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ON A CLASS OF SELF-INJECTIVE LOCALLY BOUNDED CATEGORIES

By

Zygmunt POGORZAŁY[1]

Throughout the paper K denotes a fixed algebraically closed field. Let R be a locally bounded K-category in the sense of [3]. It is well-known that every locally bounded K-category R is isomorphic to a factor category KQ_R/I_R, where KQ_R is a path category of a locally-finite quiver and I_R is some admissible ideal in KQ_R. A locally bounded K-category $R \cong KQ_R/I_R$ is said to be *triangular* if Q_R has no oriented cycles.

For a locally bounded K-category R we denote by $\mathrm{mod}(R)$ the category of all finite-dimensional right R-modules.

We are interested in self-injective locally bounded K-categories. Assume that R is a self-injective locally bounded triangular K-category which is connected. Then there is the Nakayama K-automorphism $\nu_R : R \to R$ which is induced by a permutation π_R of the isoclasses of simple right R-modules such that $\pi_R(\mathrm{top}(P)) = \mathrm{soc}(P)$ for every indecomposable projective right R-module P. Consequently, the infinite cyclic group (ν_R) generated by the Nakayama automorphism ν_R acts freely on the objects of R. We consider self-injective, locally bounded, triangular and connected K-categories R whose quotient categories $R/(\nu_R)$ are finite-dimensional K-algebras and there is no indecomposable projective R-module of length smaller than 3.

Every basic finite-dimensional K-algebra A can be considered as a locally bounded K-category, because $A \cong KQ_A/I_A$ for a finite quiver Q_A. The *repetitive category* (see [5]) of a basic finite-dimensional K-algebra A is the self-injective locally bounded K-category $\hat{A}$ whose objects are formed by the pairs $(z, x) = x_z$, $x \in ob(A)$, $z \in \boldsymbol{Z}$ and $\hat{A}(x_z, y_z) = \{z\} \times A(x, y)$, $\hat{A}(x_{z+1}, y_z) = \{z\} \times DA(y, x)$, and $\hat{A}(x_p, y_q) = 0$ if $p \neq q$, $q+1$, where DV denotes the dual space $\mathrm{Hom}_K(V, K)$. It is well-known that if A is triangular then $\hat{A}$ is triangular. Moreover, $\hat{A}/(\nu_{\hat{A}})$ is

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isomorphic to the trivial extension $T(A)$ of A by its minimal injective cogenerator bimodule $D(A)$.

The class of K-categories satisfying the above conditions was studied by several authors [1, 5, 8, 9, 11]. These categories were considered mainly as Galois covers of some classes of finite-dimensional algebras. In particular, they always were isomorphic to the repetitive categories of triangular algebras. Nevertheless there is not given any general enough structural result on such K-categories. The aim of this note is to provide such a result for the considered class of K-categories. The main result is the following.

THEOREM. *Let R be a locally bounded triangular and connected self-injective K-category whose quotient category $R/(\nu_R)$ is a finite-dimensional K-algebra and there is no indecomposable projective R-module of length smaller than* 3. *Then there is a triangular finite-dimensional connected K-algebra A such that $R \cong \hat{A}$.*

The proof of our result is rather easy. Nevertheless it is worth to stress that our proof is independent of the representation type of R.

1. ν-sections

1.1. Throughout the note let R be a locally bounded self-injective triangular and connected K-category whose quotient category $R/(\nu_R)$ is a finite-dimensional K-algebra and there is no indecomposable projective R-module of length smaller than 3. Moreover, we shall assume that $R = KQ_R/I_R$ for a bound quiver (Q_R, I_R). All considered algebras are finite-dimensional, associative K-algebras with unit 1, basic and connected.

1.2. Recall from [12] that an algebra A is said to be *weakly symmetric* if each indecomposable projective left or right A-module has a simple socle which is isomorphic to its top.

LEMMA. *$R/(\nu_R)$ is a weakly symmetric algebra.*

PROOF. Obvious.

1.3. Since the Nakayama automorphism permutes the objects of R, the group (ν_R) acts also on (Q_R, I_R). $R/(\nu_R)$ is a finite-dimensional algebra by our assumption, hence there is only finitely many (ν_R)-orbits of vertices in Q_R.

A full convex subquiver (S, I) of (Q_R, I_R) is called a *ν_R-section of* (Q_R, I_R) if it satisfies the following conditions:

(1) For every vertex x of Q_R the intersection of its (ν_R)-orbit with S consists of exactly one element.

(2) If $x \in S$ and $y \in Q_R$ are such vertices that there is an arrow α (respectively, β) in Q_R sourced at x (respectively, y) and targetted at y (respectively, x) then either y or $\nu_R^{-1}(y)$ (respectively, either $\nu_R(y)$ or y) belongs to S.

(3) $I = KS \cap I_R$.

1.4. For a bound quiver (Q_R, I_R) of R we define a *cone* C_x at a vertex $x \in Q_R$ to be the full subquiver of Q_R formed by all the vertices y of Q_R such that there exists a path of finite length in Q_R sourced at x and targetted at y. A *reduced cone* S_x at a vertex $x \in Q_R$ is the full subquiver of Q_R formed by the vertices from $C_x \backslash C_{\nu_R(x)}$.

1.5. Lemma. *Let S_x be a reduced cone at a vertex $x \in Q_R$. If $y \in S_x$ then $\nu_R^n(y) \notin S_x$ for every $n \in \boldsymbol{Z}\backslash\{0\}$.*

Proof. We prove our lemma by induction on the length $l(w)$ of the shortest path w in Q_R from x to y. If $l(w) = 0$ then $y = x$ and clearly $\nu_R^n(x) \notin S_x$ for $n < 0$, because Q_R is without oriented cycles. On the other hand $\nu_R^n(x) \notin S_x$ for $n > 0$, because there is a path in Q_R from $\nu_R(x)$ to $\nu_R^n(x)$ for every $n > 0$.

