TSUKUBA J. MATH.
Vol. 21 No. 2 (1997), 367–396

SPECTRA OF THE LAPLACIAN ON THE CAYLEY PROJECTIVE PLANE

By

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Dedicated to Professor Hideki Ozeki on his sixtieth birthday

Introduction

Let $M = G/K$ be a compact homogeneous space of a compact semi-simple Lie group G. Let V be a complex homogeneous vector bundle on M. The group G acts naturally on the space of sections $\Gamma(V)$ of V. By a theorem of Peter and Weyl, $\Gamma(V)$ is a unitary direct sum of finite dimensional representations of G. It is an important problem to decompose $\Gamma(V)$ into irreducible G-modules. By the Frobenius reciprocity theorem, the problem is divided into two parts:

1. How does an irreducible G-module decompose as a K-module (branching law)?
2. How does the fiber V_0 decompose as a K-module?

In spite of its importance there are not so many pairs (G, K) of which the branching law is investigated. For instance, see the list in Strese [7]. The branching law of the compact symmetric pair of rank one are fully explained except the case $(F_4, Spin(9))$. On the branching law of the pair $(F_4, Spin(9))$, we have a result of Lepowsky [5]. But his result is not sufficient to decompose the space of sections $\Gamma(V)$.

A section of $\bigwedge^p(T^*M^C)$ is a (complex) p-form on M. Since the Laplacian on M acting on p-forms commutes with the action of G, Δ is a scalar operator on each irreducible component of $\bigwedge^p(T^*M^C)$ and the eigenvalue is calculated by Freudenthal's formula [3]. By this program, the spectra of p-forms on spheres and complex projective spaces are calculated by Ikeda and Taniguchi [3], and the spectra of quaternion projective spaces and real Grassmann manifolds of 2-planes are calculated by Strese [8] and Tsukamoto [9]. The

Received June 20, 1995.
Revised March 11, 1996.

decomposition of the space $\bigwedge^1(M)$ of 1-forms on a compact irreducible Riemannian symmetric space M is given by Kaneda [4].

In this paper, we calculate the spectra of the Laplacian acting on p-forms $(0 \leq p \leq 5)$ on the Cayley projective plane.

The author would like to express his sincere gratitude to the referee for his valuable comments.

1. Preliminaries

Let G be a compact connected Lie group and M be a homogeneous space of G. Take a point o of M and let K be the isotropy subgroup of G at o. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Let E be a G-homogeneous complex vector bundle on M. The fiber E_o over o is a K-module. The space of smooth sections of E on M is denoted by $\Gamma(E)$. Let $C^\infty(G; E_0)$ be the space of smooth E_0-valued functions on G and put

$$C^\infty(G; E_0)_K = \{f \in C^\infty(G; E_0) | f(uk) = k^{-1}f(u) \text{ for any } u \in G \text{ and } k \in K\}.$$

We have natural actions of G on $\Gamma(E)$ and on $C^\infty(G; E_0)_K$ and a natural G-isomorphism

$$s : C^\infty(G; E_0)_K \to \Gamma(E); \quad f \to [g \cdot 0 \to gf(g)].$$

Each element of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G acts on $C^\infty(G; E_0)$ as a left invariant linear differential operator. The action of $\mathfrak{g}$ on $C^\infty(G; E_0)$ is extended to that of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ in a natural manner. An element $L \otimes X$ of $\mathrm{Hom}(E_o, E_o) \otimes U(\mathfrak{g})$ acts (as a linear differential operator) on $C^\infty(G; E_o)$ by

$$(L \otimes X)(f) = L(Xf) \quad f \in C^\infty(G; E_o).$$

Define an action of K on $\mathrm{Hom}(E_o, E_o) \otimes U(\mathfrak{g})$ by $k(L \otimes X) = (kLk^{-1}) \otimes Ad(k)X$ for $k \in K$. A K-invariant element D of $\mathrm{Hom}(E_o, E_o) \otimes U(\mathfrak{g})$ leaves the subspace $C^\infty(G; E_o)_K$ invariant and induces a G-invariant linear differential operator of $\Gamma(E)$. Conversely every G-invariant linear differential operator of $\Gamma(E)$ is obtained in the above manner.

Let T be a maximal torus of G and $\mathfrak{t}$ be its Lie algebra. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle , \rangle$ on $\mathfrak{g}$. Let (V, ρ) be a complex representation of G. For an element $\lambda \in \mathfrak{t}$, put

$$V_\lambda = \{X \in V | \rho(H)(X) = \sqrt{-1}\langle \lambda, H \rangle X \quad \text{for any } H \in \mathfrak{t}\}.$$

If $V_\lambda \neq 0$, then λ is called the *weight* and V_λ is called the *weight space*. Especially, if $(V,\rho) = (\mathfrak{g}^C, ad)$, then a weight is called a *root* and a weight space is called a *root space*. We denote by $\Sigma(G)$ the set of non-zero roots of G with respect T and $\Sigma^+(G)$ the set of all positive roots. For the sake of brevity, we normalize the inner product $\langle\,,\rangle$ so that the length of long root is equal to 2. We denote by $\mathscr{D}(G)$ the set of all equivalence classes of the complex irreduible representations of G. Let $V^G(\lambda)$ be a representation space of an element λ of $\mathscr{D}(G)$.

Take a K-invariant Hermitian inner product in E_o and extend it to a unitary structure on E. For each irreducible complex representation $V^G(\gamma)$, we define a map A_γ of $V(\gamma) \otimes \mathrm{Hom}_K(V_\gamma, E_o)$ to $C^\infty(G; E_o)_K$ by

$$A_\gamma(v \otimes L)(g) = L(g^{-1} \cdot v).$$

Then we have the following:

THEOREM 1 (Frobenius reciprocity). *The unitary representation* $\Gamma(E)$ *is the unitary direct sum*:

$$\Gamma(E) = \sum_{\gamma \in D(G)} A_\gamma(V^G(\gamma) \otimes \mathrm{Hom}_K(V^G(\gamma), E_o)).$$

Assume that T is also a maximal torus of K (namely G and K are of the same rank.) We denote by $\mathfrak{k}$ and $\mathfrak{t}$ the Lie algebras of K and T respectively. We denote by $\Sigma(K)$ the set of all non-zero roots of $\mathfrak{k}^C$ with respect to $\mathfrak{t}^C$. By our assumption $\Sigma(K)$ is contained in $\Sigma(G)$. We denote by $\Sigma^+(K)$ the set of positive roots of $\mathfrak{k}^C$.

Let $\gamma_1, \dots, \gamma_r \in \mathfrak{t}$ be the set of elements of $\Sigma^+(G) \backslash E^+(K)$. For every $\nu \in \mathfrak{t}$, we denote by $P(\nu)$ the number of non-negative integral r-tuples $(n_1, \dots, n_r)$ such that $\nu = \sum_{j=1}^r n_j \gamma_j$.

Let W be the Weyl group of G. Let $D(G)$ and $D(K)$ be the set of dominant integral linear forms for G and K respectively. Then we can identify $D(G)$ [resp. $D(K)$] with the set $\mathscr{D}(G)$ [resp. $\mathscr{D}(K)$] of all finite dimensional irreducible G- [resp. K-] modules. For each $\lambda \in D(G)$ [resp. $\mu \in D(K)$], we denote by $V^G(\lambda)$ [resp. $V^K(\mu)$] the irreducible G- [resp. K-] module which corresponds to $\lambda \in D(G)$ [resp. $\mu \in D(K)$] (i.e., the irreducible G- [resp. K-] module with highest weight λ [resp. μ]). Since K is compact, $V^G(\lambda)$ is decomposed into irreducible K-modules:

$$V^G(\lambda) = \sum_{\mu \in D(K)} m(\lambda, \mu) V^K(\mu).$$

The multiplicity $m(\lambda, \mu)$ is counted by the following:

THEOREM 2 (Kostant)

$$m(\lambda,\mu)=\sum_{\sigma\in W}(\det\sigma)P(\sigma(\lambda+\delta)-(\mu+\delta)),$$

where δ is half the sum of positive roots of $\mathfrak{g}^C$.

For the proof, we refer to Lepowsky [5], [6] or Cartier [2].

