-g(U_3,H\phi_2E_i)+2g(\phi_3E_i,HU_2)\}\phi_1E_i\\
&\quad+3g(AU_2,E_i)A\phi_2E_i-3g(AU_2,\phi_2E_i)AE_i\\
&=0.
\end{aligned}
$$

Similarly, for a case where $j=3$ we have

$$
\begin{aligned}
(3.5)\quad \mathfrak{S}R(E_i,\phi_3E_i)SU_3 &= \{g(\phi_2E_i,H\phi_3E_i)-g(\phi_1E_i,HE_i)\}U_1\\
&\quad-\{g(H\phi_2E_i,E_i)+g(H\phi_1E_i,\phi_3E_i)\}U_2\\
&\quad+g(\phi_3E_i,HU_3)E_i-g(E_i,HU_3)\phi_3E_i-2\phi_3HU_3\\
&\quad+\{g(U_1,HE_i)+g(U_2,H\phi_3E_i)-2g(\phi_2E_i,HU_3)\}\phi_1E_i\\
&\quad+\{g(U_2,HE_i)-g(U_1,H\phi_3E_i)+2g(\phi_1E_i,HU_3)\}\phi_2E_i\\
&\quad+3g(AU_3,E_i)A\phi_3E_i-3g(AU_3,\phi_3E_i)AE_i\\
&=0.
\end{aligned}
$$

By contracting from $i=1,\dots,4(m-1)$ the formulas (3.3), (3.4) and (3.5) are reduced by the followings respectively

$$
\begin{aligned}
(3.6)\qquad & -(4m-5)\phi_1HU_1+\phi_2HU_2+\phi_3HU_3\\
&\quad+3\{A\phi_1AU_1-g(AU_1,U_2)AU_3+g(AU_1,U_3)AU_2\}=0,\\
(3.7)\qquad & -(4m-5)\phi_2HU_2+\phi_3HU_3+\phi_1HU_1\\
&\quad+3\{A\phi_2AU_2-g(AU_2,U_3)AU_1+g(AU_2,U_1)AU_3\}=0,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.8)\qquad & -(4m-5)\phi_3HU_3+\phi_1HU_1+\phi_2HU_2\\
&\quad+3\{A\phi_3AU_3-g(AU_3,U_1)AU_2+g(AU_3,U_2)AU_1\}=0.
\end{aligned}
$$

Now let us denote the curvature tensor and the Ricci tensor defined in section 2 respectively by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + Q(X, Y)Z,$$

and

$$SZ = (4m+7)Z - 3\sum_k f_k(Z)U_k + HZ,$$

where

$$Q(X, Y)Z = \sum_{l=1}^{3}\{g(\phi_l Y, Z)\phi_l X - g(\phi_l X, Z)\phi_l Y + 2g(X, \phi_l Y)\phi_l Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

and

$$HZ = hAZ - A^2 Z$$

respectively, for any tangent vector fields X, Y and Z of M. In this section we want to prove the following

LEMMA 3.1. *Let M be a real hypersurface in a quaternionic projective space QP^m satisfying* (3.1). *Then*

$$g(H\mathscr{D}, \mathscr{D}^{\perp}) = 0.$$

PROOF. The formula (3.1) implies that

$$Q(X, Y)SZ + Q(Y, Z)SX + Q(Z, X)SY = 0. \tag{3.9}$$

for any tangent vector fields $X, Y \in \mathscr{D}$ and $Z \in \mathscr{D}^{\perp}$. Now let us put $X = E_i$, $Y = \phi_1 E_i$ and $Z = U_2$ in (3.9) and use the basic formulas in section 2, then the first term of the left hand side of (3.9) gives

$$\begin{aligned} Q(E_i, \phi_1 E_i)SU_2 = &-g(E_i, HU_2)\phi_1 E_i + g(\phi_1 E_i, HU_2)E_i \\ &- 8(m+1)U_3 - 2\phi_1 HU_2 - 2g(\phi_3 E_i, HU_2)\phi_2 E_i \\ &+ 2g(\phi_2 E_i, HU_2)\phi_3 E_i \\ &+ g(A\phi_1 E_i, SU_2)AE_i - g(AE_i, SU_2)A\phi_1 E_i, \end{aligned} \tag{3.10}$$