Assume that for all vertices y in S_x such that the length $l(w)$ of the shortest path from x to y is not greater than l the required condition holds.

Consider a vertex $y_0 \in S_x$ such that $l(w_0) = l + 1$ for the shortest path w_0 from x to y_0. Suppose to the contrary that there is $n \in \boldsymbol{Z}\backslash\{0\}$ such that $\nu_R^n(y_0) \in S_x$. Let $w_0 = w_1\alpha$, where α is an arrow from y_1 to y_0. It is clear that w_1 is the shortest path from x to y_1, because w_0 would not be the shortest one otherwise. Moreover, there is an arrow $\nu_R^n(\alpha)$ from $\nu_R^n(y_1)$ to $\nu_R^n(y_0)$. Thus we know from the inductive assumption that $\nu_R^n(y_1) \notin S_x$. Hence there is a path v from $\nu_R(x)$ to $\nu_R^n(y_1)$. Then we have the path $v\nu_R^n(\alpha)$ from $\nu_R(x)$ to $\nu_R^n(y_0)$ which contradicts the above assumption. Consequently, $\nu_R^n(y_0) \notin S_x$ for every $n \in \boldsymbol{Z}\backslash\{0\}$ and the lemma follows by induction.

1.6. Lemma. *Let S_x be a reduced cone at a vertex $x \in Q_R$. Then S_x is a full convex connected and finite subquiver of Q_R.*

Proof. Connectedness of S_x is clear, because every two vertices of S_x are connected by a walk passing through x. Fullness of S_x is clear by the definition of S_x. Observe that S_x is finite. Indeed, there is only finitely many (ν_R)-orbits of vertices in Q_R. Thus S_x has only finitely many vertices by Lemma 1.5. Since Q_R is locally finite, S_x is finite.

In order to show that S_x is convex, consider a path w from y_1 to y_2, where $y_1, y_2 \in S_x$. If there is a decomposition $w = w_1 w_2$ such that w_1 is targetted at z with $z \notin S_x$ then there is a path v from $\nu_R(x)$ to z. Thus vw_2 is a path from $\nu_R(x)$ to y_2 which contradicts the fact that $y_2 \in S_x$. Consequently, $z \in S_x$ and our lemma is proved.

1.7. LEMMA. *Let S_x be a reduced cone at a vertex $x \in Q_R$. If $y \in C_{\nu_R(x)}$ then there exists a natural number $n \geq 1$ such that $\nu_R^{-n}(y) \in S_x$.*

PROOF. We prove the lemma by induction on the length $l(w)$ of the shortest path w from $\nu_R(x)$ to y. If $l(w) = 0$ then $y = \nu_R(x)$ and $\nu_R^{-1}(y) = x \in S_x$.

Assume that for any vertex y in $C_{\nu_R(x)}$ with $l(w) \leq l$ there exists a natural number n such that $\nu_R^{-n}(y) \in S_x$, where w is the shortest path in Q_R from $\nu_R(x)$ to y.

Consider a vertex $y \in C_{\nu_R(x)}$ such that the length $l(w) = l + 1$ for the shortest path w in Q_R from $\nu_R(x)$ to y. Consider the decomposition $w = w_1\alpha$, where α is an arrow sourced at y_0 and targetted at y. Then $y_0 \in C_{\nu_R(x)}$ and we obtain by the inductive assumption that there is a natural number n_0 such that $\nu_R^{-n_0}(y_0) \in S_x$. Consider the vertex $\nu_R^{-n_0}(y)$. Since $\nu_R^{-n_0}(y_0) \in S_x$, there is a path u from x to $\nu_R^{-n_0}(y_0)$. Hence there is the path $u\nu_R^{-n_0}(\alpha)$ from x to $\nu_R^{-n_0}(y)$. Therefore $\nu_R^{-n_0}(y) \in C_x$. If there is no path from $\nu_R(x)$ to $\nu_R^{-n_0}(y)$ then $\nu_R^{-n_0}(y) \in S_x$. If there is a path z from $\nu_R(x)$ to $\nu_R^{-n_0}(y)$ then there is the path $\nu_R^{-1}(z)$ from x to $\nu_R^{-n_0-1}(y)$, and so $\nu_R^{-n_0-1}(y) \in C_x$. If there is a path v from $\nu_R(x)$ to $\nu_R^{-n_0-1}(y)$ then we obtain a contradiction to the fact that $\nu_R^{-n_0}(y_0)$ belongs to S_x. Indeed, in the case there is a path b from $\nu_R^{-n_0-1}(y)$ to $\nu_R^{-n_0}(y_0)$ since R is self-injective. Thus there is the path vb from $\nu_R(x)$ to $\nu_R^{-n_0}(y_0)$ which contradicts the choice of $\nu_R^{-n_0}(y_0)$. Consequently, $\nu_R^{-n_0-1}(y) \in S_x$ and the lemma is proved by induction.

1.8. LEMMA. *Let C_x be a cone at a vertex $x \in Q_R$. Then every (ν_R)-orbit of a vertex $z \in Q_R$ has a common vertex with C_x.*

PROOF. We prove the lemma by induction on the length $l(w)$ of minimal walk in Q_R connecting a vertex $z \in Q_R$ to x. Such a walk always exists since Q_R is connected. If $l(w) = 0$ then $x = z$ and the required condition holds.

Assume that for all vertices $z \in Q_R$ with $l(w) \leq l_0$ the required condition holds, where w is a minimal walk connecting z to x.