2. Cayley projective plane

2.1 ROOT AND WEIGHT SYSTEM OF F_4 AND $Spin(9)$. Thy Cayley projective plane $\mathbf{Ca}P^2$ is isomorphic to the coset space G/K for $G=F_4$, $K=Spin(9)$. Let T be a maximal torus of $Spin(9)$. We denote by $\mathfrak{g},\mathfrak{k}$ and $\mathfrak{t}$ the Lie algebras of G, K and T respectively. The complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^C$ and $\mathfrak{k}^C$. Under a suitable choise of an orthonormal base $\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}$ of $\mathfrak{t}$, the set of roots $\Sigma(G)$ [resp. $\Sigma(K)$] of $\mathfrak{g}^C$ [resp. $\mathfrak{k}^C$] with respect to $\mathfrak{t}^C$ are

$$\Sigma(G)=\left\{\begin{array}{ll}\pm\varepsilon_i\ (1\le i\le 4), & \pm\varepsilon_i\pm\varepsilon_j\ (1\le i<j\le 4),\\ (1/2)\sum_{j=1}^4 s_j\varepsilon_j\ (s_j=\pm1, 1\le j\le 4) & \end{array}\right\},$$

$$\Sigma(K)=\left\{\begin{array}{l}\pm\varepsilon_i\pm\varepsilon_j(1\le i<j\le 4),\\ (1/2)\sum_{j=1}^4 s_j\varepsilon_j(s_j=\pm1, s_1\cdot s_2\cdot s_3\cdot s_4=-1)\end{array}\right\}.$$

Define a lexicographic order $>$ in $\mathfrak{t}$ by

$$\varepsilon_1>\varepsilon_2>\varepsilon_3>\varepsilon_4>0.$$

Then the set of positive roots of $\mathfrak{g}^C$ and $\mathfrak{k}^C$ are as follows:

$$\Sigma^+(G)=\left\{\begin{array}{ll}\varepsilon_i(1\le i\le 4), & \varepsilon_i\pm\varepsilon_j\ (1\le i<j\le 4),\\ (1/2)\sum_{j=1}^4 s_j\varepsilon_j\ (s_1=1, s_2,s_3,s_4=\pm1) & \end{array}\right\},$$

$$\Sigma^+(K)=\left\{\begin{array}{l}\varepsilon_i\pm\varepsilon_j\ (1\le i<j\le 4)\\ (1/2)\sum_{j=1}^4 s_j\varepsilon_j\ (s_1=1, s_2,s_3,s_4=\pm, s_2\cdot s_3\cdot s_4=-1)\end{array}\right\},$$

and the set of dominant forms $D(G)$ [resp. $D(K)$] of G [resp. K] are

$$D(G)=\left\{\sum_{i=1}^4 a_i\varepsilon_i \middle| a_1\ge a_2\ge a_3\ge a_4\ge 0, a_1\ge a_2+a_3+a_4\right\},$$

$$D(K)=\left\{\sum_{i=1}^4 b_i\varepsilon_i \middle| b_1\ge b_2\ge b_3\ge |b_4|, b_1\ge b_2+b_3+b_4\right\}.$$

A linear form $\sum_{i=1}^{4} a_i\varepsilon_i$ is an integral form for F_4, which is also an integral form for $Spin(9)$, if and only if $2a_1$, $a_1 - a_2$, $a_2 - a_3$ and $a_3 - a_4$ are integers.

The set of simple roots of $\Sigma(G)$ with respect to $>$ is

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \quad \alpha_3 = \varepsilon_4, \quad \alpha_4 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2,$$

and the set of simple roots of $\Sigma(K)$ with respect to $>$ is

$$\beta_1 = \varepsilon_3 - \varepsilon_4, \quad \beta_2 = \varepsilon_2 - \varepsilon_3, \quad \beta_3 = \varepsilon_3 + \varepsilon_4, \quad \beta_4 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2.$$

The set of fundamental weights of $\Sigma(G)$ with respect to $>$ is

$$\lambda_1 = \varepsilon_1 + \varepsilon_2, \quad \lambda_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \lambda_3 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, \quad \lambda_4 = \varepsilon_1,$$

and the set of fundamental weights of $\Sigma(K)$ with respect to $>$ is

$$\mu = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2, \quad \mu_2 = \varepsilon_1 + \varepsilon_2, \quad \mu_3 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, \quad \mu_4 = \varepsilon_1.$$

Half the sum of positive roots of G is

$$\delta = (11/2)\varepsilon_1 + (5/2)\varepsilon_2 + (3/2)\varepsilon_3 + (1/2)\varepsilon_4.$$

2.2 Weyl group $W(F_4)$ of F_4. Take 3 subsets K_1, K_2, K_3 of $\Sigma(F_4)$ defined by

$$K_1 = \left\{ \begin{array}{l} \pm\varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le 4), \\ (1/2)\sum_{j=1}^{4} s_j\varepsilon_j \ (s_j = \pm 1, \prod s_i = -1) \end{array} \right\},$$

$$K_2 = \left\{ \begin{array}{l} \pm\varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le 4), \\ \pm\varepsilon_j \ (1 \le j \le 4) \end{array} \right\},$$

$$K_3 = \left\{ \begin{array}{l} \pm\varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le 4), \\ (1/2)\sum_{j=1}^{4} s_j\varepsilon_j \ (s_j = \pm 1, \prod s_i = 1) \end{array} \right\}.$$

Each set K_i is isomorphic to a root system of type $\mathfrak{b}_4$. We denote by S_α, the linear transformation on $\mathfrak{t}$ defined by $S_\alpha(H) = H - 2(\langle \alpha, H\rangle / \langle \alpha, \alpha \rangle)\alpha$. Let $\psi = S_{\alpha_4} \circ S_{\alpha_3}$. Then ψ acts on the set $\{K_1, K_2, K_3\}$ as a permutation:

$$\psi(K_1) = K_2, \quad \psi(K_2) = K_3, \quad \psi(K_3) = K_1.$$

Let W' be the subgroup of $W(F_4)$ generated by $\{S_\alpha | \alpha \in K_2\}$.

Lemma 1. *The Weyl group $W(F_4)$ is the semidirect product of $\{1, \psi, \psi^2\}$ and W'.*

Let U be the subgroup of $W(F_4)$ generated by $\{S_{\varepsilon_1}, S_{\varepsilon_2}, S_{\varepsilon_3}, S_{\varepsilon_4}\}$ and V be the symmetric group of 4 characters. Then the group W' is the semidirect product of V and U. The group $W(F_4)$ is the disjoint union of 3 right cosets of W' ([5]). It is easy to see that 3 right cosets W', $W'\psi$ and $W'(S_{\alpha_3} \cdot \psi^2)$ of W' are mutually disjoint subsets. We will use the following coset space decomposition of $W(F_4)$.

$$W(F_4) = W' \cup W'\psi \cup W'(S_{\alpha_3} \circ \psi^2). \tag{1}$$

2.3 DECOMPOSITION OF $\bigwedge^p(T_{eK}(\mathbf{Ca}P^2))^C$. The complexified cotangent space of the Cayley projective plane $\mathbf{Ca}P^2$ at the origin $o = eK$ is an irreducible $Spin(9)$-module with highest weight μ_4. Let $\{\varpi_i | 1 \leq i \leq 16\}$ be the set of weights of $(T_o(\mathbf{Ca}P^2))^C$. The multiplicity of each weight ω_i is equal to 1. Thus the set of weights of $\bigwedge^p(T_o(\mathbf{Ca}P^2))^C$ is as follows

$$\{\omega_{i_1} + \cdots + \omega_{i_p} | i_1 < \cdots < i_p\}.$$

Count the multiplicities of dominant integral weights and looking the table of dominant integral weight multiplicities, we can decompose $\bigwedge^p(T_o(\mathbf{Ca}P^2))^C$ into $Spin(9)$-irreducible modules.

For example, the multiplicities of dominant integral weights of $\bigwedge^2(T_o(\mathbf{Ca}P^2))^C$ are as follows:

d.i.w.	0	μ_1	μ_2	μ_3
mult.	8	4	2	1

On the other hand, the multiplicities of $V^K(\mu_3)$ are as follows (see, for example, [1]):

d.i.w.	0	μ_1	μ_2	μ_3
mult.	4	3	1	1

and the multiplicities of $V^K(\mu_2)$ are follows:

d.i.w.	0	μ_1	μ_2	μ_3
mult.	4	1	1	0

Thus we have the following decomposition:

$$\bigwedge^2(T_o(\mathbf{Ca}P^2))^C = V^K(\mu_3) \oplus V^K(\mu_2).$$

We give the table of the highest weights of irreducible *Spin*(9)-submodules of $\bigwedge^p(T_o(\mathbf{Ca}P^2))^C$.

Table 1: Highest weight of irred. *Spin*(9)-submodule of $\bigwedge^p(T_{eK}(\mathbf{Ca}P^2))\boldsymbol{C}$.

p	Highest weight			
0	0			
1	μ_4			
2	μ_2	μ_3		
3	$\mu_2+\mu_4$	$\mu_1+\mu_4$		
4	$2\mu_4$	$2\mu_1$	$\mu_1+\mu_2$	$\mu_1+2\mu_4$
	$2\mu_2$			
5	$\mu_1+\mu_4$	$\mu_2+\mu_4$	$3\mu_4$	$2\mu_1+\mu_4$
	$\mu_1+\mu_2+\mu_4$			
6	μ_2	μ_3	$\mu_1+\mu_2$	$\mu_1+\mu_3$
	$\mu_1+2\mu_4$	$\mu_2+2\mu_4$	$2\mu_1+\mu_2$	$2\mu_1+\mu_3$
7	μ_4	$\mu_1+\mu_4$	$\mu_2+\mu_4$	$\mu_3+\mu_4$
8	0	μ_1	μ_3	$2\mu_4$
	$2\mu_1$	$\mu_1+\mu_3$	$\mu_1+2\mu_4$	$2\mu_2$
	$\mu_2+\mu_3$	$2\mu_3$	$3\mu_1$	$2\mu_1+\mu_3$
	$2\mu_1+2\mu_4$	$4\mu_1$		

The multiplicity of each *Spin*(9)-submodule is 1.