where we have used the fact that

$$\phi_1 SU_2 = 4(m+1)U_3 + \phi_1 HU_2.$$

The second term of (3.9) gives

$$(3.11)\qquad \begin{aligned} Q(\phi_1 E_i, U_2)SE_i = {} & -g(U_3, HE_i)E_i + g(E_i, HE_i)U_3 \\ & - g(U_1, HE_i)\phi_2 E_i \\ & + g(\phi_2 E_i, HE_i)U_1 + g(AU_2, SE_i)A\phi_1 E_i \\ & - g(A\phi_1 E_i, SE_i)AU_2 + (4m+7)U_3, \end{aligned}$$

where we have used $SE_i = (4m+7)E_i + HE_i$. Finally the third term of (3.9) is given by the following

$$(3.12)\qquad \begin{aligned} Q(U_2, E_i)S\phi_1 E_i = {} & (4m+7)U_3 + g(\phi_1 E_i, H\phi_1 E_i)U_3 \\ & - g(U_3, H\phi_1 E_i)\phi_1 E_i \\ & - g(\phi_3 E_i, H\phi_1 E_i)U_1 + g(U_1, H\phi_1 E_i)\phi_3 E_i \\ & + g(AE_i, S\phi_1 E_i)AU_2 - g(AU_2, S\phi_1 E_i)AE_i. \end{aligned}$$

Combining (3.10), together with (3.11) and (3.12), we have

$$(3.13)\qquad \begin{aligned} 0 & = \mathfrak{S}R(E_i, \phi_1 E_i)SU_2 \\ & = \mathfrak{S}Q(E_i, \phi_1 E_i)SU_2 \\ & = \{g(\phi_2 E_i, HE_i) - g(\phi_3 E_i, H\phi_1 E_i)\}U_1 \\ & \quad + \{g(E_i, HE_i) + 6 + g(\phi_1 E_i, H\phi_1 E_i)\}U_3 \\ & \quad + \{g(\phi_1 E_i, HU_2) - g(U_3, HE_i)\}E_i \\ & \quad - \{g(E_i, HU_2) + g(U_3, H\phi_1 E_i)\}\phi_1 E_i \\ & \quad - \{2g(\phi_3 E_i, HU_2) + g(U_1, HE_i)\}\phi_2 E_i \\ & \quad + \{2g(\phi_2 E_i, HU_2) + g(U_1, H\phi_1 E_i)\}\phi_3 E_i \\ & \quad - 2\phi_1 HU_2 + \{g(A\phi_1 E_i, SU_2) - g(AU_2, S\phi_1 E_i)\}AE_i \\ & \quad + \{g(AU_2, SE_i) - g(AE_i, SU_2)\}A\phi_1 E_i \\ & \quad + \{g(AE_i, S\phi_1 E_i) - g(A\phi_1 E_i, SE_i)\}AU_2. \end{aligned}$$

From this let us take a summation given by $\sum_i = \sum_{i=1}^{4(m-1)}$, then we know the following informations

$$\sum_i g(\phi_2 E_i, HE_i) = 0$$

and

$$\sum_i g(\phi_3 E_i, H\phi_1 E_i) = 0.$$

Moreover, by contracting we also have the followings

$$\sum_i g(E_i, HE_i) = Tr\, H - \Sigma_{i=1}^3 g(U_i, HU_i),$$

$$\sum_i g(\phi_1 E_i, H\phi_1 E_i) = Tr\, H - g(\phi_1 U_2, H\phi_1 U_2) - g(\phi_1 U_3, H\phi_1 U_3),$$

$$\sum_i g(\phi_1 E_i, HU_2)E_i = -\phi_1 HU_2 - g(U_3, HU_2)U_2 + g(U_2, HU_2)U_3,$$

