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QUASILINEAR HYPERBOLIC OPERATORS WITH THE CHARACTERISTICS OF VARIABLE MULTIPLICITY

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0. Introduction

For quasilinear strictly hyperbolic operators the Cauchy problem is investigated in many papers and books (see, for instance, [1], [7], [11], [16], [21], [22]). Although there are a lot of interesting open problems for the strictly hyperbolic operators, nevertheless we reckon that it is important also to investigate quasilinear operators with characteristics of variable multiplicity.

For that type linear operators considered in Sobolev spaces very important are *Levi conditions*. These conditions are found out for many classes of such operators from the point of view of well-posedness in the Cauchy problem. At the same time almost nothing is known for quasilinear case. Moreover, even the role of hyperbolicity in the quasilinear Cauchy problem is not clear. We mean that even an analog of the Lax-Mizohata theorem for the quasilinear operators is not found out and is not proved still (see, Example 0.1 below).

Levi conditions are very closed to hyperbolicity. This is clear due to Garding hyperbolicity condition (Hadamard hyperbolicity condition, see, also [9] for Gevrey classes) for operators with constant coefficients, while for some classes of operators with variable coefficients and multiple characteristics it is noted in [23].

In [18] there is given example which hints at importance of the Levi conditions for the *stable global solvability* (see Definition 0.1, below) in the Cauchy problem for the quasilinear equations. For the second-order equations there also are given Levi conditions which are sufficient to the Cauchy problem to be stably globally solvable in the Gevrey classes.

We consider an equation

$$(0.1)\qquad D_t^m u + \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)u\})c_{j,\alpha}(t,D_t,D_x)u$$
$$= F(t,x,\{c_{k,\beta}(t,D_t,D_x)u\})$$

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with smooth coefficients $a_{j,\alpha}(t,x,z) \in C^\infty(J \times \boldsymbol{R}^{n_z})$ and right-hand side $F \in C^\infty(J \times \boldsymbol{R}^{n_z})$, where $t \in J := [0,T]$, $x \in \boldsymbol{R}^n$, and assume that all the functions are periodic in x, therefore we will write also $x \in \boldsymbol{T}^n$. Here $\boldsymbol{T}^n$ is an n-dimensional torus, though any compact could be treated with minor modifications, as well as the case $\boldsymbol{T}^n = \boldsymbol{R}^n$. Operators $c_{j,\alpha}(t,D_t,D_x)$ in (0.1) are the following

$$(0.2) \qquad c_{j,\alpha}(t,D_t,D_x)u = \begin{cases} \lambda^{m-j}(t)\Lambda^{j+|\alpha|-m}(t)D_t^jD_x^\alpha u, & \text{when } |\alpha| \neq 0; \\ D_t^ju, \quad \text{when } |\alpha| = 0, \end{cases}$$

and $c_{k,\beta}(t,D_t,D_x)u$ is defined by (0.2), too, where $j+|\alpha| \leq m$, while $k+|\beta| \leq m-1$. It is clear that lower order terms of the left-hand side of (0.1) can be included in the right-hand side.

We describe the class of the operators of (0.1) by means of a real-valued function $\lambda \in C^2([0,T])$ such that $\lambda(0) = \lambda'(0) = 0$, $\lambda'(t) > 0$ when $t \neq 0$. In the following λ' means $d\lambda/dt$. For $\lambda(t)$ we define $\Lambda(t) = \int_0^t \lambda(r)\,dr$ and assume that

$$(0.3) \qquad \begin{aligned} &c|\lambda(t)/\Lambda(t)| \leq |\lambda'(t)/\lambda(t)| \leq c_0|\lambda(t)/\Lambda(t)|, \\ &|\lambda^{(2)}(t)| \leq c_0|\lambda'(t)\lambda(t)/\Lambda(t)| \qquad \text{for all } t \in (0,T], \end{aligned}$$

with a positive constants c, c_0, where $c > (m-1)/m$.

The simplest examples of the functions, satisfying (0.3) are the following:

$$\lambda(t) = t^l, \quad \lambda(t) = \exp(-|t|^{-r}), \quad \lambda(t) = \exp(-\underbrace{\exp\cdots\exp}_{k}|t|^{-r}),$$

where l, k are integer numbers, $k \geq 2$ and $l > m-1$, while r is a positive number.

We assume that all the roots $\tau_1(t,x,w,\xi),\dots,\tau_m(t,x,w,\xi)$ of the equation

$$(0.4) \qquad \tau^m + \sum_{j+|\alpha|=m,\, j<m} a_{j,\alpha}(t,x,w)\tau^j\xi^\alpha = 0$$

are real and distinct for all $t \in [0,T]$, $x \in \boldsymbol{T}^n$, $\xi \in \boldsymbol{R}^n\backslash 0$ when w belongs to any compact set.

Thus the equation (0.1) has characteristics $\lambda(t)\tau_1(t,x,w,\xi),\dots,\lambda(t)\tau_m(t,x,w,\xi)$ which coincide at $t = 0$ while for $t \neq 0$ they are distinct. That is why equation (0.1) is said to be an equation with the characteristics of *variable multiplicity*.

For equation (0.1) consider the Cauchy problem

$$(0.5) \qquad D_t^lu(0,x) = g_l(x), \quad l = 0,\dots,m-1.$$

The main theorem of the present paper is the following

THEOREM 0.1. *Assume that for the operator* (0.1) *the above mentioned conditions are satisfied. Then there are non-negative numbers γ and M such that for $g_l(x) \in H^{\gamma+M+m-l}(\boldsymbol{T}^n)$, $l = 0, \dots, m-1$, for T_1 sufficiently small, there is a solution $u(t,x)$,*

$$D_t^l u \in C([0,T_1]; H^{M+m-l}(\boldsymbol{T}^n)), \quad l = 0, \dots, m-1, \tag{0.6}$$

to the Cauchy problem (0.1), (0.5).

For γ and M sufficiently large a solution $u(t,x)$ is unique.

In [12] is studied how much regularity of initial data (M in our notations) is needed to ensure existence of a local solution to a semi-linear wave equation.

For linear weakly hyperbolic Cauchy problem there is very developed theory which allows to prove that hyperbolicity and Levi condition are necessary for the problem to be C^∞ well-posed (See, for example, [4]). That theory is based on the closed graph theorem and on the constructions of the geometrical optics. Unfortunately, for nonlinear equations to apply that approach is very difficult (See, for details, [1]). But if we replace C^∞ well-posedness by the following *stable global solvability* concept, then, for some examples, at least, we can prove necessity of the Levi conditions as well as the Lax-Mizohata theorem.

DEFINITION 0.1. *Let S_{coeff}, S_F and S_{id} be spaces for coefficients $a_{j,\alpha}$, right-hand side F and initial data $\{g_l\}_0^{m-1}$, respectively. The Cauchy problem* (0.1), (0.5) *is said to be stably globally solvable in the space S_{sol}, in the neighbourhood of the solution $u(t,x)$ to that problem with given functions $a_{j,\alpha}$, F, g_l, if there are neighbourhoods $\Omega_{\mathrm{coeff}} \subset S_{\mathrm{coeff}}$, $\Omega_F \subset S_F$ and $\Omega_{id} \subset S_{id}$, of $a_{j,\alpha}$, F and $\{g_l\}_0^{m-1}$, respectively, and positive number T such that for every $\tilde{a}_{j,\alpha} \in \Omega_{\mathrm{coeff}}$, $\tilde{F} \in \Omega_F$ and $\{\tilde{g}_l\}_0^{m-1} \in \Omega_{id}$, a Cauchy problem*

$$D_t^m u + \sum_{j+|\alpha| \le m, j<m} \tilde{a}_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)u\}) c_{j,\alpha}(t,D_t,D_x)u$$

$$= \tilde{F}(t,x,\{c_{k,\beta}(t,D_t,D_x)u\})$$

$$D_t^l u(0,x) = \tilde{g}_l(x), \quad l = 0, \dots, m-1.$$

has a unique solution $\tilde{u}(t,x)$ defined for all $t \in [0,T]$.

The next example shows that in general without hyperbolicity not in the neighbourhood of every solution the Cauchy problem is stably globally solvable.

EXAMPLE 0.1. Consider an equation

$$u_{tt} + t^{2l} u_{xx} + t^{2l} (u_x)^2 + (u_t)^2 = 0. \tag{0.7}$$

There is a smooth solution $u = \tilde{u}(t,x)$ to the Cauchy problem with initial data $\tilde{u}(0,x) = 1$, $\tilde{u}_t(0,x) = 0$, defined for all $t \geq 0$. That is a function $\tilde{u}(t,x) = 1$. This solution is unique in C^∞ space.

Then, if $u = u(x,t)$ is a solution to (0.7), then $v(x,t) = \exp u(x,t)$ solves the following linear equation

$$v_{tt} + t^{2l} v_{xx} = 0. \tag{0.8}$$

Let us look for a real-valued solutions $v^{(n)}$ of this equation, of the form $v^{(n)}(x,t) = a^{(n)}(t)\cos(nx) + b^{(n)}(t)\sin(nx)$. Consequently, the function $w^{(n)}(t) = a^{(n)}(t) + ib^{(n)}(t)$ is a solution of the equation

$$w_{tt}^{(n)}(t) - t^{2l} n^2 w^{(n)}(t) = 0. \tag{0.9}$$

Add the initial conditions $w^{(n)}(0) = 0$, $w_t^{(n)}(0) = \rho^{(n)}$ ($\rho^{(n)}$ is real) which are equivalent to $a^{(n)}(0) = b^{(n)}(0) = 0$, $a_t^{(n)}(0) = \rho^{(n)}$, $b_t^{(n)}(0) = 0$, where the sequence $\{\rho^{(n)}\}$ will be chosen later.

Further, a function

$$W(t,n) = te^{\omega t^{l+1} n} F(1; 2-\gamma; -2\omega t^{l+1} n), \quad \omega := 1/(l+1), \quad \gamma := l\omega,$$

is a solution to (0.9) with initial data

$$W(0,n) = 0, \quad W_t'(0,n) = 1. \tag{0.10}$$

Here $F(\alpha;\gamma;z)$ is the solution of the Kummer's equation [2] and is represented in the following form

$$F(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \frac{1}{(1-e^{2\pi i(\gamma-\alpha)})(1-e^{2\pi i\alpha})} \times \int_C^{(1+,0+,1-,0-)} e^{z\zeta} \zeta^{\alpha-1} (1-\zeta)^{\gamma-\alpha-1} \, d\zeta.$$

The function $F(\alpha;\gamma;z)$ is an entire analytic function with respect to z.

Thus we have

$$w^{(n)}(t) = \rho^{(n)} te^{\omega t^{l+1} n} F(1; 2-\gamma; -2\omega t^{l+1} n). \tag{0.11}$$

According to Sec. 6.13.1 [2] the function $F(\alpha;\gamma;z)$ has the following asymptotic behavior as $\operatorname{Re} z \to -\infty$:

$$F(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}[1+O(|z|^{-1})]. \tag{0.12}$$

Let us suppose, that for some n which will be chosen later, $u(x,t)$ is a solution of (0.7) with the data $u(x,0)=1$ and $u_t(x,0)=e^{-1}\rho^{(n)}\cos(nx)$. Then $v(x,t)=\exp u(x,t)$ is a solution of (0.8) with initial data $v(x,0)=e$ and $v_t(x,0)=\rho^{(n)}\cos(nx)$. On the other hand the function $e+v^{(n)}(x,t)$ is a solution of (0.8) with the same initial values. Uniqueness in the Cauchy problem (Holmgren's theorem) for (0.8) implies, that $v(x,t)=e+v^{(n)}(x,t)$.

Furthermore, if $\rho^{(n)}=-n^{-s_1}$, then for every $s<s_1$ the sequence $\sup_{k\le s, x\in[0,2\pi]}|D_x^k e^{-1}\rho^{(n)}\cos(nx)|$ tends to 0 when n tends to infinity. At the same time for any fixed $t>0$ there is a point x such that $v^{(n)}(x,t)$ tends to $-\infty$ as $n\to\infty$. That brings blowup of the function $u(x,t)$.