Consider $z_0 \in Q_R$ such that there is a minimal walk w in Q_R connecting z_0 to x with $l(w) = l_0 + 1$. Then $w = \alpha w_1$ or $w = \alpha^{-1} w_1$, where α is an arrow sourced or targetted at z_0, respectively. If $w = \alpha w_1$ and z_0 is the source of α then there is a path v in Q_R from x to $\nu_R^n(z_1)$ for the target z_1 of α and for some $n \in \boldsymbol{Z}$ by the inductive assumption. Since R is self-injective, there is a path $\nu_R^n(\alpha)u$ in Q_R from $\nu_R^n(z_0)$ to $\nu_R^{n+1}(z_0)$. Thus there is the path vu from x to $\nu_R^{n+1}(z_0)$ in Q_R, and so $\nu_R^{n+1}(z_0) \in C_x$.

If $w = \alpha^{-1} w_1$ and z_0 is the target of α then there is a path v in Q_R from x to $\nu_R^n(z_1)$ for the source z_1 of α and for some $n \in \boldsymbol{Z}$ by the inductive assumption. On the other hand we have the arrow $\nu_R^n(\alpha)$ from $\nu_R^n(z_1)$ to $\nu_R^n(z_0)$. Hence there is the path $v\nu_R^n(\alpha)$ from x to $\nu_R^n(z_0)$ in Q_R, and so $\nu_R^n(z_0) \in C_x$. Consequently, our lemma is proved by induction.

1.9. PROPOSITION. *Let $R = KQ_R/I_R$ be a self-injective triangular and connected locally bounded K-category whose quotient category $R/(\nu_R)$ is a finite-dimensional K-algebra and there is no indecomposable projective R-module of length smaller than* 3. *Then there exists a ν_R-section of (Q_R, I_R).*

PROOF. Fix a vertex $x \in Q_R$. Consider the reduced cone S_x at the vertex x. Let $I_x = KS_x \cap I_R$. We shall show that (S_x, I_x) is a ν_R-section of (Q_R, I_R). We infer by Lemma 1.6 that S_x is a full convex connected and finite subquiver of Q_R. Applying Lemma 1.8 to the cone $C_{\nu_R(x)}$ at the vertex $\nu_R(x)$, we obtain that every (ν_R)-orbit of a vertex $z \in Q_R$ has a common vertex to $C_{\nu_R(x)}$. Furthermore, we deduce from Lemma 1.7 that every (ν_R)-orbit of a vertex z in Q_R has a common vertex to S_x. Thus we obtain from Lemma 1.5 that there is only one such a common vertex. Consequently, 1.3(1) holds for (S_x, I_x).

Suppose that a vertex z belongs to S_x and there is an arrow α in Q_R sourced at z and targeted at $y \in Q_R$. If $y \notin S_x$ then there is a path u in Q_R from $\nu_R(x)$ to y. Thus there is the path $\nu_R^{-1}(u)$ from x to $\nu_R^{-1}(y)$. Hence $\nu_R^{-1}(y) \in C_x$. If $\nu_R^{-1}(y) \notin S_x$ then there is a path v in Q_R from $\nu_R(x)$ to $\nu_R^{-1}(y)$. But R is self-injective hence there is a path $w\alpha$ in Q_R from $\nu_R^{-1}(y)$ to y. Consequently, there is the path vw from $\nu_R(x)$ to z which contradicts to the fact that $z \in S_x$. Therefore $\nu_R^{-1}(y) \in S_x$.

Now suppose that a vertex z belongs to S_x and there is an arrow β in Q_R sourced at $y \in Q_R$ and targetted at z, and suppose that there is a path βw in Q_R from y to $\nu_R(y)$. Since $z \in S_x$, there is a path u in Q_R from x to z. Thus the path uw connects x to $\nu_R(y)$ hence $\nu_R(y) \in C_x$. If $\nu_R(y) \in C_{\nu_R(x)}$ then there is a non-negative integer n such that $\nu_R^{-n}(\nu_R(y)) \in S_x$ by Lemma 1.7. Since $y \notin S_x$, $n > 1$.

But there is a path v in Q_R from x to $\nu_R^{-n}(y)$. Hence there are a path v' from $\nu_R^n(x)$ to y of the form $\nu_R^n(v)$ and a path v'' from $\nu_R(x)$ to $\nu_R^n(x)$. Thus there exists the path $v''v'\beta$ from $\nu_R(x)$ to z which contradicts that $z \in S_x$. Consequently, $\nu_R(y) \notin C_{\nu_R(x)}$, and so $\nu_R(y) \in S_x$.

In this way we have proved that 1.3(2) holds. Since 1.3(3) is obvious by the definition of I_x, the proposition is proved.

2. ν-sectional partitions

2.1. Let (S, I) be a fixed ν_R-section of (Q_R, I_R), where S is a reduced cone at a vertex $x \in Q_R$. A *collecting arrow* with respect to (S, I) is any arrow α in Q_R which does not belong to S and such that there is an arrow β in S with $\beta\alpha \notin I_R$.

2.2. LEMMA. *Let $w = \alpha_1 \cdots \alpha_n$ be a maximal nonzero path in (Q_R, I_R) whose source is a vertex $s \in S$. Then w contains exactly one collecting arrow α with respect to (S, I).*

PROOF. Suppose that $w = \alpha_1 \cdots \alpha_n$ is a maximal nonzero path in (Q_R, I_R) and $s \in S$ is its source. Since R is self-injective without indecomposable projective R-modules of length 2 then $n \geq 2$ and w connects s with $\nu_R(s)$ by the maximality of w. But if $s \in S$ then $\nu_R(s) \notin S$ by Lemma 1.5. Hence there is $i_0 \in \{1, \ldots, n\}$ such that α_{i_0} is a collecting arrow.

Now suppose that there are two collecting arrows α_{i_0}, α_{j_0} in w with $j_0 > i_0$. Since (S, I) is a full convex subquiver in (Q_R, I_R), the target of α_{i_0} cannot belong to S, because $\alpha_{i_0} \notin S$. But again α_{j_0} has the source in S by the definition of collecting arrows. Thus the target of α_{i_0} belongs to S by the convexity of S. The obtained contradiction shows the lemma.