$$\sum_i g(U_3, HE_i)E_i = HU_3 - \Sigma_{i=1}^3 g(U_3, HU_i)U_i,$$

$$\sum_i g(E_i, HU_2)\phi_1 E_i = \phi_1 HU_2 - g(U_2, HU_2)U_3 + g(U_3, HU_2)U_2,$$

$$\sum_i g(U_3, H\phi_1 E_i) = HU_3 - f_1(HU_3)U_1 - g(HU_2, U_3)U_2 - g(HU_3, U_3)U_3,$$

$$2\sum_i g(\phi_3 E_i, HU_2)\phi_2 E_i = -2\phi_1 HU_2 - 2f_3(HU_2)U_2 + 2g(HU_2, U_2)U_3,$$

and

$$\sum_i g(U_1, HE_i)\phi_2 E_i = \phi_2 HU_1 + g(U_1, HU_1)U_3 - g(U_1, HU_3)U_1.$$

Following with the proof, let us take a contraction to the latter terms of (3.13), then we have

$$\begin{aligned}
&\sum_i \{2g(\phi_2 E_i, HU_2) + g(U_1, H\phi_1 E_i)\}\phi_3 E_i \\
&\quad = 2\phi_1 HU_2 - 2f_2(HU_2)U_3 - \phi_2 HU_1 - f_1(HU_1)U_3 \\
&\qquad + 2g(HU_2, U_3)U_2 + g(HU_3, U_1)U_1,
\end{aligned}$$

$$\sum_i \{g(A\phi_1 E_i, SU_2) - g(AU_2, S\phi_1 E_i)\}AE_i$$
$$= 3A\phi_1 AU_2 + 3g(AU_2, U_3)AU_2 - 3g(AU_2, U_2)AU_3,$$
$$\sum_i \{g(AU_2, SE_i) - g(AE_i, SU_2)\}A\phi_1 E_i$$
$$= -3A\phi_1 AU_2 + 3g(AU_2, U_2)AU_3 + 3g(AU_2, U_3)A\phi_1 U_3,$$
$$\sum_i \{g(AE_i, S\phi_1 E_i) - g(A\phi_1 E_i, SE_i)\}AU_2 = 0.$$

Taking account of these formulas into (3.13), we have

$$(3.14) \qquad HU_3 = \phi_1 HU_2 - \phi_2 HU_1 + 2g(HU_2, U_3)U_2 + 2g(HU_1, U_3)U_1$$
$$- \{g(HU_1, U_1) + g(HU_2, U_2) - g(HU_3, U_3)\}U_3$$
$$- 4(m-1)\phi_1 HU_2.$$

From this it follows that

$$(3.15) \qquad \phi_3 HU_3 = -(4m-5)\phi_2 HU_2 + \phi_1 HU_1.$$

Using the similar method, we have the following

$$(3.16) \qquad \phi_2 HU_2 = -(4m-5)\phi_1 HU_1 + \phi_3 HU_3,$$

$$(3.17) \qquad \phi_1 HU_1 = -(4m-5)\phi_3 HU_3 + \phi_2 HU_2.$$

Thus summing up (3.15), (3.16) and (3.17), we have

$$\sum_{i=1}^{3} \phi_i HU_i = -(4m-5)\sum_{i=1}^{3} \phi_i HU_i + \sum_{i=1}^{3} \phi_i HU_i,$$

so that

$$\sum_{i=1}^{3} \phi_i HU_i = 0.$$

On the other hand, from (3.6) and (3.15) we know that

$$(3.18) \qquad 2\phi_3 HU_3 + 3\{A\phi_2 AU_2 - g(AU_2, U_3)AU_1 + g(AU_2, U_1)AU_3\} = 0.$$

Similarly, we can assert that

$$(3.19) \qquad 2\phi_1 HU_1 + 3\{A\phi_3 AU_3 - g(AU_3, U_1)AU_2 + g(AU_3, U_2)AU_1\} = 0,$$

and

$$(3.20) \qquad 2\phi_2 HU_2 + 3\{A\phi_1 AU_1 - g(AU_1, U_2)AU_3 + g(AU_1, U_3)AU_2\} = 0.$$