Thus, for every time interval $[0,\delta]$ and for every ε-neighbourhood of the initial data $(1,0)$ in the space C^s there exist initial data from this neighbourhood such that the solution of (0.7) does not exist in $C^2([0,\delta]\times \boldsymbol{T})$.

As it is shown in [18], one can not expect stability of global solution to weakly hyperbolic equation in Gevrey classes, if lower order terms do not satisfy some conditions, usually called *Levi conditions*. Below we use the example and arguments of [18] to show that stable global solvability does not hold in general in Sobolev spaces, too, if these conditions are not satisfied. For equation (0.1) that conditions mean a special form (0.2) of the operators $c_{j,\alpha}(t,D_t,D_x)$.

EXAMPLE 0.2. Let us consider the equation (b is real)

$$u_{tt}-t^{2j}u_{xx}-bt^k u_x - t^{2j}(u_x)^2+(u_t)^2=0. \tag{0.13}$$

If $k<j-1$, $j>1$, then the Levi condition (0.2) (see, also, [14], [15], [23], [18]) is not satisfied. If $u=u(x,t)$ is a solution, then $v(x,t)=\exp u(x,t)$ is a solution of the following linear equation

$$v_{tt}-t^{2j}v_{xx}-bt^k v_x=0. \tag{0.14}$$

Again, as in the first example we seek for real-valued solutions $v^{(n)}$ of this equation of the form $v^{(n)}(x,t)=a^{(n)}(t)\cos(nx)+b^{(n)}(t)\sin(nx)$. Consequently, function $c^{(n)}(t)=a^{(n)}(t)+ib^{(n)}(t)$ is a solution of the equation

$$c_{tt}^{(n)}(t)+t^{2j}n^2c^{(n)}(t)+ibt^k nc^{(n)}(t)=0.$$

Adding the initial conditions $c^{(n)}(0)=0$, $c_t^{(n)}(0)=\rho^{(n)}$ ($\rho^{(n)}$ is real) which are equivalent to $a^{(n)}(0)=b^{(n)}(0)=0$, $a_t^{(n)}(0)=\rho^{(n)}$, $b_t^{(n)}(0)=0$, where the sequence

$\{\rho^{(n)}\}$ will be chosen later, then according to [19], for $c^{(n)}$ the following representation holds:

$$(0.15) \qquad c^{(n)}(t) = \rho^{(n)} \sum_{m=1}^{2} a_m(t) n^{-1} \exp[C_m n^\sigma + in(-1)^m t^{j+1}/(j+1)](1+o(1)),$$

where $\sigma = (j-k-1)/(2j-k)$, $a_m(t) \neq 0$ and the real part of at least one C_m is positive.

To consider a global solution $\tilde{u}(x,t) \equiv 1$ we repeat an argument which has been used in Example 0.1. Furthermore, if $\rho^{(n)} = n^{-s_1}$, then for every $s < s_1$ the sequence $\rho^{(n)} \sup_{k \leq s, x \in P} |D_x^k \cos(nx)|$ tends to 0 when n tends to infinity. At the same time $n^{-1}\rho^{(n)} \exp(C_m n^\sigma)$ tends to infinity when $\operatorname{Re} C_m > 0$.

Thus, by (0.15) for every time interval $[0,\delta]$ and for every ε-neighbourhood of the initial data $(1,0)$ in the space C^s there exist initial data from this neighbourhood such that the solution of (0.13) does not exist in $C^2([0,\delta] \times \boldsymbol{T})$.

We note that according to Theorem 0.1 the Cauchy problem for (0.13) with $k = j-1$ is locally solvable in Sobolev spaces. (See, also, [17], where a second-order equation with linear principal part independent of x, is considered.)

REMARK 0.1. We emphasize that to get a contradiction in the above both examples we used blowup phenomenon appearing in nonlinear equations, instead of the closed graph theorem and an a priori estimate.

Local solvability, established in Theorem 0.1 leads to stable global solvability at the neighbourhood of the sufficiently smooth solutions to equation (0.1). This is an essence of the following theorem.

THEOREM 0.2. *Assume that for the operator* (0.1) *above mentioned conditions are satisfied. Then there are non-negative numbers γ and M such that the Cauchy problem* (0.1), (0.5) *is stably globally solvable at the neighbourhood of the every solution* $u \in \bigcap_{l=0}^{m-1} C^l([0,T]; H^{\gamma+M+m-l}(\boldsymbol{T}^n))$ *in the spaces* $H^{\gamma+M+m-l}(\boldsymbol{T}^n)$ *for* $g_l(x)$, $l = 0,\dots,m-1$, C^∞ *for* $a_{j,\alpha}$ *(with real roots of* (0.4)*) and* F, *while* $\bigcap_{l=0}^{m-1} C^l([0,T]; H^{M+m-l}(\boldsymbol{T}^n))$ *for the solutions* $u(t,x)$.

The last theorem is a simple consequence of Theorem 0.1. Indeed, it is enough only to take into account strict hyperbolicity of the operators of (0.1) when $t > 0$.

REMARK 0.2. We do not know whether the Cauchy problem (0.1), (0.5) is stably globally solvable at the neighbourhood of the solution $u \in \bigcap_{l=0}^{m-1} C^l([0, T_1]; H^{M+m-l}(\boldsymbol{T}^n))$ which is not smooth enough, or not.

1. Linear equation with non-smooth coefficients

In order to treat the problem (0.1), (0.5) we will deal with pseudodifferential operators with less than C^∞ symbols. We use the following symbol classes (see [21]):

DEFINITION 1.1. *We say $p(x, \xi) \in H^M S_{1,0}^m$ provided that*

$$\|D_\xi^\alpha p(\cdot, \xi)\|_{H^M(\boldsymbol{T}^n)} \leq C\langle\xi\rangle^{m-|\alpha|}, \qquad \text{for all } |\alpha| \leq M.$$

One can find main properties of such operators in Ch. IV [21]. We write some of them here for the sake of completeness.

LEMMA 1.1. *If $p(x, \xi) \in H^M S_{1,0}^0$ and $M > n/2$, then $p(x, D_x) : L^2(\boldsymbol{T}^n) \to L^2(\boldsymbol{T}^n)$. More generally, $p(x, D_x) : H^s(\boldsymbol{T}^n) \to H^s(\boldsymbol{T}^n)$ for $|s| \leq \mu$, provided that $M > n/2 + \mu$.*

LEMMA 1.2. *Given any M, m_1, m_2 there is a μ such that, if $p_j(x, D) \in OPH^\mu S_{1,0}^{m_j}$, then $p_1(x, D)p_1(x, D) \in OPH^M S_{1,0}^{m_1+m_2}$, $p_1(x, D)^* \in OPH^M S_{1,0}^{m_1}$, and*

$$[p_1(x, D), p_1(x, D)] \in OPH^M S_{1,0}^{m_1+m_2-1}.$$

Firstly we consider a linear case. For the linear system (or scalar equation) one can not use results of [20] and [23] because the coefficients are not C^∞. On the other hand one can not apply immediately a result of [15], too, because conditions of that paper differ from our ones (0.2). Nevertheless, we use approaches of [15] and [21] to prove the result of this section.

We consider the equation

$$D_t^m u + \sum_{j+|\alpha| \leq m, j<m} a_{j,\alpha}(t, x) c_{j,\alpha}(t, D_t, D_x) u = f(t, x) \tag{1.1}$$

with coefficients $a_{j,\alpha}(t, x) \in C^1([0, T]; H^{\gamma+M+s}(\boldsymbol{T}^n))$, $f \in C([0, T]; H^{\gamma+s}(\boldsymbol{T}^n))$. The operators $c_{j,\alpha}(t, D_t, D_x)$ in (1.1) are the following

$$c_{j,\alpha}(t, D_t, D_x)u = \begin{cases} \lambda^{m-j}(t)\Lambda^{j+|\alpha|-m}(t)D_t^j D_x^\alpha u, & \text{when } |\alpha| \neq 0; \\ D_t^j u, & \text{when } |\alpha| = 0, \end{cases}$$

where $j + |\alpha| \le m$. For equation (1.1) consider the Cauchy problem

$$(1.2) \qquad D_t^l u(0, x) = g_l(x), \quad l = 0, \dots, m-1,$$

with $g_l(x) \in H^{\gamma-l}(\boldsymbol{T}^n)$, $l = 0, \dots, m-1$.

THEOREM 1.1. *Assume that all the roots* $\tau_1(t, x, \xi), \dots, \tau_m(t, x, \xi)$ *of the equation*

$$(1.3) \qquad \tau^m + \sum_{j+|\alpha|=m, j<m} a_{j,\alpha}(t, x)\tau^j \xi^\alpha = 0$$

are real and distinct for all $t \in [0, T]$, $x \in \boldsymbol{T}^n$, $\xi \in \boldsymbol{R}^n \backslash 0$. *Then there are non-negative numbers* γ *and* M *such that for* $g_l(x) \in H^{\gamma+s+m-l}(\boldsymbol{T}^n)$, $l = 0, \dots, m-1$, $f \in C([0, T]; H^{\gamma+s}(\boldsymbol{T}^n))$ *for* T_1 *sufficiently small, there is a unique solution* $u(t, x)$,

$$D_t^l u \in C([0, T_1]; H^{s+m-l}(\boldsymbol{T}^n)), \quad l = 0, \dots, m-1,$$

to the Cauchy problem (1.1), (1.2). *This solution satisfies the estimate*

$$(1.4) \quad \sum_{l=0}^{m-1} \|D_t^l u(t)\|_{H^{s+m-l}(\boldsymbol{T}^n)} \le C_s \left(\sum_{l=0}^{m-1} \|g_l(x)\|_{H^{\gamma+s+m-l}(\boldsymbol{T}^n)} + \int_0^t \|f(\tau)\|_{H^{\gamma+s}(\boldsymbol{T}^n)} \, d\tau \right)$$

for all $t \in [0, T]$.

PROOF. Let

$$u_{int}(t, x) = g_0(x) + tg_1(x) + \frac{t^2}{2!} g_2(x) + \cdots + \frac{t^{m-1}}{(m-1)!} g_{m-1}(x),$$

then a function $v(t, x) = u(t, x) - u_{int}(t, x)$ solves

$$D_t^m v + \sum_{j+|\alpha| \le m, j<m} a_{j,\alpha}(t, x)c_{j,\alpha}(t, D_t, D_x)v = f_0(t, x)$$

and

$$D_t^l v(0, x) = 0, \quad l = 0, \dots, m-1,$$

with $f_0 = f - L[u_{int}]$, where L is an operator of the left-hand side of (1.1).

Let $u^{(0)}$ be a solution of the following the Cauchy problem for an ordinary differential equation

$$D_t^m u^{(0)} + \sum_{|\alpha|=0, j<m} a_{j,\alpha}(t,x)c_{j,\alpha}(t,D_t,D_x)u^{(0)} = f_0(t,x)$$

with parameter $x \in \boldsymbol{T}^n$ and initial data

$$D_t^l u^{(0)}(0,x) = 0, \quad l = 0,\dots,m-1.$$

On the other hand one can regard the last Cauchy problem as an equivalent problem for a new unknown vector-valued function $\mathscr{U}^{(0)} :=^t (\mathscr{U}_1^{(0)},\dots,\mathscr{U}_m^{(0)}) := {}^t(u^{(0)}, D_t u^{(0)},\dots,D_t^{m-1}u^{(0)})$ and a linear symmetric hyperbolic system of first order. For the solution of the Cauchy problem with vanishing initial data, for the linear symmetric hyperbolic system there is an energy estimate which leads to the following one

$$\|\mathscr{U}^{(0)}(t)\|_s \le C_s \int_0^t \|f_0(\tau)\|_s \, d\tau.$$

If we denote

$$f_1 := -\sum_{|\alpha|\neq 0} a_{j,\alpha}(t,x)c_{j,\alpha}(t,D_t,D_x)u^{(0)}(t,x),$$

then

$$L[u - u_{int} - u^{(0)}] = f_1$$

and for every given $s \in \boldsymbol{R}$ we have an estimate

$$\begin{aligned}
\|f_1(t)\|_s &\le \sum_{|\alpha|\neq 0} \|a_{j,\alpha}(t,x)c_{j,\alpha}(t,D_t,D_x)u^{(0)}(t,x)\|_s \\
&\le C_s \lambda^\varepsilon(t) \sum_{|\alpha|\neq 0} \|a_{j,\alpha}(t,x)D_x^\alpha \mathscr{U}_{j+1}^{(0)}\|_s \\
&\le C_s' \lambda^\varepsilon(t) \int_0^t \|f_0(\tau)\|_{s+m} \, d\tau, \quad \varepsilon \in (0, m-(m-1)/c),
\end{aligned}$$

provided that γ is large enough.