3. Branching Law of (F_4, *Spin*(9))

3.1 Lepowsky's result. Lepowsky proved the following

Theorem 3 (Lepowsky [5], [6]). (1) *Let* $\lambda \in \mathscr{D}(F_4)$, $\mu \in \mathscr{D}(Spin(9))$. *Suppose* λ *is the highest weight of a class* 1 *finite dimensional irreducible representation of*

F_4, so that $\lambda = a\lambda_1 (a \in \boldsymbol{Z}_+)$. Then $m(\lambda, \mu) = 1$ iff $b_2 = b_3 = -b_4$ and $b_1 + b_2 \leq a$; otherwise $m(\lambda, \mu) = 0$.

(2) *Let $\lambda = \sum_{i=1}^4 a_i \lambda_i \in \mathscr{D}(F_4)$. Then $\mu = a_2\varepsilon_1 + a_3\varepsilon_2 + a_4\varepsilon_3 - a_4\varepsilon_4 \in \mathscr{D}(Spin(9))$, and $m(\lambda, \mu) = 1$.*

Here we give a short review on a part of his proof of the above theorem. For two real numbers x, y we write $x \preceq y$ or $y \succeq x$ if $y - x$ is a non-negative integer and we write $x \prec y$ or $y \succ x$ for the relation $x \neq y$ and $x \preceq y$. The partition function P is described by the following:

LEMMA 2 (Lepowsky [5]). *Let $v = \sum_{i=1}^4 x_i \varepsilon_i$ be an integral linear form. Then $P(v)$ is the number of real quadruples (p_1, p_2, p_3, p_4) satisfying the conditions*

$$\begin{cases} p_1 + p_2 + p_3 + p_4 \geq 0, \\ p_1 + p_2 - p_3 - p_4 \geq 0, \\ p_1 - p_2 + p_3 - p_4 \geq 0, \\ p_1 - p_2 - p_3 + p_4 \geq 0, \end{cases} \tag{2}$$

$$\sum_{i=1}^4 p_i \in 2\boldsymbol{Z}, \tag{3}$$

$$p_i \preceq x_i \quad (1 \leq i \leq 4). \tag{4}$$

For the sake of completeness, we give the proof of Lemma.

PROOF. Let

$$\gamma_1 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2,$$

$$\gamma_2 = (\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2,$$

$$\gamma_3 = (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2,$$

$$\gamma_4 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)/2.$$

Then $\Sigma^+(G) \backslash \Sigma^+(K)$ consists of ε_i $(1 \leq i \leq 4)$ and γ_i $(1 \leq i \leq 4)$. Thus $P(v)$ is the number of elements of the set

$$\mathscr{P}(v) = \{(y_1, \dots, y_4, z_1, \dots, z_4) \in \boldsymbol{Z}^8 | v = \sum_{i=1}^4 y_i \varepsilon_i + \sum_{j=1}^4 z_j \gamma_j, y_1, \dots, y_4, z_1, \dots, z_4 \geq 0\}.$$

Put $p_i = x_i - y_i$ $(1 \leq i \leq 4)$. Then $(y_1, \dots, y_4, z_1, \dots, z_4)$ is contained in $\mathscr{P}(v)$ if and only if

$$(5) \qquad \begin{cases} p_1 + p_2 + p_3 + p_4 = 2z_1, \\ p_1 + p_2 - p_3 - p_4 = 2z_2, \\ p_1 - p_2 + p_3 - p_4 = 2z_3, \\ p_1 - p_2 - p_3 + p_4 = 2z_4. \end{cases}$$

Thus, if $(y_1, \dots, y_4, z_1, \dots, z_4)$ is contained in $\mathscr{P}(v)$, then $p_i's$ satisfy (2), (3) and (4). Conversely if $p_i's$ satisfy (2), (3) and (4) then $(y_1, \dots, y_4, z_1, \dots, z_4)$ defined by $y_i = x_i - p_i (1 \leq i \leq 4)$ and (5) is contained in $\mathscr{P}(v)$. Q.E.D.

For all integral linear forms v and ξ and a subset X of W, we put

$$M_v^X(\xi) = \sum_{\sigma \in X} \det(\sigma) P(\sigma(v) - \xi).$$

LEMMA 3 (Lepowsky [5]). *Let $v = \sum_{i=1}^4 x_i \varepsilon_i$ and $\xi = \sum_{i=1}^4 y_i \varepsilon_i$ be integral linear forms. If we assume that $x_1 > x_2 > x_3 > x_4 > 0$, then $M_v^U(\xi)$ is the number of real quadruples (p_1, p_2, p_3, p_4) satisfying* (2), (3) *and the condition*

$$-x_i - y_i \prec p_i \preceq x_i - y_i \quad (1 \leq i \leq 4).$$

For an integral linear form $v = \sum_{i=1}^4 x_i \varepsilon_i (x_1 > x_2 > x_3 > x_4 > 0)$, we define

$$T(v)_1 = \{t \in \boldsymbol{R} | -x_1 \prec t \preceq -x_2 \text{ or } x_2 \prec t \preceq x_1\},$$

$$T(v)_2 = \{t \in \boldsymbol{R} | -x_2 \prec t \preceq -x_3 \text{ or } x_3 \prec t \preceq x_2\},$$

$$T(v)_3 = \{t \in \boldsymbol{R} | -x_3 \prec t \preceq -x_4 \text{ or } x_4 \prec t \preceq x_3\},$$

$$T(v)_4 = \{t \in \boldsymbol{R} | -x_4 \prec t \preceq x_4\}.$$

Let $Q_\tau^v(\xi)$ be the number of real quadruples $p = (p_1, p_2, p_3, p_4)$ satisfying (2), (3) and

$$(6) \qquad p_i + y_i \in T(v)_{\tau(i)}, \quad (1 \leq i \leq 4)$$

and put

$$N_v(\xi) = \sum_{\tau \in V} (\det \tau) Q_\tau^v(\xi).$$

LEMMA 4 (Lepowsky [5]). *Let* $\nu = \sum_{i=1}^4 x_i\varepsilon_i$ *and* $\xi = \sum_{i=1}^4 y_i\varepsilon_i$ *be integral linear forms. Assume that* $x_1 > x_2 > x_3 > x_4 > 0$. *Then* $M_\nu^{W'}(\xi) = N_\nu(\xi)$.

The above lemma, with (1) and Theorem 2, implies

$$m(\lambda,\mu) = N_{\lambda+\delta}(\mu+\delta) + N_{(\lambda+\delta)^*}(\mu+\delta) - N_{(\lambda+\delta)^{**}}(\mu+\delta) \tag{7}$$

where

$$(\lambda+\delta)^* = \Psi(\lambda+\delta), (\lambda+\delta)^{**} = S_{\alpha_3}\Psi^2(\lambda+\delta).$$

We omit the proofs of Lemma 3 and Lemma 4.

For a real quadruple $p = (p_1, p_2, p_3, p_4)$, we put

$$M_1(p) = -p_2 - p_3 - p_4, \quad M_2(p) = -p_2 + p_3 + p_4,$$

$$M_3(p) = p_2 - p_3 + p_4, \quad M_4(p) = p_2 + p_3 - p_4.$$

If we put $M(p) = \max_{1\le j\le 4} M_j(p)$, then (2) is equivalent to

$$p_1 \ge M(p).$$

Since $M_1(p) - M_2(p), M_1(p) - M_3(p)$ and $M_1(p) - M_4(p)$ are even integers, (3) is equivalent to

$$p_1 - M_j(p) \equiv 0 \pmod 2 \quad \text{for some } j.$$

LEMMA 5. *Let* $\nu = \sum_{i=1}^4 x_i\varepsilon_i$ *be an integral linear form satisfying* $x_1 > x_2 > x_3 > x_4 > 0$ *and* μ *be a dominant integral form for* $Spin(9)$. *Then*

$$N_\nu(\mu+\delta) = \sum_{\tau\in V, \tau(1)=1} \det(\tau) Q_\tau^\nu(\mu+\delta).$$

PROOF. Put $\mu = \sum_{j=1}^4 b_j\varepsilon_j$ and $\mu+\delta = \sum_{j=1}^4 y_j\varepsilon_j$. Namely

$$y_1 = b_1 + 11/2, \quad y_2 = b_2 + 5/2, \quad y_3 = b_3 + 3/2, \quad y_4 = b_4 + 1/2.$$

Let $p = (p_1, p_2, p_3, p_4)$ be a real quadruple satisfying (2), (3) and (6). Assume that $\tau(1) \ne 1$. Since $p_1 + y_1 \in T(\nu)_2 \cup T(\nu)_3 \cup T(\nu)_4$ we have

$$p_1 \le x_2 - y_1 = x_2 - b_1 - 11/2.$$

There exists $j \ne 1$ such that $\tau(j) = 1$. Since $p_j + y_j \in T(\nu)_1$ we have

$$|p_j| \ge x_2 - |y_j| > x_2 - b_1 - 11/2 \ge p_1.$$

From (2), we have

$$p_1 \geq |p_2|, \quad p_1 \geq |p_3| \quad \text{and} \quad p_1 \geq |p_4|,$$

which is a contradiction. Thus $Q_\tau^\nu(\mu+\delta) = 0$ and the Lemma is proved. Q.E.D.