2.3. An *(S, I)-partition* of (Q_R, I_R) is the non-connected bound quiver $(P, I_P) = \coprod_{z \in \mathbf{Z}} (\nu_R^z(S), \nu_R^z(I))$.

LEMMA. *If an arrow α in Q_R does not belong to the (S, I)-partition (P, I_P) of (Q_R, I_R) then there exists $z_0 \in \mathbf{Z}$ such that α is a collecting arrow with respect to $(\nu_R^{z_0}(S), \nu_R^{z_0}(I))$.*

PROOF. Let α be an arrow in Q_R which does not belong to P. Then there exists a maximal nonzero path in Q_R of the form $\beta_1 \cdots \beta_r \alpha$, because R is self-injective without indecomposable projective R-modules of length smaller than 3. Now look at the vertices of the arrows β_1, α. Clearly for the source s of β_1 and the target y of α it holds $\nu_R(s) = y$. Then there is $z_0 \in \boldsymbol{Z}$ such that $s \in \nu_R^{z_0}(S)$ by the definition of (P, I_P). Observe that the target v of β_r belongs to $\nu_R^{z_0}(S)$. Indeed, if $v \notin \nu_R^{z_0}(S)$ then $\nu_R^{-1}(v) \in \nu_R^{z_0}(S)$ by 1.3(2) for the ν_R-section $(\nu_R^{z_0+1}(S), \nu_R^{z_0+1}(I))$. Thus v, $y = \nu_R(s) \in \nu_R^{z_0+1}(S)$, and so $\alpha \in \nu_R^{z_0+1}(S)$ which contradicts the choice of α. Consequently, $v \in \nu_R^{z_0}(S)$ and $\beta \in \nu_R^{z_0}(S)$ since S is convex. Hence α is a collecting arrow with respect to $(\nu_R^{z_0}(S), \nu_R^{z_0}(I))$, because $\beta_r\alpha \notin I_P$.

2.4. For a fixed ν_R-section (S, I) of (Q_R, I_R) consider the (S, I)-partition (P, I_P) of (Q_R, I_R). Define a two-sided ideal $\boldsymbol{I}_P$ in $R = KQ_R/I_R$ with respect to (P, I_P) as the ideal generated by the arrows α which do not belong to P.

Lemma. $\boldsymbol{I}_P^2 = 0$.

Proof. Clearly it is sufficient to show that if we have two paths u, $v \in \boldsymbol{I}_P$ then $uv = 0$. But if u is a path in $\boldsymbol{I}_P$ then $u = u_1\alpha_1u_2$, where $\alpha_1 \notin P$. The same holds for v, e.g. $v = v_1\alpha_2v_2$ with $\alpha_2 \notin P$. If u and v are not composable then clearly $uv = 0$. Consider the case when u and v are composable. Then we infer by Lemma 2.3 that there is $z_0 \in \boldsymbol{Z}$ such that α_1 is a collecting arrow with respect to $(\nu_R^{z_0}(S), \nu_R^{z_0}(I))$. The same holds for α_2 hence there is $z_1 \in \boldsymbol{Z}$ such that α_2 is a collecting arrow with respect to $(\nu_R^{z_1}(S), \nu_R^{z_1}(I))$. We may assume that u, v are nonzero in (Q_R, I_R). Hence, by the triangularity of R, we infer that $z_1 = z_0 + 1$. Then $u_1\alpha_1u_2v_1\alpha_2v_2$ is a path which contains two collecting arrows (with respect to different ν_R-sections). Consider the path $\alpha_1u_2v_1\alpha_2$. The source s of it is in $\nu_R^{z_0}(S)$ and the target y of it is in $\nu_R^{z_0+2}(S)$. We deduce from the self-injectivity of R that if $\alpha_1u_2v_1\alpha_2$ is nonzero in (Q_R, I_R) then there is a path $\gamma_1 \cdots \gamma_t$ from $\nu_R^{-1}(y)$ to s such that $\gamma_1 \cdots \gamma_t\alpha_1u_2v_1\alpha_2$ is nonzero in (Q_R, I_R). But $\nu_R^{-1}(y) \in \nu_R^{z_0+1}(S)$ and $s \in \nu_R^{z_0}(S)$. Since the target b of α_1 belongs to $\nu_R^{z_0+1}(S)$, we get by the convexity of $\nu_R^{z_0+1}(S)$ that $s \in \nu_R^{z_0+1}(S)$ which contradicts the above choice of α_1. Thus $\alpha_1u_2v_1\alpha_2$ is a zero path in (Q_R, I_R) and the lemma follows.

2.5. Proposition. $R/\boldsymbol{I}_P \cong \bigoplus_{z \in \boldsymbol{Z}} K(\nu_R^z(S))/\nu_R^z(I)$.

Proof. Consider a surjective functor $p : KQ_R/I_R \to \bigoplus_{z \in \boldsymbol{Z}} K(\nu_R^z(S))/\nu_R^z(I)$ defined as follows: for every vertex $q \in Q_R$, $p(q) = q$. For every path u in Q_R which does not contain a collecting arrow we put $p(u) = u$. For every path v in Q_R which contains a collecting arrow we put $p(v) = 0$. Then we extend p linearly to a functor. It is clear by the definition of p that $\boldsymbol{I}_P = \ker(p)$. Moreover, we get that p is surjective by Lemma 2.3 and the definition of a ν_R-section in (Q_R, I_R).