Indeed, the function $\lambda^{m-j}(t)\Lambda^{j+|\alpha|-m}(t)$ (with $|\alpha| \neq 0$) can be majorized by $\lambda^\varepsilon(t)(\lambda^{m-\varepsilon}(t)\Lambda^{1-m}(t))$. The non-negative function $\lambda^{m-\varepsilon}(t)\Lambda^{1-m}(t)$ has a non-negative derivative due to condition $c > (m-1)/m$ in the inequality (0.3) when we choose $\varepsilon < m-(m-1)/c$. Hence, the function $\lambda^{m-\varepsilon}(t)\Lambda^{1-m}(t)$ is bounded.

One can consider recursively defined

$$D_t^m u^{(k)} + \sum_{|\alpha|=0, j<m} a_{j,\alpha}(t,x) c_{j,\alpha}(t, D_t, D_x) u^{(k)} = f_k(t,x),$$

$$D_t^l u^{(k)}(0,x) = 0, \quad l = 0, \dots, m-1,$$

$$f_{k+1} := -\sum_{|\alpha| \neq 0} a_{j,\alpha}(t,x) c_{j,\alpha}(t, D_t, D_x) u^{(k)}(t,x),$$

$$L[u - u_{int} - u^{(0)} - \cdots - u^{(k)}] = f_{k+1},$$

$k = 1, \dots, K-1$, where

$$\|f_{k+1}(t)\|_s \leq C'_{s,k} \lambda^\varepsilon(t) \int_0^t \|f_k(\tau)\|_{s+m} \, d\tau.$$

Hence, for every given N and s one has

$$f_K \in \lambda^N C([0,T]; H^s(\boldsymbol{T}^n)) \tag{1.5}$$

provided that K, and consequently γ, are large enough.

Thus, instead of (1.1), (1.2) one can look for the solution $w(t,x)$ of the Cauchy problem

$$D_t^m w + \sum_{j+|\alpha| \leq m, j<m} a_{j,\alpha}(t,x) c_{j,\alpha}(t, D_t, D_x) w = f_K(t,x), \tag{1.6}$$

$$D_t^l w(0,x) = 0, \quad l = 0, \dots, m-1, \tag{1.7}$$

with property (1.5). Above described reduction was used in [14], [15].

Further for the solution w of (1.6) we consider a vector-valued function

$$\mathscr{U} :=^t (\mathscr{U}^1, \dots, \mathscr{U}^m), \quad \mathscr{U}^k := \lambda^{m-k}(t) \langle D_x \rangle^{m-k} D_t^{k-1} w, \quad k = 1, \dots, m, \tag{1.8}$$

where $\langle D_x \rangle$ is a pseudo-differential operator with the symbol $\langle \xi \rangle$. For the $c_{j,\alpha}(t, D_t, D_x) w$ of (1.6) we have

$$c_{j,\alpha}(t, D_t, D_x) w = \begin{cases} \dfrac{\lambda}{\Lambda} \left(\dfrac{1}{\Lambda} \right)^{m-l-|\alpha|} \langle D_x \rangle^{l-m} D_x^\alpha \mathscr{U}^l, & \text{if } |\alpha| \neq 0, \ l = j+1; \\ \lambda^{l-m} \langle D_x \rangle^{l-m} \mathscr{U}^l, & \text{if } |\alpha| = 0, \ l = j+1, \end{cases} \tag{1.9}$$

where $|\alpha| + j \leq m$. Thus we obtain

$$D_t \mathscr{U}^m + \sum_{j+|\alpha| \leq m, j<m} a_{j,\alpha}(t,x) \Lambda^{j+|\alpha|-m}(t) D_x^\alpha \langle D_x \rangle^{j-m} \lambda(t) \langle D_x \rangle \mathscr{U}^{j+1} = f, \tag{1.10}$$

$$(1.11) \qquad \mathscr{U}(0,x) = 0.$$

On the other hand according to (1.8) we have

$$(1.12) \quad D_t\mathscr{U}^l(t,x) = \lambda(t)\langle D_x\rangle\mathscr{U}^{l+1}(t,x) - i(m-l)\frac{\lambda'(t)}{\lambda(t)}\mathscr{U}^l(t,x), \quad l = 1,\dots,m-1.$$

Due to the reduction in the beginning of the section and in view of the initial value, it follows

$$(1.13) \qquad \mathscr{U}^l(t,x) = i\langle D_x\rangle\lambda^{m-l}(t)\int_0^t \lambda^{l-m+1}(\tau)\mathscr{U}^{l+1}(\tau,x)\,d\tau, \quad l = 1,\dots,m-1.$$

Hence, in (1.10) for the terms with $m-j-1-|\alpha| \geq 1$ one can conclude

$$(1.14) \qquad \langle D_x\rangle^{j+1-m}D_x^\alpha\mathscr{U}^{j+1}(t,x) = i^{m-|\alpha|-j-1}\langle D_x\rangle^{-|\alpha|}D_x^\alpha \times \lambda^{m-j-1}(t)\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\cdots\int_0^{\tau_{m-|\alpha|-j-2}} \lambda^{-|\alpha|}(\tau_{m-|\alpha|-j-1}) \times \mathscr{U}^{m-|\alpha|}(\tau_{m-|\alpha|-j-1},x)\,d\tau_{m-|\alpha|-j-1}.$$

We will use for brevity the following writing

$$(1.15) \qquad \langle D_x\rangle^{j+1-m}D_x^\alpha\mathscr{U}^{j+1}(t,x) = i^{m-|\alpha|-j-1}\langle D_x\rangle^{-|\alpha|}D_x^\alpha\lambda^{m-j-1}(t) \times I^{m-|\alpha|-j-1}\lambda^{-|\alpha|}\mathscr{U}^{m-|\alpha|},$$

$(m-j-1-|\alpha| \geq 1)$ where I denotes integration with respect to time.

The equations (1.10), (1.12) and initial condition (1.11) can be rewritten as the following, equivalent to (1.1), (1.2), Cauchy problem

$$(1.16) \quad \frac{\partial\mathscr{U}}{\partial t} = K(t,x,D_x)\mathscr{U} + K_0(t,x,D_x)\mathscr{U} + K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U} + \mathscr{F}(t,x),$$

$$(1.17) \qquad \mathscr{U}(0,x) = 0,$$

where

$$(1.18) \qquad K(t,x,\xi) \in \lambda C^1([0,T];H^MS^1), \quad K_0(t,x,\xi) \in \frac{\lambda'}{\lambda}C([0,T];H^MS^0),$$

$$(1.19) \qquad \mathscr{F} \in \lambda^N C([0,T] : H^s(\boldsymbol{T}^n)),$$

while $K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)$ is the operator-valued matrix with the following elements:

(1.20) $(K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda))_{k,l}$

$$= \begin{cases} 0 \quad \text{when } k \neq m \text{ or } l = 1, \\ \sum_{|\alpha|=m-l} \sum_{j=0}^{l-2} i^{m-|\alpha|-j-1} a_{j,\alpha}(t,x) \langle D_x \rangle^{-|\alpha|} D_x^\alpha \Lambda^{j-l}(t) \lambda^{m-j}(t) I^{l-j-1} \lambda^{l-m}, \\ \quad \text{when } l = 2,\dots,m. \end{cases}$$

Equation (1.16) contains Fuchsian type operators if $\lambda(t) = t^l$, but in general, it is more singular.

For the symbol $K(t,x,\xi)$ there is a symmetrizer $R(t,x,\xi)$ in the sense of Sec. 5, ch. IV [21], that is if $K \in H^M S_{1,0}^1$, then there is $R \in H^M S_{1,0}^0$ with $(\partial/\partial t)R \in H^M S_{1,0}^0$, such that

$$(R(t,x,D_x)\mathscr{U},\mathscr{U}) \geq c_0 \|\mathscr{U}\|_{L_2}^2, \tag{1.21}$$

$$RK + K^* R \in H^M S_{1,0}^0 \tag{1.22}$$

for M sufficiently large.

To complete the proof of Theorem 1.1 it remains to prove the following

THEOREM 1.2. *Consider the Cauchy problem* (1.16), (1.17). *For every given* μ *there are* M *and* $N \in \mathbf{N}$ *such that if* (1.18), (1.19) *are satisfied and* $R \in C^1([0,T]; H^M S_{1,0}^0)$ *satisfies* (1.21), (1.22), *then the Cauchy problem* (1.16), (1.17) *has a unique solution* $\mathscr{U} \in \lambda^N C([0,T]; H^s(\mathbf{T}^n)) \cap \lambda' \lambda^{N-1} C^1([0,T]; H^{s-1}(\mathbf{T}^n))$, *for given* $\mathscr{F} \in \lambda^N C([0,T]; H^s(\mathbf{T}^n))$, *and* $|s| \leq \mu$. *Such solution satisfies the estimates, for* $|s| \leq \mu$,

$$\|\mathscr{U}(t)\|_{H^s} \leq c\lambda^N(t) \int_0^t \lambda^{-N}(\tau) \|\mathscr{F}(\tau)\|_{H^s}\, d\tau, \quad t \in [0,T]. \tag{1.23}$$

PROOF. Consider a sequence $\{\mathscr{U}_k\}_0^\infty$ defined as follows:

$$\frac{\partial \mathscr{U}_k}{\partial t} = K(t,x,D_x)\mathscr{U}_k + K_0(t,x,D_x)\mathscr{U}_{k-1} \tag{1.24}$$

$$+ K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U}_{k-1} + \mathscr{F}(t,x),$$

$$\mathscr{U}_k(0,x) = 0, \qquad \text{for } k = 1,2,\dots, \tag{1.25}$$

while $\mathscr{U}_0 := 0$. The function $\mathscr{U}_1 \in C([0,T]; H^s(\mathbf{T}^n)) \cap C^1([0,T]; H^{s-1}(\mathbf{T}^n))$ exists due to Proposition 5.4 [21] provided that M is large enough. Moreover, according to (5.4) [21] one has

$$(1.26)\quad \|\mathscr{U}_1(t)\|_{H^s} \le C\int_0^t \|\mathscr{F}(\tau)\|_{H^s}\,d\tau \le C\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau, \quad t\in[0,T].$$

Then the function $\mathscr{U}_2 \in C([0,T];H^s(\boldsymbol{T}^n)) \cap C^1([0,T];H^{s-1}(\boldsymbol{T}^n))$ exists, too, and

$$\begin{aligned}
\|\mathscr{U}_2(t)\|_{H^s} &\le C\int_0^t \|K_0(t,x,D_x)\mathscr{U}_1 + K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U}_1 + \mathscr{F}(\tau)\|_{H^s}\,d\tau \\
&\le C\int_0^t \|K_0(t,x,D_x)\mathscr{U}_1\|_{H^s}\,d\tau \\
&\quad + C\int_0^t \|K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U}_1\|_{H^s}\,d\tau + C\int_0^t \|\mathscr{F}(\tau)\|_{H^s}\,d\tau \\
&\le CC_0\int_0^t \frac{\lambda'(\tau)}{\lambda(\tau)}\|\mathscr{U}_1\|_{H^s}\,d\tau + C\int_0^t \|\mathscr{F}(\tau)\|_{H^s}\,d\tau \\
&\quad + C\int_0^t \|K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U}_1\|_{H^s}\,d\tau.
\end{aligned}$$