On the other hand, putting $\phi_2 HU_2 + \phi_3 HU_3 = -\phi_1 HU_1$ into (3.6) and using (3.20), we have

$$-4(m-1)\phi_1 HU_1 - 2\phi_2 HU_2 = 0. \tag{3.21}$$

Similarly, we also have

$$-4(m-1)\phi_2 HU_2 - 2\phi_3 HU_3 = 0, \tag{3.22}$$

and

$$-4(m-1)\phi_3 HU_3 - 2\phi_1 HU_1 = 0. \tag{3.23}$$

Thus (3.22) and (3.21) imply $\phi_3 HU_3 = -2(m-1)\phi_2 HU_2 = 4(m-1)^2\phi_1 HU_1$. From this, together with (3.23) it follows

$$\{8(m-1)^3 + 1\}\phi_1 HU_1 = 0.$$

Similarly, we have $\phi_2 HU_2 = 0$, and $\phi_3 HU_3 = 0$. From this we complete the proof of our lemma.

4. The Proof of Main Theorem

In section 3 under the condition (3.1) we have proved that $g(H\mathscr{D}, \mathscr{D}^\perp) = 0$ for the distributions $\mathscr{D}$ and $\mathscr{D}^\perp = Span\{U_1, U_2, U_3\}$ of real hypersurfaces in QP^m, where $H = hA - A^2$. But $HA = AH$. Thus we can find an orthonormal basis of T_xM, for any $x \in M$, such that it diagonalizes simultaneously both H and A. So on this decomposition of T_xM such that $T_xM = \mathscr{D} \oplus \mathscr{D}^\perp$ the fact that $g(H\mathscr{D}, \mathscr{D}^\perp) = 0$ is equivalent to $g(A\mathscr{D}, \mathscr{D}^\perp) = 0$. Then by virtue of a theorem given by J. Berndt [1] we conclude that a real hypersurfaces satisfying (3.1) is locally congruent to one of geodesic hypersphere, a tube over QP^k, $k = 1, \ldots, n-1$ with radius $0 < r < (\pi/2)$, or a tube over CP^m with radius $0 < r < (\pi/4)$.

Firstly, let us consider the case where M is a geodesic hypersphere. Then its principal curvatures are given by $\alpha = 2\cot 2r$, $\cot r$ with multiplicities 3 and $4(m-1)$ respectively. That is, $AU_i = \alpha U_i$, $i = 1, 2, 3$ and $AX = \cot r X$ for any $X \in \mathscr{D}$. From this the Ricci tensor S, for any X in $\mathscr{D}$, is given by

$$\begin{aligned} SX &= [(4m+7) + \{(4m-1)\cot r - 3\tan r\}\cot r - \cot^2 r]X \\ &= [4m+7+(4m-1)\cot^2 r - 3 - \cot^2 r]X \\ &= [4m+4+(4m-2)\cot^2 r]X. \end{aligned}$$

On $\mathscr{D}^{\perp} = Span\{U_1, U_2, U_3\}$ we have

$$SU_i = [4m + 4 + \{(4m-1)\cot r - 3\tan r\}(\cot r - \tan r) - (\cot r - \tan r)^2]U_i$$

$$= \{4 + (4m-2)\cot^2 r + 2\tan^2 r\}U_i.$$

On the other hand, the condition (3.1) implies that

$$\mathfrak{S}R(X, Y)SU_i = R(X, Y)SU_i + R(Y, U_i)SX + R(U_i, X)SY = 0, \tag{4.1}$$

for $i = 1, 2, 3$. Thus on this geodesic hypersphere we can put $SU_i = \gamma U_i$ and $SX = \delta X$ for any X in $\mathscr{D}$. Then it can be easily verify that γ and δ could be equal to each other. This means that the geodesic hypersphere M in QP^m is Einstein. From this and the above expression of the Ricci tensor we have

$$4 + (4m-2)\cot^2 r + 2\tan^2 r - (4m+4) - (4m-2)\cot^2 r = 0.$$

That is, M is a Einstein real hypersurface in QP^m, which is congruent to a tube of radius r such that $\cot^2 r = (1/2m)$.