3. Proof of the main result

3.1. PROPOSITION. *Let $R = KQ_R/I_R$ be a self-injective triangular and connected locally bounded K-category whose quotient category $R/(\nu_R)$ is a finite-dimensional K-algebra and there is no indecomposable projective R-module of length smaller than 3. If (Q_R, I_R) contains a ν_R-section then there is an epimorphism $p : R/(\nu_R) \to A$ such that A is a triangular connected algebra and $ker(p) = \boldsymbol{I}$ is such a two-sided ideal in $R/(\nu_R)$ that $\boldsymbol{I}^2 = 0$.*

PROOF. Let (S, I) be a ν_R-section of (Q_R, I_R). Consider the (S, I)-partition (P, I_P) of (Q_R, I_R). Then we have an ideal $\boldsymbol{I}_P$ in R such $\boldsymbol{I}_P^2 = 0$ by Lemma 2.4. Moreover, $R/\boldsymbol{I}_P \cong \bigoplus_{z \in \boldsymbol{Z}} K(\nu_R^z(S))/\nu_R^z(I)$ by Proposition 2.5. It is easily seen that the group (ν_R) acts freely on $R/\boldsymbol{I}_P$ and on $\boldsymbol{I}_P$, because it acts freely on R. Then we have an epimorphism $p : R/(\nu_R) \to (R/\boldsymbol{I}_P)/(\nu_R)$ whose kernel is $\boldsymbol{I}_P/(\nu_R)$. Put $\boldsymbol{I} = \boldsymbol{I}_P/(\nu_R)$ and $A = (R/\boldsymbol{I}_P)/(\nu_R)$. We know from Lemma 2.4 that $\boldsymbol{I}^2 = 0$. A is triangular and connected, because $A \cong KS/I$. Thus the proposition follows.

3.2. If A and $\boldsymbol{I}$ are as in Proposition 3.1 then we have.

LEMMA. *$D(A) = \boldsymbol{I}$ as right A-modules.*

PROOF. We shall prove our lemma considering KS/I as a subcategory of R, where (S, I) is a fixed ν_R-section of (Q_R, I_R). Then consider the two-sided ideal $\boldsymbol{J}$ in R generated by the collecting arrows in Q_R with respect to (S, I). We infer by Propositions 2.5, 3.1 that $\boldsymbol{I}_P = \bigoplus_{z \in \boldsymbol{Z}} \nu_R^z(\boldsymbol{J})$ and $R/\boldsymbol{I}_P = \bigoplus_{z \in \boldsymbol{Z}} \nu_R^z(KS/I)$. Since $\boldsymbol{I}^2 = 0$, $\boldsymbol{I}$ is a right A-module. Thus $\boldsymbol{I}$ is a submodule of $D(A)$, because $\mathrm{soc}_{R/(\nu_R)}(\boldsymbol{I}) = \mathrm{soc}_{R/(\nu_R)}(R/(\nu_R)) = \mathrm{soc}_{R/(\nu_R)}(D(A))$. Suppose to the contrary that $\boldsymbol{I} \neq D(A)$. Then there is a morphism from $D(A)$ to A which is a nonzero morphism from $\nu_R(D(KS/I))$ to KS/I which does not factorize through $\boldsymbol{J}$. Thus we have a path u in $(\nu_R(S), \nu_R(I))$ which is nonzero, sourced at s and targetted at y with $s \in S \cap \nu_R(S)$, $y \in \nu_R(S)$ which contradicts to the fact that (S, I) is a ν_R-section of (Q_R, I_R) by 1.3(1). Therefore $D(A) = \boldsymbol{I}$.

3.3. The following fact was proved in [6].

LEMMA. *Let $\boldsymbol{I}$ be such a two-sided ideal in a self-injective finite-dimensional K-algebra Λ that $\boldsymbol{I}^2 = 0$ and $\Lambda/\boldsymbol{I}$ is triangular. If $\boldsymbol{I}$ is injective as a right $\Lambda/\boldsymbol{I}$-module, then for any isomorphism $\varphi : \boldsymbol{I} \to D(\Lambda/\boldsymbol{I})$ of right $\Lambda/\boldsymbol{I}$-modules there is a $\Lambda/\boldsymbol{I}$-bimodule isomorphism $\varphi' : \boldsymbol{I} \to D(\Lambda/\boldsymbol{I})$.*

3.4. The following proposition in a weaker form was shown in [7]. We repeat the modified version of its proof for the convenience of the reader.

PROPOSITION. *Let R_1, R_2 be triangular connected self-injective locally bounded K-categories whose quotient categories $R_1/(\nu_{R_1})$, $R_2/(\nu_{R_2})$ are finite-dimensional K-algebras. If $R_1/(\nu_{R_1}) \cong R_2/(\nu_{R_2})$ then $R_1 \cong R_2$.*

PROOF. Under the assumptions of the proposition fix some representatives $\{P_x\}_{x\in X}$ of the isomorphism classes of indecomposable projective R_1-modules and some representatives $\{Q_y\}_{y\in Y}$ of the isomorphism classes of indecomposable projective R_2-modules. Then $R_1 \cong \mathrm{End}_{R_1}(\bigoplus_{x\in X} P_x)^{op}$ and $R_2 \cong \mathrm{End}_{R_2}\left(\bigoplus_{y\in Y} Q_y\right)^{op}$. Let $F_{\lambda,t} : \mathrm{mod}(R_t) \longrightarrow \mathrm{mod}(R_t/(\nu_{R_t}))$, $t = 1, 2$, be the push-down functors induced by the actions of (ν_{R_t}) on R_t (see [3, 2]). It is well-known that indecomposable projective $R_t/(\nu_{R_t})$-modules and their radicals are contained in the image of $F_{\lambda,t}$, $t = 1, 2$. Moreover, $F_{\lambda,t}$ preserves projectives and their radicals.