To estimate last integral of the right-hand side we remind assumption (0.3) where $c > (m-1)/m$, $m \ge 2$. For every $k \ge 2$ and every $l \ge 0$ it follows an inequality

$$(1.27)\quad \int_0^t \lambda^k(\tau)\Lambda^l(\tau)\,d\tau \le C_{k,l}\lambda^{k-1}(t)\Lambda^{l+1}(t), \quad C_{k,l} = \frac{m}{mk+ml+1-k} < 1,$$

so that $C_{k,l} \to 0$ when $k \to \infty$. In particular $C_{k,l} \le 2/(k+2l+1)$. Using (1.27) we estimate

$$\begin{aligned}
&\int_0^t \|K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U}_1\|_{H^s}\,d\tau \\
&\le C\sum_{l=2}^m\sum_{j=0}^{l-2}\int_0^t \Lambda^{j-l}(\tau)\lambda^{m-j}(\tau)\int_0^\tau d\tau_1\int_0^{\tau_1} d\tau_2\cdots\int_0^{\tau_{l-j-2}} d\tau_{l-j-1} \\
&\quad \times \lambda^{l-m}(\tau_{l-j-1})\|\mathscr{U}_1^l(\tau_{l-j-1})\|_{H^s} \\
&\le C\sum_{l=2}^m\sum_{j=0}^{l-2}\int_0^t \Lambda^{j-l}(\tau)\lambda^{m-j}(\tau)\int_0^\tau d\tau_1\int_0^{\tau_1} d\tau_2\cdots\int_0^{\tau_{l-j-2}} d\tau_{l-j-1} \\
&\quad \times \lambda^{N+l-m}(\tau_{l-j-1})\int_0^{\tau_{l-j-1}} \lambda^{-N}(\tau_{l-j})\|\mathscr{F}(\tau_{l-j})\|_{H^s}\,d\tau_{l-j}
\end{aligned}$$

$$\leq C\left(\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau\right)\sum_{l=2}^{m}\sum_{j=0}^{l-2}\int_0^t \Lambda^{j-l}(\tau)\lambda^{m-j}(\tau)$$

$$\times\int_0^\tau d\tau_1\int_0^{\tau_1} d\tau_2\cdots\int_0^{\tau_{l-j-2}}\lambda^{N+l-m}(\tau_{l-j-1})\,d\tau_{l-j-1}$$

$$\leq CC_N\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau,$$

where $C_N < 1$ and is decreasing function of N and m, while C is independent of N.

Thus

$$\|\mathscr{U}_2(t)\|_{H^s} \leq C\left(1+\frac{C_0}{N}+CC_N\right)\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau. \tag{1.28}$$

Moreover, $\mathscr{U}_2 \in \lambda^N C([0,T];H^s(\boldsymbol{T}^n))$ and $(\partial/\partial t)\mathscr{U}_2 \in \lambda'\lambda^{N-1}C([0,T];H^{s-1}(\boldsymbol{T}^n))$.

If we assume that

$$\|\mathscr{U}_k(t)\|_{H^s} \leq A_k(t)\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau \tag{1.29}$$

with a monotone continuous function $A_k(t)$, then by analogy with deriving (1.28), we obtain

$$A_{k+1}(t) \leq \left(\frac{CC_0}{N}+CC_N\right)A_k(t)+C$$

where $k=1,\dots,A_1=C$. According to condition of the theorem one can choose N large enough. Then for N sufficiently large the sequence $\{A_k\}$ is bounded. Hence,

$$\|\mathscr{U}_k(t)\|_{H^s} \leq C_\infty\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau \qquad \text{for all } t\in[0,T], \tag{1.30}$$

with some constant C_∞.

Moreover, due to our choice of N the sequence $\{\mathscr{U}_k\}_0^\infty$ is fundamental in the space $C([0,T];H^s(\boldsymbol{T}^n))\cap C^1([0,T];H^{s-1}(\boldsymbol{T}^n))$. Indeed, from the linear equations for $\mathscr{U}_{k+1}$ and $\mathscr{U}_k$ we obtain for difference $\mathscr{U}_{k+1}-\mathscr{U}_k$:

$$
\begin{aligned}
(1.31)\quad & \|\mathscr{U}_{k+1}(t)-\mathscr{U}_k(t)\|_{H^s} \\
& \le C\int_0^t \|K_0(\tau,x,D_x)(\mathscr{U}_k-\mathscr{U}_{k-1})\|_{H^s}\,d\tau \\
& \quad + C\int_0^t \|K_{int,pd}(\tau,x,D_x,\Lambda,\lambda,I,\lambda)(\mathscr{U}_k-\mathscr{U}_{k-1})\|_{H^s}\,d\tau \\
& \le \left(\frac{CC_0}{N}+CC_N\right)\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|(\mathscr{U}_k-\mathscr{U}_{k-1})(\tau)\|_{H^s}\,d\tau \le \cdots \\
& \le \left(\frac{CC_0}{N}+CC_N\right)^k\lambda^N(t)\int_0^t \lambda^{-N}(\tau)\|\mathscr{F}(\tau)\|_{H^s}\,d\tau, \quad k=1,2,\dots,
\end{aligned}
$$

where $((CC_0/N)+CC_N)<1$. The estimates for the derivatives $D_t(\mathscr{U}_{k+1}-\mathscr{U}_k)$ follows from (1.24), (1.25).

To prove uniqueness suppose that $\mathscr{U}$ solves

$$
(1.32)\qquad \frac{\partial\mathscr{U}}{\partial t}=K(t,x,D_x)\mathscr{U}+K_0(t,x,D_x)\mathscr{U}+K_{int,pd}(t,x,D_x,\Lambda,\lambda,I,\lambda)\mathscr{U},
$$

$$
(1.33)\qquad \mathscr{U}(0,x)=0.
$$

Then, according to Nersesian's lemma (see below Lemma 1.3)

$$
(1.34)\qquad \|\mathscr{U}(t)\|_{H^s}\le\left(\frac{CC_0}{N}+CC_N\right)^k\lambda^N(t)\int_0^t\lambda^{-N}(\tau)\|\mathscr{U}(\tau)\|_{H^s}\,d\tau,
$$

for any k, provided that conditions of that lemma are satisfied. It remains to take into account our choice N large enough and that T is small enough. The theorem is proved. □

Completion of the proof of Theorem 1.1. It remains only to prove uniqueness in the Cauchy problem (1.1), (1.2). To this end assume that $D_t^l u\in C([0,T_1];H^{s+m-l}(\boldsymbol{T}^n))$, $l=0,\dots,m-1$, and that

$$
D_t^m u+\sum_{j+|\alpha|\le m,\,j<m} a_{j,\alpha}(t,x)c_{j,\alpha}(t,D_t,D_x)u=0,
$$

$$
D_t^l u(0,x)=0,\quad l=0,\dots,m-1.
$$

It follows

$$
D_t^m u+\sum_{|\alpha|=0,\,j<m} a_{j,\alpha}(t,x)D_t^j u=-\sum_{j+|\alpha|\le m,\,|\alpha|\ne0} a_{j,\alpha}(t,x)c_{j,\alpha}(t,D_t,D_x)u.
$$

That means

$$\sum_{l=0}^{m-1} \|D_t^l u(t)\|_{H^s} \le C_s \int_0^t \Bigg\| \sum_{j+|\alpha|\le m, |\alpha|\ne 0} a_{j,\alpha}(\tau,x) c_{j,\alpha}(\tau, D_t, D_x) u(\tau,x) \Bigg\|_{H^s} d\tau$$

$$\le C_s \sum_{j+|\alpha|\le m, |\alpha|\ne 0} \int_0^t \lambda^{|\alpha|}(\tau) \left(\frac{\lambda(\tau)}{\Lambda(\tau)}\right)^{m-j-|\alpha|} \|a_{j,\alpha}(\tau,x) D_t^j D_x^\alpha u(\tau,x)\|_{H^s} d\tau$$

$$\le C_s \int_0^t \sum_{l=0}^{m-1} \|D_t^l u(\tau,x)\|_{H^{s+m}} \lambda(\tau) \left(\frac{\lambda(\tau)}{\Lambda(\tau)}\right)^{m-1} d\tau$$

$$\le C_s \lambda^{\varepsilon}(t) \int_0^t \sum_{l=0}^{m-1} \|D_t^l u(\tau,x)\|_{H^{s+m}} d\tau$$

due to condition (0.3) and to Lemma 1.1, provided that $D_t^l D_x^\alpha u \in H^s$, $l + |\alpha| \le m$, and that γ is large enough. Here s can be chosen negative as well. If we continue this procedure then for $|s|$ large enough, we get the right asymptotic behaviour at $t = 0$ which brings uniqueness, due to Theorem 1.2. The theorem is proved. □

REMARK 1.1. The constant C_s of the estimate (1.4) depends on finitely many seminorms of K, R and $(\partial/\partial t)R$, in $H^M S_{1,0}^1$ and $H^M S_{1,0}^0$, and of $RK + K^*R$ in $H^M S_{1,0}^1$, on T and on μ and on n. Further, M depends on μ and on n, but not on the order m of the system.

REMARK 1.2. *One can prove Theorem* 1.1 *with*

$$c_{j,\alpha}(t, D_t, D_x)u = \lambda^{m-j}(t) \Lambda^{j+|\alpha|-m}(t)\, |\ln \lambda(t)|^{m-j-|\alpha|} D_t^j D_x^\alpha u,$$

for coefficients $a_{j,\alpha}$ with $m - j - |\alpha| \ge 2$ instead of given in the beginning of this section. At that case N will depend on these coefficients, too.

For the sake of completeness we give here

LEMMA 1.3 (Nersesian) [14]. *Let us given the differential inequality*

$$y'(t) \le K(t)y(t) + f(t) \quad \textit{for all } t \in (0, T], \tag{1.35}$$

$T > 0$, where the functions $K = K(t)$ and $f = f(t)$ belong to $C(0, T]$. Under the assumptions

(1.36) $$\int_0^{\varepsilon} K(\tau)\,d\tau = \infty, \quad \int_{\varepsilon}^{T} K(\tau)\,d\tau < \infty \quad \textit{for every } \varepsilon \in (0,T),$$

(1.37) $$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{t} f(s)\exp\left(\int_s^t K(\tau)\,d\tau\right) ds \quad \textit{exists for every } t \in (0,T],$$

every solution $y \in C([0,T]) \cap C^1((0,T])$ *of* (1.35) *which posses the property*

(1.38) $$y(\varepsilon)\exp\left(\int_{\varepsilon}^{T} K(\tau)\,d\tau\right) = o(1) \quad \textit{as } \varepsilon \to 0,$$

satisfies an inequality

(1.39) $$y(t) \le \int_0^t e^{\int_s^t K(\tau)\,d\tau} f(s)\,ds \quad \text{for all } t \in (0,T].$$

2. Reduction to nonlinear term with right asymptotic at zero

In this section we carry out for nonlinear equation (0.1) the analogy of the procedure of the first part of the proof of Theorem 1.1. Let

$$u_{int}(t,x) = g_0(x) + tg_1(x) + \frac{t^2}{2!}g_2(x) + \cdots + \frac{t^{m-1}}{(m-1)!}g_{m-1}(x)$$

then a function $v(t,x) = u(t,x) - u_{int}(t,x)$ solves

(2.1) $$D_t^m v + \sum_{j+|\alpha|\le m,\, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(v+u_{int}(t,x))\})c_{j,\alpha}(t,D_t,D_x)v$$
$$= G(t,x,\{c_{k,\beta}(t,D_t,D_x)(v+u_{int}(t,x))\}).$$

Here the following notaion

(2.2) $$G(t,x,\{c_{k,\beta}(t,D_t,D_x)(v+u_{int}(t,x))\}) := F(t,x,\{c_{k,\beta}(t,D_t,D_x)(v+u_{int}(t,x))\})$$
$$- \sum_{j+|\alpha|\le m,\, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(v+u_{int}(t,x))\})c_{j,\alpha}(t,D_t,D_x)u_{int}(t,x)$$

is used. Moreover,

(2.3) $$D_t^l v(0,x) = 0, \quad l = 0,\dots,m-1.$$

Consider the solution $u^{(0)}(t,x)$ to the Cauchy problem

(2.4)
$$D_t^m u^{(0)} + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)} + u_{int}(t,x))\})c_{j,\alpha}(t,D_t,D_x)\varepsilon_{j,\alpha}u^{(0)}$$
$$= G(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)} + u_{int}(t,x))\}),$$

(2.5)
$$D_t^l u^{(0)}(0,x) = 0, \quad l = 0,\dots,m-1,$$

where $\varepsilon_{l,\gamma} = 1$ when $|\gamma| = 0$, while $\varepsilon_{l,\gamma} = 0$ when $|\gamma| \neq 0$, $l = k, j$, $\gamma = \alpha, \beta$. For every given s the solution $u^{(0)}$ exists uniquely in $\bigcap_{k-0}^m C^k([0,T_0];H^{s-k}(\boldsymbol{T}^n))$, $T_0 > 0$, provided that the constant γ of Theorem 0.1 is large enough.