3.2 FURTHER CALCULATION OF $N_\nu(\lambda)$. Let $\nu = \sum_{i=1}^4 x_i \varepsilon_i$ be an integral form with $x_1 > x_2 > x_3 > x_4 > 0$. We put

$$\begin{aligned}
T(\nu)_1^0 &= \{t | x_2 \prec t \preceq x_1\} \\
&= \{x_1 + 1 - i_1 | i_1 \in \boldsymbol{Z}, 1 \leq i_1 \leq x_1 - x_2\}, \\
T(\nu)_2^0 &= \{t | x_3 \prec t \preceq x_2\} \\
&= \{x_2 + 1 - i_2 | i_2 \in \boldsymbol{Z}, 1 \leq i_2 \leq x_2 - x_3\}, \\
T(\nu)_2^1 &= \{t | -x_2 \prec t \preceq -x_3\} \\
&= \{-x_3 + 1 - i_2 | i_2 \in \boldsymbol{Z}, 1 \leq i_2 \leq x_2 - x_3\}, \\
T(\nu)_3^0 &= \{t | x_4 \prec t \preceq x_3\} \\
&= \{x_3 + 1 - i_3 | i_3 \in \boldsymbol{Z}, 1 \leq i_3 \leq x_3 - x_4\}, \\
T(\nu)_3^1 &= \{t | -x_3 \prec t \preceq -x_4\} \\
&= \{-x_4 + 1 - i_3 | i_3 \in \boldsymbol{Z}, 1 \leq i_3 \leq x_3 - x_4\}, \\
T(\nu)_4 &= \{t | -x_4 \prec t \preceq x_4\} \\
&= \{x_4 + 1 - i_4 | i_4 \in \boldsymbol{Z}, 1 \leq i_4 \leq 2x_4\}.
\end{aligned}$$

Let $p = (p_1, p_2, p_3, p_4)$ be a real quadruple satisfying (2), (3) and (6). Since $p_1 > 0$ and $\tau(1) = 1$ (see the proof of Lemma 5), we have $p_1 + (b_1 + 11/2) \in T_1^0(\nu)$. Let $\mathscr{S}^\nu_{\tau,s_2,s_3}(\xi)(\tau \in V, \tau(1) = 1, s_2, s_3 = 0, 1)$ be the set of real quadruples $p = (p_1, p_2, p_3, p_4)$ satisfying

$$\begin{aligned}
&p_1 + y_1 \in T(\nu)_1^0, \\
&p_{\tau^{-1}(2)} + y_{\tau^{-1}(2)} \in T(\nu)_2^{s_2}, \\
&p_{\tau^{-1}(3)} + y_{\tau^{-1}(3)} \in T(\nu)_3^{s_3}, \\
&p_{\tau^{-1}(4)} + y_{\tau^{-1}(4)} \in T(\nu)_4
\end{aligned}$$

and

$$\mathscr{T}^{\nu}_{\tau,s_2,s_3}(\xi) = \{p \in \mathscr{S}^{\nu}_{\tau,s_2,s_3}(\xi) | p_1 \geq M(p),\ p_1 + M(p) \equiv 0 \pmod 2\}.$$

If we denote by $Q^{\nu}_{\tau,s_2,s_3}(\xi)$ the number of elements of $\mathscr{T}^{\nu}_{\tau,s_2,s_3}(\xi)$, then we have

$$N_{\nu}(\xi) = \sum_{\tau \in V, \tau(1)=1} \sum_{s_2,s_3=0}^{1} \operatorname{sgn}(\tau) Q^{\nu}_{\tau,s_2,s_3}(\xi). \tag{8}$$

Let

$$I_{\nu} = ([1, x_1 - x_2] \times [1, x_2 - x_3] \times [1, x_3 - x_4] \times [1, 2x_4]) \cap \boldsymbol{Z}^4$$

and p_{τ,s_2,s_3} be a mapping of I_{ν} to $\boldsymbol{R}^4$ defined by

$$(p_{\tau,s_2,s_3}(i))_1 = x_1 - y_1 + 1 - i_1,$$

$$(p_{\tau,s_2,s_3}(i))_{\tau^{-1}(2)} = \begin{cases} x_2 - y_{\tau^{-1}(2)} + 1 - i_2, & \text{if } s_2 = 0, \\ -x_3 - y_{\tau^{-1}(2)} + 1 - i_2, & \text{if } s_2 = 1, \end{cases}$$

$$(p_{\tau,s_2,s_3}(i))_{\tau^{-1}(3)} = \begin{cases} x_3 - y_{\tau^{-1}(3)} + 1 - i_3, & \text{if } s_3 = 0, \\ -x_4 - y_{\tau^{-1}(3)} + 1 - i_3, & \text{if } s_3 = 1, \end{cases}$$

$$(p_{\tau,s_2,s_3}(i))_{\tau^{-1}(4)} = x_4 - y_{\tau^{-1}(4)} + 1 - i_4.$$

Then p_{τ,s_2,s_3} gives a bijection of I_{ν} onto $\mathscr{S}^{\nu}_{\tau,s_2,s_3}(\xi)$. Hereafter we consider $\mathscr{T}^{\nu}_{\tau,s_2,s_3}(\xi)$ as a subset of I_{ν} by the bijection p_{τ,s_2,s_3}.

Put

$$\tau_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix},$$

$$\tau_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

From the following table we can see that $M(P_{\tau_0,1,0}(i))$ is equal to $M(P_{\tau_3,1,0}(i))$ for any $i \in I_{\nu}$. Thus we have

$$Q^{\nu}_{\tau_0,1,0}(\xi) = Q^{\nu}_{\tau_3,1,0}(\xi).$$

(τ, s_2, s_3)	$(\tau_0, 1, 0)$	$(\tau_3, 1, 0)$
$M_1(p)$	$-x_4 + y_2 + y_3 + y_4$ $-3 + i_2 + i_3 + i_4$	$-x_4 + y_2 + y_3 + y_4$ $-3 + i_2 + i_3 + i_4$
$M_2(p)$	$2x_3 + x_4 + y_2 - y_3 - y_4$ $+1 + i_2 - i_3 - i_4$	$2x_3 + x_4 + y_2 - y_3 - y_4$ $+1 + i_2 - i_3 - i_4$
$M_3(p)$	$(-2x_3 + x_4 - y_2 + y_3 - y_4$ $+1 - i_2 + i_3 - i_4)$	$(-x_4 - y_2 + y_3 - y_4$ $+1 - i_2 - i_3 + 4)$
$M_4(p)$	$(-x_4 - y_2 - y_3 + y_4$ $+1 - i_2 - i_3 + i_4)$	$(-2x_3 + x_4 - y_2 - y_3 + y_4$ $+1 - i_2 + i_3 - i_4)$

† $p = p_{\tau,s_2,s_3}(i)$
‡ parenthesized entry is smaller than or equal to another entry in the same column

Similarly, we have

$$Q^{\nu}_{\tau_0,1,1}(\xi) = Q^{\nu}_{\tau_5,1,1}(\xi), \quad Q^{\nu}_{\tau_1,0,1}(\xi) = Q^{\nu}_{\tau_4,0,1}(\xi), \quad Q^{\nu}_{\tau_1,1,0}(\xi) = Q^{\nu}_{\tau_5,1,0}(\xi),$$
$$Q^{\nu}_{\tau_1,1,1}(\xi) = Q^{\nu}_{\tau_3,1,1}(\xi), \quad Q^{\nu}_{\tau_2,0,0}(\xi) = Q^{\nu}_{\tau_5,0,0}(\xi), \quad Q^{\nu}_{\tau_2,1,0}(\xi) = Q^{\nu}_{\tau_4,1,0}(\xi),$$
$$Q^{\nu}_{\tau_2,1,1}(\xi) = Q^{\nu}_{\tau_4,1,1}(\xi).$$

Thus by (8), we have

$$N_{T(\nu)}(\xi) = Q^{\nu}_{\tau_0,0,0}(\xi) + Q^{\nu}_{\tau_1,0,0}(\xi) - Q^{\nu}_{\tau_3,0,0}(\xi) - Q^{\nu}_{\tau_4,0,0}(\xi)$$
$$+ Q^{\nu}_{\tau_0,0,1}(\xi) + Q^{\nu}_{\tau_2,0,1}(\xi) - Q^{\nu}_{\tau_3,0,1}(\xi) - Q^{\nu}_{\tau_5,0,1}(\xi).$$

Now we consider $\mathscr{T}^{\nu}_{\tau_0,0,0}(\xi)$ and $\mathscr{T}^{\nu}_{\tau_3,0,0}(\xi)$. From $M_3(p_{\tau_3,0,0}(i)) - M_4(p_{\tau_3,0,0}(i)) = 2(x_3 - x_4 + y_3 - y_4 - i_3 + i_4) \geq 2$, we have

$$M(p_{\tau_3,0,0}(i)) = \max\{M_1(p_{\tau_3,0,0}(i)), M_2(p_{\tau_3,0,0}(i)), M_3(p_{\tau_3,0,0}(i))\}.$$