For the case where M is congruent to a tube over QP^k, $k = 1, 2, \ldots, m-1$. Its principal curvatures are also given by $\cot r$, $-\tan r$ and $2\cot 2r$ with their multiplicities $4m - 4k - 4$, $4k$ and 3, respectively. Thus h is given by

$$h = Tr\,A = (4m - 4k - 4)\cot r - 4k\tan r + 3(\cot r - \tan r)$$

$$= (4m - 4k - 1)\cot r - (4k+3)\tan r.$$

Now let us take principal vectors such that $X \in V_{\cot r}$, $Y \in V_{-\tan r}$ and $U_i \in \mathscr{D}^{\perp}$, where the distribution $\mathscr{D}$ is given by $\mathscr{D} = V_{\cot r} \oplus V_{-\tan r}$. Then we have the following

$$SX = (4m+7)X + \{(4m - 4k - 1)\cot r - (4k+3)\tan r\}\cot r\,X - \cot^2 r X \tag{4.2}$$

$$= \{(4m - 4k + 4) + (4m - 4k - 2)\cot^2 r\}X,$$

$$SY = (4m+7)Y - \{(4m - 4k - 1)\cot r - (4k+3)\tan r\}\tan r\,Y - \tan^2 r Y \tag{4.3}$$

$$= \{4k + 8 + (4k+2)\tan^2 r\}Y,$$

and

(4.4) $$SU_i = (4m+4)U_i + (\cot r - \tan r)\{(4m-4k-1)\cot r - (4k+3)\tan r - (\cot r - \tan r)\}U_i$$
$$= \{4 + (4m-4k-2)\cot^2 r + (4k+2)\tan^2 r\}U_i.$$

Thus if we put $SX = \gamma X$, $SY = \delta Y$ for any $X \in V_{\cot r}$ and $Y \in V_{-\tan r}$, and $SU_i = \beta U_i$, then the condition (4.1) implies that $\gamma = \beta = \delta$. Thus substracting (4.2) and (4.3) from (4.4) respectively, then it follows respectively that

$$(4k+2)\tan^2 r = 4m - 4k$$

and

$$(4m-4k-2)\cot^2 r = 4k+4.$$

These imply $(4m-4k)(4k+4) = (4m-4k-2)(4k+2)$. Thus $8m = -4$. This makes also a contradiction. Thus this case does not appear.

Finally let us consider for the case where M is congruent to a tube over CP^n. Then its principal curvatures are given by $\cot r$, $-\tan r$, $2\cot 2r$ and $-2\tan 2r$ with multiplicities $2(m-1)$, $2(m-1)$, 1 and 2 respectively. Then the trace of the second fundamental form A is given by

$$h = 2(m-1)(\cot r - \tan r) + 2\cot 2r - 4\tan 2r$$
$$= (2m-1)(\cot r - \tan r) - 4\tan 2r.$$

Now let us denote by its corresponding principal curvature vectors $X \in V_{\cot r}$, $Y \in V_{-\tan r}$, $U_1 \in V_{2\cot 2r}$, and U_2, $U_3 \in V_{-2\tan 2r}$. Then we have the following

$$SX = (4m+7)X + \{(2m-1)(\cot r - \tan r) - 4\tan 2r\}\cot r X - \cot^2 r X$$
$$= \{2m + 8 + 2(m-1)\cot^2 r - 4\tan 2r \cot r\}X,$$
$$SY = (4m+7)Y - \{(2m-1)(\cot r - \tan r) - 4\tan 2r\}\tan r Y - \tan^2 r Y$$
$$= \{2m + 8 + 2(m-1)\tan^2 r + 4\tan 2r \tan r\}Y,$$
$$SU_1 = (4m+4)U_1 + (\cot r - \tan r)\{(2m-1)(\cot r - \tan r) - 4\tan 2r\}U_1 - 4\cot^2 2r U_1,$$
$$SU_k = (-4m + 8 + 4\tan^2 2r)U_k, \quad k = 2, 3.$$