Further, consider the solution $u^{(1)}(t,x)$ to the Cauchy problem

(2.6)
$$D_t^m u^{(1)} + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(1)} + u^{(0)}(t,x) + u_{int}(t,x))\})$$
$$\times c_{j,\alpha}(t,D_t,D_x)(\varepsilon_{j,\alpha}u^{(1)} + u^{(0)}(t,x))$$
$$= G(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(1)} + u^{(0)}(t,x) + u_{int}(t,x))\})$$
$$- G(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)}(t,x) + u_{int}(t,x))\})$$
$$+ \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)}(t,x) + u_{int}(t,x))\})$$
$$\times c_{j,\alpha}(t,D_t,D_x)\varepsilon_{j,\alpha}u^{(0)}(t,x),$$

(2.7)
$$D_t^l u^{(1)}(0,x) = 0, \quad l = 0,\dots,m-1.$$

The solution $u^{(1)}$ exists uniquely in $\bigcap_{k-0}^m C^k([0,T_1];H^{s-k}(\boldsymbol{T}^n))$, $T_1 > 0$, provided that γ is large enough.

Then we continue step by step and at last step we consider the solution $u^{(n)}(t,x)$ to the Cauchy problem

(2.8)
$$D_t^m u^{(n)} + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}\left(t,x,\left\{c_{k,\beta}(t,D_t,D_x)\left(\varepsilon_{k,\beta}u^{(n)} + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$
$$\times c_{j,\alpha}(t,D_t,D_x)\left(\varepsilon_{j,\alpha}u^{(n)} + \sum_{k=0}^{n-1} u^{(k)}(t,x)\right)$$

$$= G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta}u^{(n)} + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$- G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta}u^{n-1} + \sum_{k=0}^{n-2} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$+ \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta}u^{n-1} + \sum_{k=0}^{n-2} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times c_{j,\alpha}(t, D_t, D_x)\left(\varepsilon_{j,\alpha}u^{n-1} + \sum_{k=0}^{n-2} u^{(k)}(t,x)\right),$$

(2.9) $$D_t^l u^{(n)}(0,x) = 0, \quad l = 0, \ldots, m-1.$$

The solution $u^{(n)}$ exists uniquely in $\bigcap_{k-0}^{m} C^k([0,T_n]; H^{s-k}(\boldsymbol{T}^n))$, $T_n > 0$, provided that γ is large enough.

At last consider the solution $w(t,x)$ to the Cauchy problem

(2.10)

$$D_t^m w + \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(w + \sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times c_{j,\alpha}(t, D_t, D_x)w$$

$$= G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(w + \sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$- G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta}u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$- \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(w + \sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times c_{j,\alpha}(t, D_t, D_x) \sum_{k=0}^{n} u^{(k)}(t,x)$$

$$+ \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta}u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times c_{j,\alpha}(t, D_t, D_x)\left(\varepsilon_{j,\alpha}u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x)\right),$$

$$D_t^l w(0,x) = 0, \quad l = 0, \dots, m-1. \tag{2.11}$$

Thus we have proved the following

LEMMA 2.1. *Assume that the functions* $u^{(k)}(t,x)$, $k = 0, \dots, n$ *are given by* (2.4) *to* (2.9). *If* $w(t,x)$ *solves* (2.10), (2.11) *then*

$$u(t,x) = w(t,x) + \sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x) \tag{2.12}$$

solves (0.1), (0.5) *and vice versa.*

The following lemma shows a benefit of the representation (2.12).

LEMMA 2.2. *For every given* N *numbers* n *and* γ *can be chosen such that if* $g_l(x) \in H^{\gamma+m-l}(T^n)$, $l = 0, \dots, m-1$, *then the right-hand side of* (2.10)

$$\begin{aligned}
f_n := {} & G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
& - G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
& - \sum_{j+|\alpha| \le m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
& \times c_{j,\alpha}(t, D_t, D_x) \sum_{k=0}^{n} u^{(k)}(t,x) \\
& + \sum_{j+|\alpha| \le m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\right.\right. \\
& \qquad \left.\left.\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
& \times c_{j,\alpha}(t, D_t, D_x)\left(\varepsilon_{j,\alpha} u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x)\right)
\end{aligned}$$

at $w = 0$ *has the following behaviour*

$$f_n = O(\lambda^N(t)) \qquad \text{as } t \to 0. \tag{2.13}$$

Moreover if w with

$$D_t^l w \in C([0,T]; H^{M-l}(\boldsymbol{T}^n)) \cap L^\infty([0,T]; H^{M+m-l}(\boldsymbol{T}^n)), \quad l = 0, \dots, m-1,$$

solves (2.10), (2.11) *with above mentioned* $g_l(x)$, *then*

$$\sum_{k=0}^{m-1} \|D_t^k w(t)\|_{H^{m-k-1}(\boldsymbol{T}^n)} = O(\lambda^N(t)) \qquad \text{as } t \to 0. \tag{2.14}$$

Proof. In contrast to the beginning of the proof of Theorem 1.1, we have in (2.4), (2.6), (2.8) instead of linear ordinary differential equations, nonlinear ones. Therefore we rewrite, for example (2.6), (2.7) as a Cauchy problem for a quasilinear symmetric hyperbolic system of the first order

$$\frac{\partial \mathscr{U}}{\partial t} = g(t, x, \mathscr{U}), \quad \mathscr{U}(0, x) = 0, \tag{2.15}$$

for $\mathscr{U} := {}^t(\mathscr{U}_1, \dots, \mathscr{U}_m) := {}^t(u^{(1)}, \partial_t u^{(1)}, \dots, \partial_t^{m-1} u^{(1)})$. This system can be handled similarly to (5.1.1) of Section 5.1 [22]. The only difference is that in our case the function $g(t, x, \mathscr{U})$ is not C^∞ smooth in its argument x. Nevertheless, the last circumstance does not bring any new difficulties. As a matter of fact to get a solution $\mathscr{U} \in C^1([0, T_1]; H^s(\boldsymbol{T}^n))$, $s > 1 + n/2$, it is enough to assume that $\partial_t^i \partial_{\mathscr{U}}^j g(t, x, \mathscr{U})$ belongs to $C([0, T_1] \times \boldsymbol{R}^m; H^s(\boldsymbol{T}^n))$ for all $i \le 1$ and all j (See Prop. 5.1.D [22]). But we need some additional estimates. Therefore we write

$$g(t, x, \mathscr{U}) = g_0(t, x) + G(t, x, \mathscr{U})\mathscr{U}.$$

Then, by means of Freidrichs mollifier J_ε, $\varepsilon \in (0, 1]$, consider a solution $\mathscr{U}_\varepsilon$ to the Cauchy problem

$$\frac{\partial \mathscr{U}_\varepsilon}{\partial t} = J_\varepsilon g_0(t, x) + J_\varepsilon G(t, x, J_\varepsilon \mathscr{U}_\varepsilon) J_\varepsilon \mathscr{U}_\varepsilon, \quad \mathscr{U}_\varepsilon(0, x) = 0.$$

According to discussion in Section 5.1 [22] (with $A_j(t, x, u) \equiv 0$ in (5.1.1) [22]) the solution $\mathscr{U}_\varepsilon$ exists for t in the interval independent of $\varepsilon \in (0, 1]$, and has a limit as $\varepsilon \to 0$ solving (2.15). Then, by means of Moser estimates and Sobolev imbedding theorem, by means of Bihari's [3] and Gronwall's inequality (see pp. 110–113 [22]) we obtain that with some positive B the following inequality

$$\|\mathscr{U}_\varepsilon(t)\|_s \le C_s \int_0^t \|g_0(\tau, x)\|_s \, d\tau$$

holds for all $t \in [0, B)$, $\varepsilon \in (0, 1]$. It follows an estimate

$$\|\mathscr{U}(t)\|_s \le C \int_0^t \|g_0(\tau,x)\|_s \, d\tau, \quad t \in [0,B), \tag{2.16}$$

for the solution $\mathscr{U}$.

Thus, to estimate $\sum_{k=0}^{m-1} \|D_t^k u^{(1)}(t)\|_{H^{m-1+s}(T^n)}$ we consider the following two integrals:

$$\begin{aligned}\int_0^t &\|G(t,x,\{c_{k,\beta}(t,D_t,D_x)(u^{(0)}(t,x)+u_{int}(t,x))\}) \\ &- G(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)}(t,x)+u_{int}(t,x))\})\|_s \, d\tau,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\int_0^t \Bigg\| &- \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(u^{(0)}(t,x)+u_{int}(t,x))\}) \\ &\times c_{j,\alpha}(t,D_t,D_x)(u^{(0)}(t,x)) \\ &+ \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)}(t,x)+u_{int}(t,x))\}) \\ &\times c_{j,\alpha}(t,D_t,D_x)\varepsilon_{j,\alpha}u^{(0)}(t,x)\Bigg\|_s \, d\tau.\end{aligned} \tag{2.18}$$

One can write

$$\begin{aligned}&G(t,x,\{c_{k,\beta}(t,D_t,D_x)(u^{(0)}(t,x)+u_{int}(t,x))\}) \\ &\quad - G(t,x,\{c_{k,\beta}(t,D_t,D_x)(\varepsilon_{k,\beta}u^{(0)}(t,x)+u_{int}(t,x))\}) \\ &= \int_0^1 \sum_{k,\beta,|\beta|\ne 0} G^{(k,\beta)}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\theta u^{(0)}(t,x) \\ &\quad + u_{int}(t,x))\})\, d\theta c_{k,\beta}(t,D_t,D_x)u^{(0)}(t,x).\end{aligned} \tag{2.19}$$

Now one has only to take into account the existence of multipliers $\lambda^{m-k}(t)\Lambda^{k+|\beta|-m}(t)$ and repeat the arguments of the beginning of the proof of Theorem 1.1:

$$\begin{aligned}\int_0^t &\|G(\tau,x,\{c_{k,\beta}(\tau,D_\tau,D_x)(u^{(0)}(\tau,x)+u_{int}(\tau,x))\}) \\ &- G(\tau,x,\{c_{k,\beta}(\tau,D_\tau,D_x)(\varepsilon_{k,\beta}u^{(0)}(\tau,x)+u_{int}(\tau,x))\})\|_s \, d\tau\end{aligned}$$

$$\leq C_s \int_0^t \int_0^1 \sum_{k,\beta,|\beta|\neq 0} \|G^{(k,\beta)}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\theta u^{(0)}(t,x)$$

$$+ u_{int}(t,x))\})\, d\theta\, c_{k,\beta}(t,D_t,D_x)u^{(0)}(t,x)\|_s\, d\tau$$

$$\leq C_s \int_0^t \int_0^1 \sum_{k,\beta,|\beta|\neq 0} \|G^{(k,\beta)}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\theta u^{(0)}(t,x)$$

$$+ u_{int}(t,x))\})\, d\theta\, D_t^k D_x^\beta u^{(0)}(\tau,x)\|_s \lambda^{m-k}(\tau)\Lambda^{k+|\beta|-m}(\tau)\, d\tau$$

$$\leq C_s' \lambda^{\varepsilon}(t) \int_0^1 \sum_{k,\beta,|\beta|\neq 0} \|G^{(k,\beta)}(t,x,\{c_{k,\beta}(t,D_t,D_x)(\theta u^{(0)}(t,x)$$

$$+ u_{int}(t,x))\})\, d\theta D_t^k D_x^\beta u^{(0)}(\tau,x)\|_s.$$

The integral (2.18) can be estimated in the same way. Moreover, the all other functions $u^{(k)}$ can be considered in the same manner. Thus, the asymptotic behaviour (2.13) can be achieved step by step.