Since

$$M_1(p_{\tau_3,0,0}(i)) = M_1(p_{\tau_0,0,0}(i)),$$
$$M_2(p_{\tau_3,0,0}(i)) = M_2(p_{\tau_0,0,0}(i)),$$
$$M_3(p_{\tau_3,0,0}(i)) - M_3(p_{\tau_0,0,0}(i)) = 2(x_3 - x_4 - i_3 + i_4) \geq 2,$$
$$M_3(p_{\tau_3,0,0}(i)) - M_4(p_{\tau_0,0,0}(i)) = 2(y_3 - y_4) > 0,$$

hold for any $i \in I_\nu$, we have $M(p_{\tau_0,0,0}(i)) \leq M(p_{\tau_3,0,0}(i))$ and $M(p_{\tau_0,0,0}(i)) \equiv M(p_{\tau_3,0,0}(i)) \pmod 2$ for any $i \in I_\nu$. Namely $\mathscr{T}^{\nu}_{\tau_3,0,0}(\xi)$ is a subset of $\mathscr{T}^{\nu}_{\tau_0,0,0}(\xi)$ and $\mathscr{T}^{\nu}_{\tau_0,0,0}(\xi) \backslash \mathscr{T}^{\nu}_{\tau_3,0,0}(\xi)$ is the set of integral quadruples $i = (i_1, i_2, i_3, i_4) \in I_\nu$ satisfying

$$\begin{cases} M_3(p_{\tau_3,0,0}(i)) > x_1 - y_1 + 1 - i_1 \geq M(p_{\tau_0,0,0}(i)), \\ x_1 - y_1 + 1 - i_1 \equiv M_2(p_{\tau_0,0,0}(i)) \pmod 2. \end{cases}$$

Similarly $\mathscr{T}^{\nu}_{\tau_1,0,0}(\xi)$ is a subset of $\mathscr{T}^{\nu}_{\tau_4,0,0}(\xi)$ and $\mathscr{T}^{\nu}_{\tau_4,0,0}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_1,0,0}(\xi)$ is the set of integral quadruples $i = (i_1, i_2, i_3, i_4) \in I$ satisfying

$$\begin{cases} M_3(p_{\tau_1,0,0}(i)) > x_1 - y_1 + 1 - i_1 \geq M(p_{\tau_4,0,0}(i)), \\ x_1 - y_1 + 1 - i_1 \equiv M_2(p_{\tau_1,0,0}(i)) \pmod 2. \end{cases}$$

From $M_2(p_{\tau_4,0,0}(i)) - M_3(p_{\tau_4,0,0}(i)) = 2(x_2 - x_3 + y_2 - y_3 - i_2 + i_3) \geq 2$, we have

$$M(p_{\tau_4,0,0}(i)) = \max\{M_1(p_{\tau_4,0,0}(i)), M_2(p_{\tau_4,0,0}(i)), M_4(p_{\tau_4,0,0}(i))\}.$$

Since

$$M_3(p_{\tau_3,0,0}(i)) = M_3(p_{\tau_1,0,0}(i)),$$

$$M_1(p_{\tau_4,0,0}(i)) = M_1(p_{\tau_0,0,0}(i)),$$

$$M_2(p_{\tau_4,0,0}(i)) - M_2(p_{\tau_0,0,0}(i)) = 2(x_2 - x_3 - i_2 + i_3) \geq 2,$$

$$M_2(p_{\tau_4,0,0}(i)) - M_3(p_{\tau_0,0,0}(i)) = 2(y_2 - y_3) \geq 2,$$

$$M_4(p_{\tau_4,0,0}(i)) = M_4(p_{\tau_0,0,0}(i)),$$

hold for any $i \in I_\nu$, we have $M(p_{\tau_4,0,0}(i)) \geq M(p_{\tau_0,0,0}(i))$. Thus $(\mathscr{T}^{\nu}_{\tau_0,0,0}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_4,0,0}(\xi)) \supseteq (\mathscr{T}^{\nu}_{\tau_3,0,0}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_1,0,0}(\xi))$ and $(\mathscr{T}^{\nu}_{\tau_0,0,0}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_4,0,0}(\xi))\backslash\mathscr{T}^{\nu}_{\tau_3,0,0}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_1,0,0}(\xi))$ is the set of integral quadruples $i = (i_1, i_2, i_3, i_4) \in I_\nu$ satisfying

$$\begin{cases} \min\{M_2(p_{\tau_4,0,0}(i)), M_3(p_{\tau_3,0,0}(i))\} > x_1 - y_1 - i_1 + 1 \geq M(p_{\tau_0,0,0}(i)), \\ x_1 - y_1 - i_1 + 1 - M(p_{\tau_0,0,0}(i)) \equiv 0 \pmod 2. \end{cases}$$

By a similar argument, we can show that

$$\mathscr{T}^{\nu}_{\tau_0,0,1}(\xi) \supseteq \mathscr{T}^{\nu}_{\tau_5,0,1}(\xi),$$

$$\mathscr{T}^{\nu}_{\tau_3,0,1}(\xi) \supseteq \mathscr{T}^{\nu}_{\tau_2,0,1}(\xi),$$

$$\mathscr{T}^{\nu}_{\tau_3,0,1}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_2,0,1}(\xi) \supseteq \mathscr{T}^{\nu}_{\tau_0,0,1}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_5,0,1}(\xi)$$

and $\mathscr{T}^{\nu}_{\tau_3,0,1}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_2,0,1}(\xi) \supseteq \mathscr{T}^{\nu}_{\tau_0,0,1}(\xi)\backslash\mathscr{T}^{\nu}_{\tau_5,0,1}(\xi)$ is the set of integral quadruples $j = (j_1, j_2, j_3, j_4) \in I_\nu$ satisfying

$$\begin{cases} \min\{M_2(p_{\tau_2,0,1}(j)), M_3(p_{\tau_0,0,1}(j)) > x_1 - y_1 - j_1 + 1 \geq M(p_{\tau_3,0,1}(j)), \\ x_1 - y_1 - j_1 + 1 - M(p_{\tau_2,0,1}(j)) \equiv 0 \pmod 2. \end{cases}$$

If we put $j_1 = i_1, j_2 = i_2, j_3 = x_3 - x_4 + 1 - i_3$ and $j_4 = 2x_4 + 1 - i_4$, then we have

$$M_2(p_{\tau_2,0,1}(j)) = M_2(p_{\tau_4,0,0}(i)),$$

$$M_3(p_{\tau_0,0,1}(j)) = M_3(p_{\tau_3,0,0}(i)),$$

$$M_1(p_{\tau_3,0,1}(j)) > M_2(p_{\tau_3,0,1}(j)),$$

$$M_1(p_{\tau_3,0,1}(j)) > M_1(p_{\tau_0,0,0}(i)),$$

$$M_1(p_{\tau_3,0,1}(j)) > M_2(p_{\tau_0,0,0}(i)),$$

$$M_3(p_{\tau_3,0,1}(j)) = M_3(p_{\tau_0,0,0}(i)),$$

$$M_4(p_{\tau_3,0,1}(j)) = M_4(p_{\tau_0,0,0}(i))$$

for any $i \in I_\nu$. Thus $((\mathscr{T}^\nu_{\tau_3,0,1}(\xi) \backslash (\mathscr{T}^\nu_{\tau_2,0,1}(\xi) \backslash (\mathscr{T}^\nu_{\tau_0,0,1}(\xi) \backslash \mathscr{T}^\nu_{\tau_5,0,1}(\xi))$ is a subset of $((\mathscr{T}^\nu_{\tau_0,0,0}(\xi) \backslash \mathscr{T}^\nu_{\tau_4,0,0}(\xi) \backslash \mathscr{T}^\nu_{\tau_3,0,0}(\xi) \backslash \mathscr{T}^\nu_{\tau_1,0,0}(\xi)))$ and $N_\nu(\mu+\delta)$ is the number of integral quadruples $i = (i_1, i_2, i_3, i_4)$ satisfying

$$\text{(9)} \qquad \begin{cases} \min\{h_1, h_2, h_3\} > x_1 - b_1 - i_1 - 9/2 \geq \max\{l_1, l_2, l_3, l_4\}, \\ x_1 - b_1 - i_1 - 9/2 \equiv h_1 \pmod 2 \end{cases}$$

where we put

$$h_1 = M_1(p_{\tau_3,0,1}(j)) = -x_2 + x_3 + x_4 + b_2 + b_3 + b_4 + i_2 - i_3 - i_4 + 7/2,$$

$$h_2 = M_2(p_{\tau_4,0,0}(i)) = x_2 - x_3 + x_4 + b_2 - b_3 - b_4 - i_2 + i_3 - i_4 + 3/2,$$

$$h_3 = M_3(p_{\tau_3,0,0}(i)) = x_2 + x_3 - x_4 - b_2 + b_3 - b_4 - i_2 - i_3 + i_4 - 1/2,$$

$$l_1 = M_1(p_{\tau_0,0,0}(j)) = -x_2 - x_3 - x_4 + b_2 + b_3 + b_4 + i_2 + i_3 + i_4 + 3/2,$$

$$l_2 = M_2(p_{\tau_0,0,0}(j)) = -x_2 + x_3 + x_4 + b_2 - b_3 - b_4 + i_2 - i_3 - i_4 + 3/2,$$

$$l_3 = M_3(p_{\tau_0,0,0}(j)) = x_2 - x_3 + x_4 - b_2 + b_3 - b_4 - i_2 + i_3 - i_4 - 1/2,$$

$$l_4 = M_4(p_{\tau_0,0,0}(j)) = x_2 + x_3 - x_4 - b_2 - b_3 + b_4 - i_2 - i_3 + i_4 - 5/2.$$