Fix some $x_0 \in X$. Let $LF_{\lambda,1}(P_{x_0}) \cong F_{\lambda,2}(Q_{y_0})$ for a fixed $y_0 \in Y$, where $L : \mathrm{mod}(R_1/(\nu_{R_1})) \longrightarrow \mathrm{mod}(R_2/(\nu_{R_2}))$ is the equivalence induced by a fixed isomorphism from $R_1/(\nu_{R_1})$ onto $R_2/(\nu_{R_2})$. Let $R_{1,1}$ be the subcategory of R_1 formed by P_{x_0} and the P_x, $P_{x'}$ such that the following conditions are satisfied:

(a) there is a nonzero morphism $f_x : P_x \to P_{x_0}$ in $\mathrm{mod}(R_1)$ of the form $f_x = f^* f'_x$, where $f'_x : P_x \to \mathrm{rad}(P_{x_0})$ satisfies $\pi_{x_0} f'_x \neq 0$ for the canonical epimorphism $\pi_{x_0} : \mathrm{rad}(P_{x_0}) \to \mathrm{top}(\mathrm{rad}(P_{x_0}))$, and $f^* : \mathrm{rad}(P_{x_0}) \to P_{x_0}$ is the identity monomorphism;

(b) there is a nonzero morphism $h_{x'} : P_{x_0} \to P_{x'}$ of the form $h_{x'} = h''_{x'} h'_{x'}$, where $h'_{x'} : P_{x_0} \to \mathrm{rad}(P_{x'})$ satisfies $\pi_{x'} h'_{x'} \neq 0$ for the canonical epimorphism $\pi_{x'} : \mathrm{rad}(P_{x'}) \to \mathrm{top}(\mathrm{rad}(P_{x'}))$, and $h''_{x'} : \mathrm{rad}(P_{x'}) \to P_{x'}$ is the identity monomorphism.

If P, P' are objects of $R_{1,1}$ then $\mathrm{Hom}_{R_{1,1}}(P, P')$ is the subspace of $\mathrm{Hom}_{R_1}(P, P')$ generated by the isomorphisms between P and P' and the morphisms of the form $a = a_1 a_2$, where $a_1 = h_{x'}$ for some x' and a_2 is an automorphism of P_{x_0}, or $a_2 = f_x$ for some x and a_1 is an automorphism of P_{x_0}, or else $a_1 = h_{x'}$ for some x' and $a_2 = f_x$ for some x. Since R_1 is locally bounded K-category, $R_{1,1}$ is finite.

Let $R_{2,1}$ be the subcategory of R_2 formed by Q_{y_0} and the Q_y, $Q_{y'}$ such that the following conditions are satisfied:

(a) there is a nonzero morphism $r_y : Q_y \to Q_{y_0}$ of the form $r_y = r^* r'_y$, where $r'_y : Q_y \to \mathrm{rad}(Q_{y_0})$ satisfies $\kappa_{y_0} r'_y \neq 0$ for the canonical epimorphism

$\kappa_{y_0} : \mathrm{rad}(Q_{y_0}) \to \mathrm{top}(\mathrm{rad}(Q_{y_0}))$, and $r^* : \mathrm{rad}(Q_{y_0}) \to Q_{y_0}$ is the identity monomorphism;

(b) there is a nonzero morphism $s_{y'} : Q_{y_0} \to Q_{y'}$ of the form $s_{y'} = s''_{y'} s'_{y'}$, where $s'_{y'} : Q_{y_0} \to \mathrm{rad}(Q_{y'})$ satisfies $\kappa_{y'} s'_{y'} \neq 0$ for the canonical epimorphism $\kappa_{y'} : \mathrm{rad}(Q_{y'}) \to \mathrm{top}(\mathrm{rad}(Q_{y'}))$, and $s''_{y'} : \mathrm{rad}(Q_{y'} \to Q_{y'}$ is the identity monomorphism.

If Q, Q' are objects of $R_{2,1}$ then $\mathrm{Hom}_{R_{2,1}}(Q, Q')$ is the subspace of $\mathrm{Hom}_{R_2}(Q, Q')$ generated by the isomorphisms between Q and Q' and the morphisms of the form $w = w_1 w_2$, where $w_1 = s_{y'}$ for some y' and w_2 is an automorphism of Q_{y_0}, or $w_2 = r_y$ for some y and w_1 is an automorphism of Q_{y_0}, or else $w_1 = s_{y'}$ for some y' and $w_2 = r_y$ for some y. Since R_2 is locally bounded K-category, $R_{2,1}$ is finite.

Observe that if $P_{x_1} \in R_{1,1}$ and $\mathrm{Hom}_{R_{1,1}}(P_{x_1}, P_{x_0}) \neq 0$ then there is a uniquely determined $Q_{y_1} \in R_{2,1}$ with $\mathrm{Hom}_{R_{2,1}}(Q_{y_1}, Q_{y_0}) \neq 0$ and $LF_{\lambda,1}(P_{x_1}) \cong F_{\lambda,2}(Q_{y_1})$. Indeed, if there are $Q_{y_1}, Q_{y_2} \in R_{2,1}$ with $\mathrm{Hom}_{R_{2,1}}(Q_{y_l}, Q_{y_0}) \neq 0$, $l = 1, 2$, and $LF_{\lambda,1}(P_{x_1}) \cong F_{\lambda,2}(Q_{y_l})$, then there is $z \in \boldsymbol{Z}$ such that ${}^{\nu^z_{R_2}}(Q_{y_1}) \cong Q_{y_2}$. Furthermore, there are $0 \neq r_{y_l} : Q_{y_l} \to Q_{y_0}$, $l = 1, 2$, such that r_{y_l} factorize through $\mathrm{rad}(Q_{y_0})$ by the definition of $R_{2,1}$. Hence $\mathrm{top}(Q_{y_l})$, $l = 1, 2$, are direct summands in $\mathrm{top}(\mathrm{rad}(Q_{y_0}))$. Then in case $z > 0$ we get that there is a sequence $Q'_1, \ldots, Q'_z$ of indecomposable projective R_2-modules such that $\mathrm{soc}(Q'_m) \cong \mathrm{top}(Q'_{m-1})$, $m = 2, \ldots, z$, and $\mathrm{top}(Q_{y_1}) \cong \mathrm{soc}(Q'_1)$, $\mathrm{top}(Q'_z) \cong \mathrm{soc}(Q_{y_2})$. But $\mathrm{top}(Q_{y_0})$ is contained in the support of Q'_1 hence R_2 is not triangular which contradicts our assumption. Similarly we obtain a contradiction if $z < 0$. Thus $z = 0$ and $Q_{y_1} = Q_{y_2}$. Dually one proves that if $P_{x'_1} \in R_{1,1}$ and $\mathrm{Hom}_{R_{1,1}}(P_{x_0}, P_{x'_1}) \neq 0$ then there exists the uniquely determined $Q_{y'_1} \in R_{2,1}$ with $\mathrm{Hom}_{R_{2,1}}(Q_{y_0}, Q_{y'_1}) \neq 0$ and $LF_{\lambda,1}(P_{x'_1}) \cong F_{\lambda,2}(Q_{y'_1})$.