To prove (2.14) one can write once more (2.10), (2.11) in the form

$$\frac{\partial \mathscr{W}}{\partial t} = G(t,x,\mathscr{W}) + G_0(t,x,\lambda(t)D_x\mathscr{W})\lambda(t)D_x\mathscr{W}, \quad \mathscr{W}(0,x) = 0,$$

then according to estimate (2.16),

$$\|\mathscr{W}(t)\|_s \leq C_s \int_0^t \|G_0(\tau,x,\lambda(\tau)D_x\mathscr{W})\lambda(\tau)D_x\mathscr{W}(\tau,x)\|_s\, d\tau, \quad t \in [0,B).$$

It remains to apply it step by step. The lemma is proved. □

Remark 2.1. Above described reduction is responsible for the loss of regularity, counted by γ, in Theorem 0.1, and connected with multiplicity of characteristics.

Remark 2.2. For the following is important, that the crucial constant in the differential inequality will be obtained below, is changing on small quantity in every step of above described reduction.

3. Reduction to the "first-order" system

According to the results of the previous section to solve the problem (0.1), (0.5) is equivalent to solve the problem (2.10), (2.11) for a function w. Further, for the solution w of (2.10) we consider a vector-valued function

$$\mathscr{U} := {}^t(\mathscr{U}^1, \dots, \mathscr{U}^m), \quad \mathscr{U}^k := \lambda^{m-k}(t)\langle D_x\rangle^{m-k} D_t^{k-1} w, \quad k = 1, \dots, m, \tag{3.1}$$

where the symbol of the operator $\langle D_x\rangle$ is $\langle\xi\rangle$. For $c_{k,\beta}(t, D_t, D_x)w$ of (2.10) we have

$$c_{k,\beta}(t, D_t, D_x)w = \begin{cases} \dfrac{\lambda}{\Lambda}\left(\dfrac{1}{\Lambda}\right)^{m-l-|\beta|} \langle D_x\rangle^{l-m} D_x^\beta \mathscr{U}^l, & \text{if } |\beta| \neq 0,\ l = k+1; \\ \lambda^{l-m}\langle D_x\rangle^{l-m}\mathscr{U}^l, & \text{if } |\beta| = 0,\ l = k+1, \end{cases} \tag{3.2}$$

where $|\beta| + l \leq m$. We will use a notation

$$c_{k,\beta}(t, D_t, D_x)w = \tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l. \tag{3.3}$$

Thus we obtain

$$\begin{aligned} D_t\mathscr{U}^m &+ \sum_{j+|\alpha| \leq m, j<m} \\ &a_{j,\alpha}\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\ &\times \Lambda^{j+|\alpha|-m}(t) D_x^\alpha \langle D_x\rangle^{j-m}\lambda(t)\langle D_x\rangle \mathscr{U}^{j+1} \\ &= G\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\ &- G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta}u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\ &- \sum_{j+|\alpha| \leq m, j<m} \\ &a_{j,\alpha}\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\ &\times c_{j,\alpha}(t, D_t, D_x) \sum_{k=0}^{n} u^{(k)}(t,x) \end{aligned} \tag{3.4}$$

$$+ \sum_{j+|\alpha| \le m, j<m}$$

$$a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times\, c_{j,\alpha}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n} u^{(k)}(t,x)\right),$$

(3.5) $$\mathscr{U}(0,x) = 0, \quad l = 0, \ldots, m-1.$$

In what follows we use (1.12) to (1.15) and for the sake of brevity the following writing

$$\langle D_x\rangle^{1-m} D_x^{\beta} \mathscr{U}^l(t,x) = i^{m-l-|\beta|} \langle D_x\rangle^{-|\beta|} D_x^{\beta} I^{m-l-|\beta|} \lambda^{-|\beta|} \mathscr{U}^{m-|\beta|}.$$

Thus

$$\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l = C_{l\beta}(t, Int, D_x)\mathscr{U},$$

where the operator-valued matrix is defined by

(3.6)
$$C_{l\beta}(t, Int, D_x)\mathscr{U} = \begin{cases} \dfrac{\lambda}{\Lambda}\left(\dfrac{1}{\Lambda}\right)^{m-l-|\beta|} \langle D_x\rangle^{-|\beta|} D_x^{\beta} I^{m-l-|\beta|} \lambda^{-|\beta|} \mathscr{U}^{m-|\beta|}, & \text{if } |\beta| \ne 0; \\ \lambda^{l-m} I^{m-l-1} \mathscr{U}^m, \quad \text{if } |\beta| = 0. \end{cases}$$

Lemma 3.1. *Let* $|\beta| \ne 0$ *and assume that* $\|\mathscr{U}^{m-|\beta|}(t)\|_{H^s} \le C\lambda^N(t)$. *Then*

(3.7) $$\|\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l(t)\|_{H^s} \le C_N \frac{\lambda(t)}{\Lambda^2(t)} \lambda^{m-l}(t) \int_0^t \lambda^{l-m+1}(\tau) \|\mathscr{U}^{m-|\beta|}(\tau)\|_{H^s}\, d\tau.$$

Proof. It is simple consequence of (1.13). □

On the other hand one can easily obtain the following

(3.8) $$G\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$- G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$
\begin{aligned}
&= G\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
&\quad - G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
&\quad + G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
&\quad - G\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
&= \sum_{l,\beta}\left(\int_0^1 G'_{(k,\beta)}\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l\right.\right.\right. \\
&\quad \left.\left.\left. + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) d\tau\right) \tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l \\
&\quad + \sum_{|\beta| \neq 0} (c_{k,\beta}(t, D_t, D_x) u^{(n)}(t,x)) \int_0^1 G'_{(k,\beta)}\left(t, x, \left\{\tau c_{k,\beta}(t, D_t, D_x) u^{(n)}(t,x)\right.\right. \\
&\quad \left.\left. + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) d\tau.
\end{aligned}
$$

Analogously

(3.9)

$$
\begin{aligned}
&\sum_{j+|\alpha| \le m, j<m} a_{j,\alpha}\left(t, x, \left\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l + c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
&\quad \times c_{j,\alpha}(t, D_t, D_x) \sum_{k=0}^{n} u^{(k)}(t,x) \\
&\quad - \sum_{j+|\alpha| \le m, j<m} a_{j,\alpha}\left(t, x, \left\{c_{k,\beta}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right) \\
&\quad \times c_{j,\alpha}(t, D_t, D_x)\left(\varepsilon_{k,\beta} u^{(n)}(t,x) + \sum_{k=0}^{n} u^{(k)}(t,x)\right)
\end{aligned}
$$

$$= \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}\Bigg(t, x, \Bigg\{ \tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l$$

$$+ c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\Bigg\}\Bigg)$$

$$\times\, c_{j,\alpha}(t, D_t, D_x)\sum_{k=0}^{n} u^{(k)}(t,x)$$

$$- \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}\left(t, x, \left\{ c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times\, c_{j,\alpha}(t, D_t, D_x)\sum_{k=0}^{n} u^{(k)}(t,x)$$

$$+ \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}\left(t, x, \left\{ c_{k,\beta}(t, D_t, D_x)\left(u^{(n)}(t,x) + \sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\right)\right\}\right)$$

$$\times\, c_{j,\alpha}(t, D_t, D_x)\sum_{k=0}^{n-1} u^{(k)}(t,x)$$

$$- \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}\Bigg(t, x, \Bigg\{ c_{k,\beta}(t, D_t, D_x)\Bigg(\varepsilon_{k,\beta} u^{(n)}(t,x)$$

$$+ \sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\Bigg)\Bigg\}\Bigg)$$

$$\times\, c_{j,\alpha}(t, D_t, D_x)\sum_{k=0}^{n} u^{(k)}(t,x)$$

$$= \sum_{l,\beta,l=k+1} \tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l \int_0^1 \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha(k,\beta)}\Bigg(t, x, \Bigg\{ \tau\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l$$

$$+ c_{k,\beta}(t, D_t, D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x) + u_{int}(t,x)\right)\Bigg\}\Bigg) d\tau$$

$$\times\, c_{j,\alpha}(t, D_t, D_x)\sum_{k=0}^{n} u^{(k)}(t,x)$$

$$+\sum_{|\beta|\neq 0} c_{k,\beta}(t,D_t,D_x)u^{(n)}(t,x)\int_0^1 \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha(k,\beta)}\left(t,x,\left\{\tau c_{k,\beta}(t,D_t,D_x)u^{(n)}(t,x)\right.\right.$$

$$\left.\left.+\,c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^{n-1} u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)d\tau\, c_{j,\alpha}(t,D_t,D_x)\sum_{k=0}^{n} u^{(k)}(t,x).$$

Finally we can rewrite (3.4) to (3.6) as a Cauchy problem for first order integro-differential system for a vector-valued function $\mathscr{U}$ in the following way

$$\frac{\partial \mathscr{U}}{\partial t}=K(t,x,\{C_{l\beta}(t,Int,D_x)\mathscr{U}\},D_x)\mathscr{U}+K_0(t)\mathscr{U} \tag{3.10}$$

$$+\tilde{F}(t,x,\{C_{l\beta}(t,Int,D_x)\mathscr{U}\},D_x)\mathscr{U}+\tilde{F}_n(t,x),$$

$$\mathscr{U}(0,x)=0. \tag{3.11}$$

Here $K=K(t,\mathscr{U})$ is a family of pseudodifferential operators in $OPS^1(\boldsymbol{T}^n)$ depending on t and $\mathscr{U}$, $K_0(t)$ is a diagonal matrix with elements $(K_0(t))_{ll}=\{(m-l)\lambda'(t)/\lambda(y)\}_{l=1}^m$ while

$$\tilde{F}(t,x,\{C_{l\beta}(t,Int,D_x)\mathscr{U}\},D_x)\mathscr{V} \tag{3.12}$$

$$=-\sum_{j+|\alpha|\leq m-1}\Lambda(t)^{j+|\alpha|-m}\lambda(t)a_{j,\alpha}\left(t,x,\left\{\tilde{C}_{l\beta}(t,D_x)\mathscr{U}^l\right.\right.$$

$$\left.\left.+\,c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)\langle D_x\rangle^{j-m+1}D_x^\alpha\mathscr{V}^{j+1}$$

$$+\sum_{l,\beta}(\tilde{C}_{l\beta}(t,D_x)\mathscr{V}^l)\int_0^1 G'_{(k,\beta)}\left(t,x,\left\{\tilde{C}_{l\beta}(t,D_x)\mathscr{U}^l\right.\right.$$

$$\left.\left.+\,c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)d\tau$$

$$-\sum_{l,\beta}(\tilde{C}_{l\beta}(t,D_x)\mathscr{V}^l)\int_0^1\sum_{j+|\alpha|\leq m, j<m}$$

$$\times\, a'_{j,\alpha(k,\beta)}\left(t,x,\left\{\tau\tilde{C}_{l\beta}(t,D_x)\mathscr{U}^l\right.\right.$$

$$\left.\left.+\,c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^{n} u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)d\tau$$