Thus we have the following

THEOREM 4. *The multiplicity $m(\lambda, \mu)$ of $V^K(\mu)$ in $V^G(\lambda)$ is given by*

$$\text{(10)} \qquad m(\lambda, \mu) = N_{\lambda+\delta}(\mu+\delta) + N_{(\lambda+\delta)^*}(\mu+\delta) - N_{(\lambda+\delta)^{**}}(\mu+\delta)$$

where

$$\lambda+\delta=(a_1+11/2)\varepsilon_1+(a_2+5/2)\varepsilon_2+(a_3+3/2)\varepsilon_3+(a_4+1/2)\varepsilon_4$$

$$(\lambda+\delta)^*=\frac{a_1+a_2+a_3-a_4+9}{2}\varepsilon_1+\frac{a_1+a_2-a_3+a_4+7}{2}\varepsilon_2+\frac{a_1-a_2+a_3+a_4+5}{2}\varepsilon_3+\frac{a_1-a_2-a_3-a_4+1}{2}\varepsilon_4$$

$$(\lambda+\delta)^{**}=\frac{a_1+a_2+a_3+a_4+10}{2}\varepsilon_1+\frac{a_1+a_2-a_3-a_4+6}{2}\varepsilon_2+\frac{a_1-a_2+a_3-a_4+4}{2}\varepsilon_3+\frac{a_1-a_2-a_3+a_4+2}{2}\varepsilon_4$$

and $N_\nu(\mu+\delta)$ *is the number of integral quadruples*

$$i=(i_1,i_2,i_3,i_4)\in([1,x_1-x_2]\times[1,x_2-x_3]\times[1,x_3-x_4]\times[1,2x_4])\cap \boldsymbol{Z}^4$$

satisfying

$$(11)\quad\begin{cases} b_1+b_2+b_3+b_4+i_1+i_2-i_3-i_4-x_1-x_2+x_3+x_4+8>0,\\ b_1+b_2-b_3-b_4+i_1-i_2+i_3-i_4-x_1+x_2-x_3+x_4+6>0,\\ b_1-b_2+b_3-b_4+i_1-i_2-i_3+i_4-x_1+x_2+x_3-x_4+4>0,\\ -b_1-b_2-b_3-b_4-i_1-i_2-i_3-i_4+x_1+x_2+x_3+x_4-6\geq 0,\\ -b_1-b_2+b_3+b_4-i_1-i_2+i_3+i_4+x_1+x_2-x_3-x_4-6\geq 0,\\ -b_1+b_2-b_3+b_4-i_1+i_2-i_3+i_4+x_1-x_2+x_3-x_4-4\geq 0,\\ -b_1+b_2+b_3-b_4-i_1+i_2+i_3-i_4+x_1-x_2-x_3+x_4-2\geq 0,\\ \sum_{l=1}^4 x_l+\sum_{l=1}^4 b_l+\sum_{l=1}^4 i_l\equiv 0 \pmod 2. \end{cases}$$

PROPOSITION 1. *Let* $\nu=\sum_{i=1}^4 x_i\varepsilon_i$ *be an integral weight with* $x_1>x_2>x_3>x_4>0$ *and* $\mu=\sum_{i=1}^4 b_i\varepsilon_i$ *be a dominant integral weight of Spin*(9) *with* $b_3+b_4=0$. *Then* $N_\nu(\mu+\delta)$ *is the number of integral quadruples*

$$i=(i_1,i_2,i_3,i_4)\in([1,x_1-x_2]\times[1,x_2-x_3]\times[1,x_3-x_4]\times[1,2x_4])\cap \boldsymbol{Z}^4$$

satisfying

$$\begin{cases} x_1+x_2-x_3-x_4-b_1-b_2-i_1-i_2+i_3+i_4-6=0,\\ x_2-x_4-b_2-b_4-i_2+i_4-2\geq 0,\\ -x_2+x_3+b_2-b_3+i_2-i_3+1\geq 0,\\ -x_2+x_4+b_2-b_4+i_2-i_4+2\geq 0. \end{cases}$$

PROPOSITION 2. *Let* $\nu=\sum_{i=1}^4 x_i\varepsilon_i$ *be an integral weight with* $x_1>x_2>x_3>x_4>0$ *and* $\mu=\sum_{i=1}^4 b_i\varepsilon_i$ *be a dominant integral weight of Spin*(9) *with* $b_3=b_4$. *Then* $N_\nu(\mu+\delta)$ *is the number of integral quadruples*

$$i = (i_1, i_2, i_3, i_4) \in ([1, x_1 - x_2] \times [1, x_2 - x_3] \times [1, x_3 - x_4] \times [1, 2x_4]) \cap \boldsymbol{Z}^4$$

satisfying

$$\begin{cases} x_1 - x_2 - x_3 + x_4 - b_1 + b_2 - i_1 + i_2 + i_3 - i_4 - 2 = 0, \\ -x_2 + x_4 + b_2 + b_4 + i_2 - i_4 + 2 \geq 0, \\ -x_3 + x_4 + b_2 - b_3 + i_3 - i_4 + 1 \geq 0, \\ x_2 + x_3 - b_2 - b_3 - i_2 - i_3 - 2 \geq 0, \\ x_2 - x_4 - b_2 + b_4 - i_2 + i_4 - 2 \geq 0. \end{cases}$$

Proof of Proposition 1. Let $i = (i_1, i_2, i_3, i_4)$ be an integral quadruple satisfying (9). Since $h_1 - l_2 = 2$, we have

$$\min\{h_1, h_2, h_3\} = h_1 > x_1 - b_1 - i_1 - 9/2 = l_2 = \max\{l_1, l_2, l_3, l_4\}.$$

Inequalities $h_2 \geq h_1$, $l_2 \geq l_1$ hold for any $i \in I_\nu$. From

$$x_1 - b_1 - i_1 - 9/2 - l_2 = x_1 + x_2 - x_3 - x_4 - b_1 - b_2 - i_1 - i_2 + i_3 + i_4 - 6 = 0,$$

$$h_3 - h_1 = 2(x_2 - x_4 - b_2 - b_4 - i_2 + i_4 - 2) \geq 0,$$

$$l_2 - l_3 = 2(-x_2 + x_3 + b_2 - b_3 + i_2 - i_3 + 1) \geq 0,$$

$$l_2 - l_4 = 2(-x_2 + x_4 + b_2 - b_4 + i_2 - i_4 + 2) \geq 0$$

we obtain the Proposition. Q.E.D.

4. Spectra of $\Delta^p(\mathbf{Ca}P^2)$ $(0 \leq p \leq 5)$

4.1 Multiplicity of irreducible $Spin(9)$-submodules. In order to calculate the spectra of the Laplacian $\Delta^p(\mathbf{Ca}P^2)$ $(0 \leq p \leq 5)$, we calculate the multiplicity $m(\lambda, \mu)$ of $V^{Spin(9)}(\mu)$ in $V^{F_4}(\lambda)$ for some classes of the dominant integral weight μ of K.

Case $\mu = m\mu_1 + n\mu_4$.

Lemma 6. *Let* $\lambda = \sum_{i+1}^4 a_i \varepsilon_i \in \mathscr{D}(F_4)$ *and* $\mu = m\mu_1 + n\mu_4 \in \mathscr{D}(Spin(9))$. *Then the multiplicity* $m(\lambda, \mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is equal to*

$$\begin{aligned} &\# \left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq j \leq \min(-a_3 + a_4 + m + 1, 2a_4 + 1) \\ a_2 + a_3 + j - 1 \leq m + n \leq a_1 - a_4 + j - 1 \end{array} \right\} \\ (12) \quad & - \# \left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq j \leq \min(-a_3 - a_4 + m, a_1 - a_2 - a_3 - a_4 + 1) \\ a_2 + a_3 + j - 1 \leq m + n \leq a_2 + a_3 + 2a_4 + j - 1 \end{array} \right\}. \end{aligned}$$

PROOF OF LEMMA 6. Let $\lambda = \sum_{i=1}^{4} x_i\varepsilon_i$ be an integral weight with $x_1 > x_2 > x_3 > x_4 > 0$ and $\mu = m\mu_1 + n\mu_4 = (m/2+n)\varepsilon_1 + (m/2)(\varepsilon_2+\varepsilon_3-\varepsilon_4)$ be a dominant integral weight of $Spin(9)$. By Proposition 1, $N_\lambda(\mu+\delta)$ is the number of integral quadruples

$$i = (i_1, i_2, i_3, i_4) \in ([1, x_1-x_2] \times [1, x_2-x_3] \times [1, x_3-x_4] \times [1, 2x_4]) \cap \boldsymbol{Z}^4$$

satisfying

$$(13)\quad \begin{cases} x_1+x_2-x_3-x_4-m-n-i_1-i_2+i_3+i_4-6=0 & (13.1)\\ x_2-x_4-i_2+i_4-2 \geq 0 & (13.2)\\ -x_2+x_3+i_2-i_3+1 \geq 0 & (13.3)\\ -x_2+x_4+m+i_2-i_4+2 \geq 0 & (13.4) \end{cases}$$

From (13.1) and (13.3), we have $i_2 = x_2-x_3$, $i_3 = 1$ and $i_1 = i_4+x_1-x_4-m-n-5$. The integral quadruple $i = (i_4+x_1-x_4-m-n-5, x_2-x_3, 1, i_4)$ satisfies (13.2) automatically and satisfies (13.4) if and only if $i_4 < -x_3+x_4+m+2$. The integral quadruple i is contained in $[1, x_1-x_2] \times [1, x_2-x_3] \times [1, x_3-x_4] \times [1, 2x_4]$ if and only if

$$x_2-x_4-5 \leq m+n-i_4 \leq x_1-x_4-6, \quad 1 \leq i_4 \leq 2x_4.$$

Thus $N_\lambda(\mu+\delta)$ is equal to

$$\#\left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq j \leq \min(2x_4, -x_3+x_4+m+2) \\ x_2-x_4-5 \leq m+n-j \leq x_1-x_4-6 \end{array} \right\}.$$