On the other hand, let us put $X \in V_{\cot r}$, $\phi_2 X \in V_{-\tan r}$ in (3.1). Then we have

$$\begin{aligned}
&R(X, \phi_2 X)SU_1 + R(\phi_2 X, U_1)SX + R(U_1, X)S\phi_2 X \\
&\quad = \{(4m-4) + (2m-2)(\cot r - \tan r)^2\} R(X, \phi_2 X) U_1 \\
&\qquad + \{(2m+8) + (2m-2)\cot^2 r - 4\tan 2r \cot r\} R(\phi_2 X, U_1) X \\
&\qquad + \{(2m+8) + (2m-2)\tan^2 r + 4\tan 2r \tan r\} R(U_1, X)\phi_2 X \\
&\quad = 2\{(4m-4) + 2(m-1)(\cot r - \tan r)^2\} U_3 \\
&\qquad - 2\{(m+4) + (m-1)\cot^2 r - 2\tan 2r \cot r\} U_3 \\
&\qquad - 2\{(m+4) + (m-1)\tan^2 r + 2\tan 2r \tan r\} U_3 \\
&\quad = 0.
\end{aligned}$$

So it follows

$$\begin{aligned}
&2\{4(m-1) + 2(m-1)(\cot r - \tan r)^2\} - 2(2m+8) - 4(m-1)(\cot^2 r + \tan^2 r) \\
&\quad + 4\tan 2r(\cot r - \tan r) = 0.
\end{aligned}$$

Thus $-4m - 8 = 0$. This is impossible. Thus this case also can not occur.

Summing up this result, we conclude that a real hypersurface in QP^m satisfying (3.1) is Einstein and it is congruent to a geodesic hypersphere, that is a tube over one point with radius r such that $\cot^2 r = (1/2m)$. This completes the proof of our assertion.

Remark. But if we consider the above situation for the shape operator A of M in a quaternionic projective space QP^m, we can verify that QP^m do not admit any real hypersurfaces satisfying the corresponding condition. Using the same method as in the proof of Theorem 1, we can assert this as follows:

Theorem 2. *There do not exist any real hypersurfaces M in a quaternionic projective space QP^m, $m \geq 2$, satisfying $\mathfrak{S}R(X, Y)AZ = 0$ for any X, $Y \in \mathscr{D}$ and $Z \in \mathscr{D}^{\perp}$, where $\mathfrak{S}$ denotes the cyclic sum of X, Y and Z and R is the curvature tensor of M.*

Corollary 3. *There do not exist any real hypersurfaces M in QP^m, $m \geq 2$, satisfying $\mathfrak{S}R(X, Y)AZ = 0$ for any X, Y and Z tangent to M, where $\mathfrak{S}$ denotes the cyclic sum of X, Y and Z.*

References

[1] Berndt J., Real hypersurfaces in quaternionic space forms, J. Reine Angew. Math. **419** (1991), 9–26.
[2] Ishihara S., Quaternion Kaehler manifolds, J. Diff. Geometry **9** (1974), 483–500.
[3] Martinez A. and Pérez J. D., Real hypersurfaces in quaternionic projective space, Ann. Math. Pura Appl. **145** (1986), 355–384.
[4] Pérez J. D., On certain real hypersurfaces of quaternionic projective space II, Alg. Groups Geom. **10** (1993), 13–24.

Department of Mathematics
Kyungpook National University
Taegu, 702-701, KOREA

Departmento de Geometría y Topología
Facultad de Ciencias
Universidad de Granada
18071-Granada, SPAIN

Department of Mathematics
Kyungpook National University
Taegu, 702-701, KOREA