$$\times\, c_{j,\alpha}(t,D_t,D_x)\sum_{k=0}^{n} u^{(k)}(t,x),$$

$$(3.13)\quad \tilde{F}_n(t,x) = \sum_{l,\beta} c_{k,\beta}(t,D_t,D_x)u^{(n)}(t,x)\int_0^1 G'_{(k,\beta)}\Bigg(t,x,\Bigg\{\tau c_{k,\beta}(t,D_t,D_x)u^{(n)}(t,x)$$

$$+ c_{k,\beta}(t,D_t,D_x)\Bigg(\sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\Bigg)\Bigg\}\Bigg)d\tau.$$

$$- \sum_{|\beta|\neq 0} c_{k,\beta}(t,D_t,D_x)u^{(n)}(t,x)$$

$$\times \int_0^1 \sum_{j+|\alpha|\leq m, j<m} a_{j,\alpha(k,\beta)}\Bigg(t,x,\Bigg\{\tau c_{k,\beta}(t,D_t,D_x)u^{(n)}(t,x)$$

$$+ c_{k,\beta}(t,D_t,D_x)\Bigg(\sum_{k=0}^{n-1} u^{(k)}(t,x) + u_{int}(t,x)\Bigg)\Bigg\}\Bigg)d\tau$$

$$\times c_{j,\alpha}(t,D_t,D_x)\sum_{k=0}^{n} u^{(k)}(t,x).$$

4. Proof of Theorem 0.1

To prove Theorem 0.1 we use the following iterative method. For a given $\mathscr{U}(t,x)$ defined on $[0,T]\times \boldsymbol{T}^n$ with $\mathscr{U}(0,x)=0$ we define $Q\mathscr{U} := \mathscr{V}$ to be the solution to the system

$$(4.1)\quad \frac{\partial \mathscr{V}}{\partial t} = K(t,x,\{C_{l\beta}(t,I\lambda,D_x)\mathscr{U}\},D_x)\mathscr{V} + K_0(t)\mathscr{V}$$

$$+ \tilde{F}(t,x,\{C_{l\beta}(t,I\lambda,D_x)\mathscr{U}\},D_x)\mathscr{V} + \tilde{F}_n(t,x),$$

$$(4.2)\quad \mathscr{V}(0,x) = 0,$$

where

$$(4.3)\quad \tilde{F}_n \in \lambda^N C([0,T];H^s(\boldsymbol{T}^n)).$$

For the sake of simplicity of notations we consider the case of equation (0.1) with terms $c_{k,\beta}(t,D_t,D_x)u$, $k+|\beta| = m-1$, only. Moreover we assume that $a_{j,\alpha} = 0$ for $j+|\alpha|\leq m-2$. Thus we consider the following most important special case of equation (0.1):

$$(4.4)\quad D_t^m u + \sum_{m-1\leq j+|\alpha|\leq m, j<m} a_{j,\alpha}(t,x,\{c_{k,\beta}(t,D_t,D_x)u\})c_{j,\alpha}(t,D_t,D_x)u$$

$$= F(t,x,\{c_{k,\beta}(t,D_t,D_x)u\}),$$

where $k+|\beta|=m-1$. At that rate system (4.1) becomes

$$\frac{\partial \mathscr{V}}{\partial t}=K\left(t,x,\left\{\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^{k+1}\right\},D_x\right)\mathscr{V}+K_0(t)\mathscr{V} \tag{4.5}$$
$$+\tilde{F}\left(t,x,\left\{\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^{k+1}\right\},D_x\right)\mathscr{V}+\tilde{F}_n(t,x),$$

where

$$c_\beta(D_x)=\langle D_x\rangle^{-|\beta|}D_x^\beta \tag{4.6}$$

is a zero-order pseudodifferential operator, while

(4.7)
$$\tilde{F}\left(t,x,\left\{\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^{k+1}\right\},D_x\right)\mathscr{V}$$
$$=-\frac{\lambda(t)}{\Lambda(t)}\sum_{j+|\alpha|=m-1}a_{j,\alpha}\left(t,x,\left\{\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^{k+1}\right.\right.$$
$$\left.\left.+c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^n u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)\langle D_x\rangle^{-|\alpha|}D_x^\alpha\mathscr{V}^{j+1}$$
$$+\sum_{k+|\beta|=m-1}\left(\frac{\lambda'(t)}{\lambda(t)}\langle D_x\rangle^{-|\beta|}D_x^\beta\mathscr{V}^{k+1}\right)\int_0^1 G'_{(k,\beta)}\left(t,x,\left\{\tau\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^{k+1}\right.\right.$$
$$\left.\left.+c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^n u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)d\tau$$
$$-\sum_{k+|\beta|=m-1}\left(\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{V}^{k+1}\right)\int_0^1\sum_{j+|\alpha|\le m,j<m}a'_{j,\alpha(k,\beta)}\left(t,x,\left\{\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^{k+1}\right.\right.$$
$$\left.\left.+c_{k,\beta}(t,D_t,D_x)\left(\sum_{k=0}^n u^{(k)}(t,x)+u_{int}(t,x)\right)\right\}\right)d\tau\, c_{j,\alpha}(t,D_t,D_x)\sum_{k=0}^n u^{(k)}(t,x).$$

If we introduce a diagonal matrix of zero-order pseudodifferential operators $C_\beta(t,D_x)$ by

$$(C_\beta(t,D_x)\mathscr{U})_k:=\delta_{m,k+|\beta|}\frac{\lambda'(t)}{\lambda(t)}c_\beta(D_x)\mathscr{U}^k, \tag{4.8}$$

then for (4.1) we get more simple representation

$$\frac{\partial \mathscr{V}}{\partial t} = K(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{V} + K_0(t)\mathscr{V} + \tilde{F}(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{V} + \tilde{F}_n(t, x), \tag{4.9}$$

where

(4.10)

$$\begin{aligned}
&\tilde{F}(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{V} \\
&= -\frac{\lambda(t)}{\Lambda(t)} \sum_{j+|\alpha|=m-1} a_{j,\alpha}\Bigg(t, x, \Bigg\{\tilde{C}_{l\beta}(t, D_x)\mathscr{U}^l \\
&+ c_{k,\beta}(t, D_t, D_x)\Bigg(\sum_{k=0}^{n} u^{(k)}(t, x) + u_{int}(t, x)\Bigg)\Bigg\}\Bigg)\langle D_x\rangle^{-|\alpha|} D_x^\alpha \mathscr{V}^{j+1} \\
&+ \sum_{k+|\beta|=m-1} \left(\frac{\lambda'(t)}{\lambda(t)}\langle D_x\rangle^{-|\beta|} D_x^\beta \mathscr{V}^{k+1}\right) \int_0^1 G'_{(k,\beta)}\Bigg(t, x, \Bigg\{\tau\frac{\lambda'(t)}{\lambda(t)} c_\beta(D_x)\mathscr{U}^{k+1} \\
&+ c_{k,\beta}(t, D_t, D_x)\Bigg(\sum_{k=0}^{n} u^{(k)}(t, x) + u_{int}(t, x)\Bigg)\Bigg\}\Bigg) d\tau \\
&- \sum_{k+|\beta|=m-1} \left(\frac{\lambda'(t)}{\lambda(t)} c_\beta(D_x)\mathscr{V}^{k+1}\right) \int_0^1 \sum_{j+|\alpha|\le m, j<m} a'_{j,\alpha(k,\beta)}\Bigg(t, x, \Bigg\{\frac{\lambda'(t)}{\lambda(t)} c_\beta(D_x)\mathscr{U}^{k+1} \\
&+ c_{k,\beta}(t, D_t, D_x)\Bigg(\sum_{k=0}^{n} u^{(k)}(t, x) + u_{int}(t, x)\Bigg)\Bigg\}\Bigg) d\tau \\
&\times c_{j,\alpha}(t, D_t, D_x)\sum_{k=0}^{n} u^{(k)}(t, x).
\end{aligned}$$

The Cauchy problem (4.4), (4.2) can be handled in the same way as it is done in Sec. 5 Ch. IV [21]. The only difference is that instead of Gronwall's inequality one has to apply Nersesian's lemma.

Suppose that

$$\mathscr{U} \in \lambda^N C([0, T]; H^M(\boldsymbol{T}^n)) \quad \text{and} \quad (\partial/\partial t)\mathscr{U} \in \lambda'\lambda^{N-1} C([0, T]; H^{M-1}(\boldsymbol{T}^n)). \tag{4.11}$$

First of all, take M and N large enough so that there is a unique solution $\mathscr{V} \in \lambda^N C([0, T]; H^n(\boldsymbol{T}^n))$ and $(\partial/\partial t)\mathscr{V} \in \lambda'\lambda^{N-1} C([0, T]; H^n(\boldsymbol{T}^n))$. Say this happens if

$$M \ge M_1 \quad N \ge N_1. \tag{4.12}$$

To obtain more precise estimates on $\mathscr{V}$, it is convenient to obtain equations for various derivatives of $\mathscr{V}$. Indeed, set $\tilde{F}_n := F$ and

$$\mathscr{V}_{0\alpha} = D_x^\alpha \mathscr{V}, \quad \mathscr{V}_{1\alpha} = \frac{\partial}{\partial t} D_x^\alpha \mathscr{V}.$$

Similarly define $\mathscr{U}_{0\alpha}$, $\mathscr{U}_{1\alpha}$. Applying the chain rule to (4.1) yields

$$\begin{aligned}(4.13)\qquad \frac{\partial}{\partial t}\mathscr{V}_{0\alpha} &= K(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{V}_{0\alpha} + K_0(t)\mathscr{V}_{0\alpha}\\ &+ \tilde{F}(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{V}_{0\alpha} + \tilde{F}_{0\alpha}(t,x)\\ &+ \sum_{\substack{\gamma+\delta+\sigma=\alpha,\sigma<\alpha\\ \delta_1+\cdots+\delta_\mu=\delta}} C^K_{\sigma\gamma\delta_1\cdots\delta_\mu}\{C_\beta(t,D_x)\mathscr{U}\}_{0\delta_1}\cdots\{C_\beta(t,D_x)\mathscr{U}\}_{0\delta_\mu}\\ &\times K_{\gamma\mu}(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{V}_{0\sigma}\\ &+ \sum_{\substack{\gamma+\delta+\sigma=\alpha,\sigma<\alpha\\ \delta_1+\cdots+\delta_\mu=\delta}} C^F_{\sigma\gamma\delta_1\cdots\delta_\mu}\{C_\beta(t,D_x)\mathscr{U}\}_{0\delta_1}\cdots\{C_\beta(t,D_x)\mathscr{U}\}_{0\delta_\mu}\\ &\times \tilde{F}(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)_{\gamma\mu}\mathscr{V}_{0\sigma}.\end{aligned}$$

Here $K_{\gamma\mu} = D_x^\gamma D_z^\mu K(t,x,z,D_x)$, $\tilde{F}_{\gamma\mu} = D_x^\gamma D_z^\mu \tilde{F}(t,x,z,D_x)$. Now replace $\mathscr{V}_{j\sigma}$ by $P_{j\sigma}(\mathscr{V}^M)$, where

$$(4.14)\qquad \mathscr{V}^M = \{\mathscr{V}_{0\alpha}^M, \mathscr{V}_{1\tilde{\beta}}^{M-1} : 0 \le |\alpha| \le M, 0 \le |\tilde{\beta}| \le M-1\} := (\mathscr{V}_0^M, \mathscr{V}_1^{M-1})$$

and

$$(4.15)\qquad P_{j\sigma}(\mathscr{V}^M) = \langle D_x\rangle^{-(M-j-|\sigma|)} \sum_{|\beta|=M-j} c_{\sigma\beta}(x,D_x)\mathscr{V}_{j\beta}, \quad j=0,1,$$

$c_{\sigma\beta}(x,\xi) \in S^0$ being appropriately chosen. Thus $P_{j\sigma} \in S^{-(M-j-|\sigma|)}$. Similarly, replace $\mathscr{U}_{j\delta}$ by $P_{j\delta}(\tilde{\mathscr{U}})$ and

$$(4.16)\qquad F^M = (F_0^M, F_1^{M-1}) := (\tilde{F}_{0\alpha}, \tilde{F}_{1\tilde{\beta}}), \quad |\alpha| = M, \quad |\tilde{\beta}| = M-1.$$