Now we define a functor $F_1 : R_{1,1} \to R_{2,1}$ putting $F_1(P_{x_0}) = Q_{y_0}$, and for all possible x_1, x'_1 we put $F_1(P_{x_1}) = Q_{y_1}$, $F_1(P_{x'_1}) = Q_{y'_1}$. If P, $P' \in R_{1,1}$ then $\mathrm{Hom}_{R_{1,1}}(P, P')$ either consists of isomorphisms (if $P = P'$) or is generated by the above a. If $P = P'$ then $\mathrm{Hom}_{R_{1,1}}(P, P) \cong K \cdot \mathrm{id}_P \cong K \cdot \mathrm{id}_{F_{\lambda,1}(P)}$ as K-spaces and $\mathrm{Hom}_{R_{2,1}}(F_1(P), F_1(P)) \cong K \cdot \mathrm{id}_{F_1(P)} \cong K \cdot \mathrm{id}_{F_{\lambda,2}(F_1(P))}$. Then, since L induces a K-space isomorphism, $K \cdot \mathrm{id}_{F_{\lambda,1}(P)} \cong K \cdot \mathrm{id}_{F_{\lambda,2}(F_1(P))}$, for every $f \in \mathrm{Hom}_{R_{1,1}}(P, P)$ there is exactly one $r \in \mathrm{Hom}_{R_{2,1}}(F_1(P), F_1(P))$ such that $LF_{\lambda,1}(f) = F_{\lambda,2}(r)$. Thus we put $F_1(f) = r$. If $P \neq P'$ then we define F_1 for the morphisms of the form $a = a''a'$, where $a' : P \to \mathrm{rad}(P')$ satisfies $\pi a' \neq 0$ for the canonical epimorphism $\pi : \mathrm{rad}(P') \to \mathrm{top}(\mathrm{rad}(P'))$ and $a'' : \mathrm{rad}(P') \to P'$ is the inclusion monomorphism. If $a : P \to P'$ is such a morphism then there is the uniquely determined $r : F_1(P) \to F_1(P')$ in $\mathrm{Hom}_{R_{2,1}}(F_1(P), F_1(P'))$ such that $LF_{\lambda,1}(a) = F_{\lambda,2}(r)$. Indeed, if r_1, r_2 satisfy $LF_{\lambda,1}(a) = F_{\lambda,2}(r_1) = F_{\lambda,2}(r_2)$ then there are r_1', $r_2' : F_1(P) \to \mathrm{rad}(F_1(P'))$ such that $\pi' r_1'$, $\pi' r_2' \neq 0$ for the canonical projection $\pi' : \mathrm{rad}(F_1(P')) \to \mathrm{top}(\mathrm{rad}(F_1(P')))$. Furthermore, for the inclusion $r'' : \mathrm{rad}(F_1(P')) \to F_1(P')$ we have $r_1 = r'' r_1'$, $r_2 = r'' r_2'$. But if r_1', r_2' are different then $F_{\lambda,2}(r_1') \neq F_{\lambda,2}(r_2')$, because R_2 is triangular and $F_{\lambda,2}$ is induced by the action of (ν_{R_2}). Thus $F_{\lambda,2}(r_1) \neq F_{\lambda,2}(r_2)$ for $r_1 \neq r_2$. Consequently, $r_1 = r_2$ if $F_{\lambda,2}(r_1) = F_{\lambda,2}(r_2)$. Then we put $F_1(a) = r$. If $a = a_1 a_2$ is a composition of either an isomorphism and a morphism of the above form or two morphisms of the above form then we put $F_1(a) = F_1(a_1)F_1(a_2)$. Finally we extend F_1 linearly to a K-functor. It is clear by the above considerations that we obtained a functor $F_1 : R_{1,1} \to R_{2,1}$ which is dense and fully faithful. Thus F_1 yields an equivalence of categories.

Assume now that we defined a subcategory $R_{1,n}$ in R_1 such that for every pair P, P' of objects from $R_{1,n}$ it holds either $P = P'$ and $\mathrm{Hom}_{R_{1,n}}(P, P')$ consists only of automorphisms or $P \neq P'$ and $\mathrm{Hom}_{R_{1,n}}(P, P')$ is generated by the morphisms of the form $a = a_s \cdots a_2 a_1$ such that:

(i) $a_l : P_l \to P_{l+1}$ for some objects $P_1, \dots, P_{s+1}$ of $R_{1,n}$, where $P_1 = P$, $P_{s+1} = P'$;

(ii) $a_l = a_l'' a_l'$, $l = 1, \dots, s$, $a_l' : P_l \to \mathrm{rad}(P_{l+1})$ satisfies $\pi_{l+1} a_l' \neq 0$ for the canonical epimorphism $\pi_{l+1} : \mathrm{rad}(P_{l+1}) \to \mathrm{top}(\mathrm{rad}(P_{l+1}))$;

(iii) $a_l'' : \mathrm{rad}(P_{l+1}) \to P_{l+1}$ is the inclusion for $l = 1, \dots, s$.