We have

$$\begin{aligned} &N_{\lambda+\delta}(\mu+\delta) - N_{(\lambda+\delta)^{**}}(\mu+\delta) \\ &= \#\left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq j \leq \min(-a_3+a_4+m+1, 2a_4+1) \\ a_2-a_4-3 \leq m+n-j \leq a_1-a_4-1 \end{array} \right\} \\ &\quad - \#\left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq j \leq \min(-a_3+a_4+m+1, 2a_4+1) \\ a_2-a_4-3 \leq m+n-j \leq a_2+a_3-2 \end{array} \right\} \\ &\quad - \#\left\{ k \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq k \leq -a_3+a_4+m+1, \\ 2a_4+2 \leq k \leq a_1-a_2-a_3-a_4+2 \\ a_2-a_4-3 \leq m+n-k \leq a_2+a_3-2 \end{array} \right\} \end{aligned}$$

$$= \#\left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq j \leq \min(-a_3 + a_4 + m + 1, 2a_4 + 1) \\ a_2 + a_3 - 1 \leq m + n - j \leq a_1 - a_4 - 1 \end{array} \right\}$$

$$- \#\left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq k \leq \min(-a_3 - a_4 + m, a_1 - a_2 - a_3 - a_4 + 1) \\ a_2 + a_4 - 2 \leq m + n - k \leq a_2 + a_3 + 2a_4 - 1 \end{array} \right\}$$

We obtain the Lemma by adding

$$N_{(\lambda+\delta)^*}(\mu + \delta) = \#\left\{ j \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} 1 \leq k \leq \min(-a_3 - a_4 + m, a_1 - a_2 - a_3 - a_4 + 1) \\ a_2 + a_4 - 2 \leq m + n - j \leq a_2 + a_3 - 2 \end{array} \right\}$$

to the both side of the above. Q.E.D.

THEOREM 5. *Let* $\lambda = \sum_{i=1}^{4} a_i \varepsilon_i \in \mathscr{D}(F_4)$ *and* $\mu = n\mu_4 \in \mathscr{D}(Spin(9))$. *If* $a_3 = a_4$ *and* $a_2 + a_3 \leq n \leq a_1 - a_3$ *then the multiplicity* $m(\lambda, \mu)$ *is equal to 1 otherwise* $m(\lambda, \mu)$ *is equal to* 0.

THEOREM 6. *Let* $\lambda = \sum_{i=1}^{4} a_i \varepsilon_i \in \mathscr{D}(F_4)$. *If* $\mu = \mu_1 + n\mu_4 \in \mathscr{D}(Spin(9))$, *then the multiplicity* $m(\lambda, \mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given as follows*:

Condition			$m(\lambda, \mu)$
$a_3 - a_4$			
1	$a_4 \geq 0$	$a_2 + a_3 - 1 \leq n \leq a_1 - a_4 - 1$	1
0	$a_4 \geq \frac{1}{2}$	$a_2 + a_3 \leq n \leq a_1 - a_3 - 1$	2
		$n = a_2 + a_3 - 1$ or $n = a_1 - a_4$	1
	$a_4 = 0$	$a_2 \leq n \leq a_1 - 1$	1
otherwise			0

THEOREM 7. *Let* $\lambda = \sum_{i=1}^{4} a_i \varepsilon_i \in \mathscr{D}(F_4)$. *If* $\mu = 2\mu_1 + n\mu_4 \in \mathscr{D}(Spin(9))$, *then the multiplicity* $m = m(\lambda, \mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given as follows:*

Condition			$m(\lambda,\mu)$
a_3-a_4			
2		$a_2+a_3-2\le n\le a_1-a_4-2$	1
1	$a_4\ge\frac{1}{2}$	$n=a_2+a_3-2$ or a_1-a_4-1	1
		$a_2+a_3-1\le n\le a_1-a_4-2$	2
	$a_4=0$	$a_2\le n\le a_1-2$	1
0	$a_4\ge 1$, $a_1>a_2+a_3+a_4$	$a_2+a_3\le n\le a_1-a_4-2$	3
		$n=a_2+a_3-1$ or a_1-a_4-1	2
		$n=a_2+a_3-2$ or a_1-a_4	1
	$a_4\ge 1$, $a_1=a_2+a_3+a_4$	$a_2+a_3-2\le n\le a_1-a_4$	1
	$a_4=\frac{1}{2}$, $a_1>a_2+a_3+a_4$	$a_2+\frac{1}{2}\le n\le a_1-\frac{5}{2}$	2
		$n=a_2-\frac{1}{2}$ or $a_1-\frac{3}{2}$	1
	$a_4=0, a_1\ge a_2+1$	$a_2\le n\le a_1-2$	1
otherwise			0

PROOF OF THEOREM 5. The second term of (12) is equal to 0, for $\min(-a_3-a_4+m, a_1-a_2-a_3-a_4+1)=-a_3-a_4\le 0$. Since $\min(-a_3+a_4+1, 2a_4+1)\le 1$ (the equality holds if and only if $a_3=a_4$) the first term of (12) is less than or equal to 1. The first term of (12) is equal to 1 if and only if $a_3=a_4$ and $a_2+a_3\le n\le a_1-a_4$. Thus we obtain the theorem. Q.E.D.

Similarly we can prove Theorem 6 and Theorem 7. So we omit the proof of them.

CASE $\mu=m\mu_3+n\mu_4$. In this case, we can prove the following by a similar manner to the proof of lemma 6.

LEMMA 7. *Let $\lambda=\sum_{i=1}^4 a_i\varepsilon_i\in\mathscr{D}(F_4)$ and $\mu=m\mu_3+n\mu_4\in\mathscr{D}(Spin(9))$. Then the multiplicity $m(\lambda,\mu)$ of $V^{Spin(9)}(\mu)$ in $V^{F_4}(\lambda)$ is equal to*

$$\#\left\{ j\in \boldsymbol{N} \,\middle|\, \begin{array}{l} 1\le j\le a_2-a_3+1 \\ a_2-a_4-m+1\le j\le a_2+a_4-m+1 \\ a_3+a_4-1\le m+n-j\le a_1-a_2-1 \end{array} \right\}$$

$$-\#\left\{ j \in N \,\middle|\, \begin{array}{l} 1 \le j \le a_2 - a_3 + 1 \\ a_2 + a_4 - m + 2 \le j \le a_1 - a_3 - m + 2 \\ a_3 - a_4 - 2 \le m + n - j \le a_3 + a_4 - 2 \end{array} \right\}.$$

Using the above lemma, we can prove the following theorem.

THEOREM 8. *Let* $\lambda = \sum_{i=1}^{4} a_i\varepsilon_i \in \mathscr{D}(F_4)$. *If* $\mu = \mu_3 + n\mu_4 \in \mathscr{D}(Spin(0))$, *then the multiplicity* $m(\lambda, \mu)$ *of* $V^{Spin(9))}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given as follows*:

Condition			$m(\lambda,\mu)$
$a_3 - a_4$			
1		$a_2 + a_3 - 2 \le n \le a_1 - a_4 - 2$	1
0	$a_2 = a_3 = a_4 > 0$	$a_2 - 1 \le n \le a_1 - a_2 - 1$	1
	$a_2 > a_3 = a_4 > 0$	$n = a_2 + a_4 - 2$ or $n = a_1 - a_3 - 1$	1
		$a_2 + a_3 - 1 \le n \le a_1 - a_3 - 2$	2
	$a_2 > a_3 = a_4 = 0$	$a_2 - 1 \le n \le a_1 - 2$	1
otherwise			0

CASE $\mu = m\mu_2 + n\mu_4 (m = 1, 2)$.