We rewrite the system (4.13) as

$$\begin{aligned}(4.17)\qquad \frac{\partial}{\partial t}\mathscr{V}_0^M &= K(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{V}_0^M + K_0(t)\mathscr{V}_0^M\\ &+ \tilde{F}(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{V}_0^M + F_0^M(t,x) + \Phi_{0K}^M + \Phi_{0F}^M,\end{aligned}$$

where due to (4.11) we have

$$(4.18)\qquad \mathscr{U}^M = (\mathscr{U}_0^M, \mathscr{U}_1^M), \quad \mathscr{U}_0^M \in \lambda^N C([0,T]; L^2(\boldsymbol{T}^n)),$$
$$\mathscr{U}_1^M \in \lambda'\lambda^{N-1} C([0,T]; L^2(\boldsymbol{T}^n)).$$

LEMMA 4.1. *The mapping* $\tilde{Q} : \tilde{\mathscr{U}} \mapsto \tilde{\mathscr{V}}$ *defined by* (4.17) *maps the set*

$$(4.19)\qquad \left\{ \tilde{\mathscr{U}} \in (\lambda^N \times \lambda'\lambda^{N-1}) C([0,T]; L^2(\boldsymbol{T}^n)) : \frac{1}{\lambda^N} \|\mathscr{U}_{0\alpha}\| + \frac{1}{\lambda'\lambda^{N-1}} \|\mathscr{U}_{1\tilde{\beta}}\| \le A_0 \right\}$$

into itself for $T \le T_0$ *with* T_0 *sufficiently small and* N *sufficiently large.*

PROOF. Indeed, for $m \times K$ tuple system (4.17) one can construct a symmetrizer $R(t,x,z,D_x)$ for the block diagonal matrix operator $K(t,x,z,D_x)$ and substituting $\{C_\beta(t,D_x)\mathscr{U}\}$ for z, gives $R(t,x,(\lambda'/\lambda)C(D_x)P_{00}\tilde{\mathscr{U}},D_x) \in OPH^M S^0_{1,0}$, where $C(D_x)$ is a zero-order pseudodifferential operator corresponding to $c_\beta(D_x)$ in (4.8). One can write

$$(4.20)\qquad \frac{d}{dt}(R\mathscr{V}_0^M, \mathscr{V}_0^M)$$
$$= (R\mathscr{V}_0^M, (K\mathscr{V}_0^M + K_0\mathscr{V}_0^M + \tilde{F}\mathscr{V}_0^M + F_0^M + \Phi_{0K}^M + \Phi_{0F}^M))$$
$$+ (R(K\mathscr{V}_0^M + K_0\mathscr{V}_0^M + \tilde{F}\mathscr{V}_0^M + F_0^M + \Phi_{0K}^M + \Phi_{0F}^M), \mathscr{V}_0^M)$$
$$+ \left(\frac{\partial R}{\partial t}\mathscr{V}_0^M, \mathscr{V}_0^M\right)$$
$$\le C_0 \frac{\lambda'(t)}{\lambda(t)} B(\|\mathscr{U}_0^M\|_{L^2})(R\mathscr{V}_0^M, \mathscr{V}_0^M) + B(\|\mathscr{U}_1^{n/2+1}\|_{L^2})(R\mathscr{V}_0^M, \mathscr{V}_0^M)$$
$$+ \|\mathscr{V}_0^M\|_{L^2} \|F_0^M\|_{L^2},$$

where B is some function of its argument which we need not specify. It follows

$$(4.21)\qquad \|\mathscr{V}_0^M(t)\|_{L^2} \le C\lambda^{C_0 B(\|\mathscr{U}_0^M\|_{L^2})}(t) e^{B(\|\mathscr{U}^M\|_{L^2})t}$$
$$\times \int_0^t \lambda^{-C_0 B(\|\mathscr{U}_0^M\|_{L^2})}(s) e^{-B(\|\mathscr{U}^M\|_{L^2})s} \|F_0^M(s)\|_{L^2}\, ds$$

for all $t \in [0,T]$. We can choose $N > C_0 B(\|\mathscr{U}_0^M\|_{L^2})$ when T is small enough. (See, also, Remark 2.2.) From system (4.17) keeping in mind the relation $(\partial/\partial t)\mathscr{V}_0^{M-1} = \mathscr{V}_1^{M-1}$ we derive an estimate for $\|(\partial/\partial t)\mathscr{V}_0^{M-1}\|_{L^2}$:

$$(4.22)\quad \left\| \frac{\partial}{\partial t} \mathscr{V}_0^{M-1}(t) \right\|_{L^2}$$

$$\leq C\lambda(t) B'(\|\mathscr{U}_0^{n/2+1}\|_{L^2}) \|\mathscr{V}_0^M(t)\|_{L^2} + C\frac{\lambda'(t)}{\lambda(t)} \|\mathscr{V}_0^{M-1}(t)\|_{L^2}$$

$$+ C\frac{\lambda'(t)}{\lambda(t)} B'(\|\mathscr{U}_0^{n/2+1}\|_{L^2}) \|\mathscr{V}_0^{M-1}(t)\|_{L^2} + \|F_0^{M-1}(t)\|_{L^2}$$

$$+ C\lambda(t) B'(\|\mathscr{U}_0^M\|_{L^2}) \|\mathscr{V}_0^M(t)\|_{L^2} + C\frac{\lambda'(t)}{\lambda(t)} B'(\|\mathscr{U}_0^M\|_{L^2}) \|\mathscr{V}_0^{M-1}(t)\|_{L^2}$$

$$\leq C\lambda'(t) \lambda^{C_0 B''(\|\mathscr{U}^M\|_{L^2})-1}(t) e^{B''(\|\mathscr{U}^M\|_{L^2})t}$$

$$\times \int_0^t \lambda^{-C_0 B''(\|\mathscr{U}^M\|_{L^2})}(s) e^{-B(\|\mathscr{U}^M\|_{L^2})s} \|F^M(s)\|_{L^2}\, ds$$

for all $t \in [0, T]$. Thus

$$(4.23)\quad \frac{1}{\lambda^N(t)} \|\mathscr{V}_0^M(t)\|_{L^2} + \frac{1}{\lambda'(t)\lambda^{N-1}(t)} \left\| \frac{\partial}{\partial t} \mathscr{V}_0^{M-1}(t) \right\|_{L^2} \leq C_M \int_0^t \lambda^{-N}(s) \|F^M(s)\|_{L^2}\, ds.$$

To check convergence of $\mathscr{V}_k = Q^k \mathscr{U}$ we estimate the difference between $\mathscr{V}^M = Q^{(M)} \mathscr{U}^M$ and $\tilde{\mathscr{V}}^M = Q^{(M)} \tilde{\mathscr{U}}^M$. From the equation we get

$$(4.24)\quad \frac{\partial}{\partial t}(\mathscr{V}_0^M - \tilde{\mathscr{V}}_0^M) = K(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{V}_0^M$$

$$- K(t, x, \{C_\beta(t, D_x)\tilde{\mathscr{U}}\}, D_x)\tilde{\mathscr{V}}_0^M$$

$$+ K_0(t)(\mathscr{V}_0^M - \tilde{\mathscr{V}}_0^M) + \tilde{F}(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{V}_0^M$$

$$- \tilde{F}(t, x, \{C_\beta(t, D_x)\tilde{\mathscr{U}}\}, D_x)\tilde{\mathscr{V}}_0^M$$

$$+ (\Phi_{0K}^M - \tilde{\Phi}_{0K}^M) + (\Phi_{0F}^M - \tilde{\Phi}_{0F}^M).$$

If we denote $\mathscr{W}_0^M = \mathscr{V}_0^M - \tilde{\mathscr{V}}_0^M$ then

$$(4.25)\quad \frac{\partial}{\partial t}\mathscr{W}_0^M = K(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{W}_0^M$$

$$+ (K(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x) - K(t, x, \{C_\beta(t, D_x)\tilde{\mathscr{U}}\}, D_x))\tilde{\mathscr{V}}_0^M$$

$$+ K_0(t)\mathscr{W}_0^M + \tilde{F}(t, x, \{C_\beta(t, D_x)\mathscr{U}\}, D_x)\mathscr{W}_0^M$$

$$+ (\tilde{F}(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x) - \tilde{F}(t,x,\{C_\beta(t,D_x)\tilde{\mathscr{U}}\},D_x))\tilde{\mathscr{V}}_0^M$$

$$+ (\Phi_{0K}^M - \tilde{\Phi}_{0K}^M) + (\Phi_{0F}^M - \tilde{\Phi}_{0F}^M)$$

$$= K(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{W}_0^M + K_0(t)\mathscr{W}_0^M$$

$$+ \tilde{F}(t,x,\{C_\beta(t,D_x)\mathscr{U}\},D_x)\mathscr{W}_0^M + \Delta,$$

where according to Lemma 5.5 [21] one has

$$\|\Delta(t)\|_{L^2} \le C_K\|\mathscr{U}_0^M - \tilde{\mathscr{U}}_0^M\|_{L^2}(\|\tilde{\mathscr{V}}_0^M(t)\|_{H^1} + C_{1K}) + C_F\frac{\lambda'(t)}{\lambda(t)}\|\mathscr{U}_0^M - \tilde{\mathscr{U}}_0^M\|_{L^2}(\|\tilde{\mathscr{V}}_0^M(t)\|_{L^2} + C_{1F}) \tag{4.26}$$

and

$$\|\tilde{\mathscr{V}}_0^M(t)\|_{H^1} \le \|\tilde{\mathscr{V}}_0^{M+1}(t)\|_{L^2} \le C_{M+1}\lambda^N(t)\int_0^t \lambda^{-N}(s)\|F^{M+1}(s)\|_{L^2}\,ds. \tag{4.27}$$

Then

$$\begin{aligned}\|\mathscr{W}_0^M(t)\|_{L^2} &\le C_M\lambda^N(t)\int_0^t \lambda^{-N}(s)\|\Delta(s)\|_{L^2}\,ds\\ &\le C_M\lambda^N(t)\int_0^t \lambda^{-N}(s)\Big\{C_K'\|\mathscr{U}_0^M(s) - \tilde{\mathscr{U}}_0^M(s)\|_{L^2}\\ &\qquad + C_F\frac{\lambda'(s)}{\lambda(s)}\|\mathscr{U}_0^M(s) - \tilde{\mathscr{U}}_0^M(s)\|_{L^2}\Big\}ds\\ &\le C_M'\lambda^N(t)\int_0^t \lambda^{-N}(s)\|\mathscr{U}_0^M(s) - \tilde{\mathscr{U}}_0^M(s)\|_{L^2}\,ds\\ &\qquad + C_MC_F'\lambda^N(t)\int_0^t \lambda^{-N-1}(s)\lambda'(s)\|\mathscr{U}_0^M(s) - \tilde{\mathscr{U}}_0^M(s)\|_{L^2}\,ds.\end{aligned} \tag{4.28}$$

From the last inequality it is clear that $Q^{(M)^k}\mathscr{U}^M$ will converge to a limit as $k \to \infty$ in $\lambda C([0,T];L^2(\boldsymbol{T}^n))$, provided that T is picked small, while N is picked so large, in addition to above requirements, that $N > C_MC_F'$. The limit $\mathscr{X}^M$ must be of the form $\{\mathscr{X}_{0\alpha},\mathscr{X}_{1\beta} : |\alpha| \le M, |\beta| \le M-1\}$ for some

$$\mathscr{X} \in \lambda C([0,T];H^M(\boldsymbol{T}^n)) \cap \lambda'/\lambda C^1([0,T];H^{M-1}(\boldsymbol{T}^n)),$$

and $\mathscr{X}$ must solve the problem. The rest of the proof is the quite repetition of the proof of Theorem 5.6 [21], therefore we drop it. Theorem 0.1 is proved. $\square$

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