Moreover, assume that we have defined a subcategory $R_{2,n}$ of R_2 satisfying the above conditions for morphisms, and a functor $F_n : R_{1,n} \to R_{2,n}$ which is a K-linear equivalence such that it maps the generators of $\mathrm{Hom}_{R_{1,n}}(P, P')$ onto the generators of $\mathrm{Hom}_{R_{2,n}}(F_n(P), F_n(P'))$.

Define a subcategory $R_{1,n+1}$ of R_1 in the following way. The objects of $R_{1,n+1}$ are those of $R_{1,n}$ and the objects P of R_1 such that either there is a nonzero morphism $a : P \to P'$ with $P' \in R_{1,n}$ and $a = a'' a'$, where $a' : P \to \mathrm{rad}(P')$ satisfies $\pi' a' \neq 0$ for the canonical projection $\pi' : \mathrm{rad}(P') \to \mathrm{top}(\mathrm{rad}(P'))$ and $a'' : \mathrm{rad}(P') \to P'$ is the inclusion, or there is a nonzero morphism $h : P' \to P$ with $P' \in R_{1,n}$ and $h = h'' h'$, where $h' : P' \to \mathrm{rad}(P)$ satisfies $\pi h' \neq 0$ for the canonical epimorphism $\pi : \mathrm{rad}(P) \to \mathrm{top}(\mathrm{rad}(P))$ and $h'' : \mathrm{rad}(P) \to P$ is the inclusion. For every two objects P, P'' from $R_{1,n+1}$ the morphism space $\mathrm{Hom}_{R_{1,n+1}}(P, P'')$ is generated by the isomorphisms between P and P'' and the compositions $a = a_s \cdots a_2 a_1$ which satisfy conditions (i)–(iii) above. In the same way we define a subcategory $R_{2,n+1}$ of R_2. Then repeating the arguments used for $R_{1,1}$ and $R_{2,1}$ we get that for every $P \in R_{1,n+1}$ such that there is a nonzero morphism $a : P \to P'$ with $P' \in R_{1,n}$ there is the uniquely determined object $Q \in R_{2,n+1}$ such that there is a nonzero morphism $r : Q \to F_n(P')$ in $R_{2,n+1}$ and $LF_{\lambda,1}(P) \cong F_{\lambda,2}(Q)$.

Furthermore, for every object $P \in R_{1,n+1}$ such that there is a nonzero morphism $h : P' \to P$ in $R_{1,n+1}$ with $P' \in R_{1,n}$ there is the uniquely determined object $Q \in R_{2,n+1}$ such that there is a nonzero morphism $r : F_n(P') \to Q$ in $R_{2,n+1}$ and $LF_{\lambda,1}(P) \cong F_{\lambda,2}(Q)$. Moreover, we have also the same uniqueness for generating morphisms $a : P \to P''$ with P, $P'' \in R_{1,n+1}$. Thus we define $F_{n+1} : R_{1,n+1} \to R_{2,n+1}$ in the following way. For every $P \in R_{1,n+1} \backslash R_{1,n}$ we put $F_{n+1}(P) = Q$, where Q is a uniquely determined object of $R_{2,n+1}$ as above. For every $P' \in R_{1,n}$ we put $F_{n+1}(P') = F_n(P')$. For every pair P, $P'' \in R_{1,n+1}$; if $a : P \to P''$ is a generator of $\mathrm{Hom}_{R_{1,n+1}}(P, P'')$ then we put $F_{n+1}(a) = r$, where r is a uniquely determined generator of $\mathrm{Hom}_{R_{2,n+1}}(F_{n+1}(P), F_{n+1}(P''))$. It is clear that for a generating morphism $a : P \to P''$ with P, $P'' \in R_{1,n}$ it holds $F_{n+1}(a) = F_n(a)$. If $a : P \to P''$ is an isomorphism then we put $F_{n+1}(a) = r$, where $LF_{\lambda,1}(a) = F_{\lambda,2}(r)$. Finally we extend F_{n+1} for the compositions of generating morphisms and isomorphisms $a = a_s \cdots a_1$ by putting $F_{n+1}(a) = F_{n+1}(a_s) \cdots F_{n+1}(a_1)$. Then we extend F_{n+1} to a K-linear functor. In this way we obtain a functor $F_{n+1} : R_{1,n+1} \to R_{2,n+1}$ which is dense and fully faithful. Thus F_{n+1} yields an equivalence of categories.

Consequently, we construct inductively a functor $F : R_1 \to R_2$ which is dense and fully faithful since R_1, R_2 are connected locally bounded K-categories. Thus the proposition follows.

PROOF OF THEOREM. We prove that $R \cong \hat{A}$, where $A \cong KS/I$ for a ν_R-section (S, I) of (Q_R, I_R). Since $D(A) = \boldsymbol{I}$ as right A-modules by Lemma 3.2, where $\boldsymbol{I}$ is the two-sided ideal in $R/(\nu_R)$ chosen in Proposition 3.1, we get by Lemma 3.3 that the structures of A-bimodules on $D(A)$ and on $\boldsymbol{I}$ coincide. Since A is triangular, the second Hochschild cohomology group vanishes (see [4, 10]). Thus $R/(\nu_R) \cong T(A)$. Then applying Proposition 3.4 we obtain that $R \cong \hat{A}$.

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Zygmunt Pogorzały
Faculty of Mathematics and Informatics
Nicholas Copernicus University
ul. Chopina 12/18, 87-100 Toruń
Poland
e-mail: zypo@mat.uni.torun.pl