THEOREM 9. *Let* $\lambda = \sum_{i=1}^{4} a_i\varepsilon_i \in \mathscr{D}(F_4)$. *If* $\mu = \mu_2 + n\mu_4 \in \mathscr{D}(Spin(9))$, *then the multiplicity* $m(\lambda, \mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given as follows*:

Condition			$m(\lambda,\mu)$
$a_3 - a_4$			
1		$a_2 + a_4 \le n \le a_1 - a_3 - 1$	2
		$n = a_2 + a_4 - 1$ or $n = a_1 - a_3$	1
0	$a_2 > a_3 = a_4 \ge \frac{1}{2}$	$a_2 + a_4 - 1 \le n \le a_1 - a_3 - 1$	2
	$a_2 = a_3 = a_4 \ge \frac{1}{2}$	$a_2 + a_4 - 1 \le n \le a_1 - a_3 - 1$	1
	$a_2 > a_3 = a_4 = 0$	$a_2 - 1 \le n \le a_1 - 1$	1
otherwise			0

THEOREM 10. *Let* $\lambda = \sum_{i=1}^{4} a_i\varepsilon_i \in \mathscr{D}(F_4)$. *If* $\mu = 2\mu_2 + n\mu_4 \in \mathscr{D}(Spin(9))$, *then the multiplicity* $m(\lambda,\mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given as follows*:

Condition			$m(\lambda,\mu)$
$a_3 - a_4$			
2	$a_1 > a_2 + a_3 + a_4$	$n = a_2 + a_4 - 2$ or $a_1 - a_3$	1
		$n = a_2 + a_4 - 1$ or $a_1 - a_3 - 1$	2
		$a_2 + a_4 \leq n \leq a_1 - a_3 - 2$	3
	$a_1 = a_2 + a_3 + a_4$	$a_2 + a_4 - 2 \leq n \leq a_1 - a_3$	1
1	$a_2 - a_3 \geq 1, a_4 \geq \frac{1}{2}$	$n = a_2 + a_4 - 2$ or $a_1 - a_3 - 1$	2
		$a_2 + a_4 - 1 \leq n \leq a_1 - a_3 - 2$	4
	$a_2 - a_3 \geq 1, a_4 = 0$	$n = a_2 - 2$ or $a_1 - 2$	1
		$a_2 - 1 \leq n \leq a_1 - 3$	2
	$a_2 = a_3, a_4 \geq \frac{1}{2}$	$n = a_2 + a_4 - 2$ or $a_1 - a_3 - 1$	1
		$a_2 + a_4 - 1 \leq n \leq a_1 - a_3 - 2$	2
0	$a_2 - a_3 \geq 2, a_4 \geq 1$	$a_2 + a_4 - 2 \leq n \leq a_1 - a_3 - 2$	3
	$a_2 - a_3 \geq 2, a_4 = \frac{1}{2}$	$a_2 + a_4 - 2 \leq n \leq a_1 - a_3 - 2$	2
	$a_2 - a_3 = 1, a_4 = \frac{1}{2}$	$a_2 + a_4 - 2 \leq n \leq a_1 - a_3 - 2$	2
	$a_2 - a_3 = 1, a_4 \geq 1$	$a_2 + a_4 - 2 \leq n \leq a_1 - a_3 - 2$	1
	$a_2 = a_3, a_4 \geq 1$	$a_2 + a_4 - 2 \leq n \leq a_1 - a_3 - 2$	1
	$a_4 = 0$	$a_2 - 2 \leq n \leq a_1 - 2$	1
otherwise			0

PROOF OF THEOREM 9. Let $\mu = \mu_2 + n\mu_4 \in \mathscr{D}(Spin(9))$. Then, for an integral weight $\nu = \sum_{i=1}^{4} x_i\varepsilon_i (x_1 > x_2 > x_3 > x_4)$, $N_\nu(\mu+\delta)$ is the number of integral quadruples

$$i = (i_1, i_2, i_3, i_4) \in I_\nu = ([1, x_1 - x_2] \times [1, x_2 - x_3] \times [1, x_3 - x_4] \times [1, 2x_4]) \cap \boldsymbol{Z}^4$$

satisfying

$$(14) \qquad \begin{cases} x_1 + x_2 - x_3 - x_4 - n - i_1 - i_2 + i_3 + i_4 - 8 = 0, \\ x_2 - x_4 - 3 = i_2 - i_4, \\ i_2 - i_3 \geq x_2 - x_3 - 2. \end{cases}$$

The integral quadruples i satisfying $i_2 \leq x_2 - x_3, i_3 \geq 1$ and (14) are as follows;

$$i^{(1)} = (x_1 - x_2 - n - 4, x_2 - x_3, 1, -x_3 + x_4 + 3),$$

$$i^{(2)} = (x_1 - x_2 - n - 4, x_2 - x_3 - 1, 1, -x_3 + x_4 + 2),$$

$$i^{(3)} = (x_1 - x_2 - n - 3, x_2 - x_3, 2, -x_3 + x_4 + 3).$$

Since

$$N_\nu(\mu + \delta) = \#\{k | i^{(k)} \in I_\nu\},$$

by examining the condition $i^{(k)} \in I_\nu$, we obtain the following

$x_3 - x_4$	Condition		$N_\nu(\lambda + \delta)$
2		$x_2 - x_3 - 3 \leq n \leq x_1 - x_3 - 5$	2
		$n = x_2 - x_3 - 4$ or $x_1 - x_3 - 4$	1
1	$x_2 - x_3 \geq 2, x_4 \geq 1$	$x_2 - x_3 - 4 \leq n \leq x_1 - x_3 - 5$	2
	$x_2 - x_3 \geq 2, x_4 \geq \frac{1}{2}$	$x_2 - x_3 - 4 \leq n \leq x_1 - x_3 - 5$	1
	$x_2 - x_3 = 1, x_4 = 1$	$x_2 - x_3 - 4 \leq n \leq x_1 - x_3 - 5$	1
otherwise			0

For each case $\nu = \lambda + \delta$, $\nu = (\lambda + \delta)^*$ and $\nu = (\lambda + \delta)^{**}$, we have

$$N_{\lambda+\delta}(\mu + \delta) = \begin{cases} 2, & \text{if } a_3 - a_4 = 1, a_2 - a_3 - 2 \leq n \leq a_1 - a_3 - 1, \\ 1, & \text{if } a_3 - a_4 = 1, n = a_2 - a_3 - 3 \text{ or } a_1 - a_3, \\ 2, & \text{if } a_2 > a_3 = a_4 \geq \frac{1}{2}, a_2 - a_3 - 3 \leq n \leq a_1 - a_3 - 1, \\ 1, & \text{if } a_2 > a_3 = a_4 = 0, a_2 - a_3 - 3 \leq n \leq a_1 - a_3 - 1, \\ 1, & \text{if } a_2 = a_3 = a_4 > 0, a_2 - a_3 - 3 \leq n \leq a_1 - a_3 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_{(\lambda+\delta)^*}(\mu + \delta) = \begin{cases} 1, & \text{if } a_3 = a_4 = 0, n = a_2 - a_3 - 3 \text{ or } a_2 - a_4 - 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$N_{(\lambda+\delta)^{**}}(\mu + \delta) = \begin{cases} 2, & \text{if } a_3 - a_4 = 1, a_2 - a_3 - 2 \leq n \leq a_2 + a_4 - 2, \\ 1, & \text{if } a_3 - a_4 = 1, n = a_2 - a_3 - 3 \text{ or } a_2 + a_4 - 1, \\ 2, & \text{if } a_2 > a_3 = a_4, a_2 - a_3 - 3 \leq n \leq a_2 + a_4 - 1, \\ 1, & \text{if } a_2 = a_3 = a_4, a_2 - a_3 - 3 \leq n \leq a_2 + a_4 - 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we obtain the theorem from (10). Q.E.D.

Though they are much complicated, we can prove theorem 10 and the following theorem 11 by similar manner to the proof of theorem 9. So we omit the proof of them.

CASE $\mu = \mu_1 + \mu_2 + n\mu_4$.

THEOREM 11. *Let* $\lambda = \sum_{i=1}^{4} a_i \varepsilon_i \in \mathscr{D}(F_4)$. *If* $\mu = \mu_1 + \mu_2 + n\mu_4 \in \mathscr{D}(Spin(9))$, *then the multiplicity* $m(\lambda, \mu)$ *of* $V^{Spin(9)}(\mu)$ *in* $V^{F_4}(\lambda)$ *is given as follows*:

Condition			$m(\lambda, \mu)$
$a_3 - a_4$			
2		$n = a_2 + a_4 - 1$ or $a_1 - a_3$	1
		$a_2 + a_4 \leq n \leq a_1 - a_3 - 1$	2
1	$a_2 > a_3 > 1$, $a_1 > a_2 + a_3 + a_4$	$n = a_2 + a_4 - 2$ or $a_1 - a_3$	1
		$n = a_2 + a_4 - 1$ or $a_1 - a_3 - 1$	4
		$a_2 + a_4 \leq n \leq a_1 - a_3 - 1$	5
	$a_2 > a_3 > 1$, $a_1 = a_2 + a_3 + a_4$	$n = a_2 + a_4 - 2$ or $a_1 - a_3$	1
		$n = a_2 + a_4 - 1$	3
	$a_2 > a_3 = 1$, $a_1 > a_2 + a_3 + a_4$	$n = a_2 - 1$ or $a_1 - 2$	2
		$a_2 - 1 \leq n \leq a_1 - 3$	3
	$a_2 = a_3 > 1$, $a_1 = a_2 + a_3 + a_4$	$n = a_2 - 1$	1
	$a_2 = a_3 > 1$ $a_1 > a_2 + a_3 + a_4$	$n = a_2 + a_4 - 2$ or $a_1 - a_3$	1
		$n = a_2 + a_4 - 1$ or $a_1 - a_3 - 1$	3
		$n = a_2 + a_4 \leq a_1 - a_3 - 2$	4
	$a_2 = a_3 > 1$, $a_1 = a_2 + a_3 + a_4$	$n = a_2 + a_4 - 2$ or $a_1 - a_3$	1
		$n = a_1 - a_3 - 1$	2
	$a_2 = a_3 = 1$	$1 \leq n = a_1 - 3$	2
	$a_2 = a_3 = 1, a_1 > 3$	$n = 0$ or $a_1 - 2$	1

Condition			$m(\lambda,\mu)$
$a_3 - a_4$			
0	$a_2 > a_3 > \frac{1}{2}$	$n = a_2 + a_4 - 2$ or $a_1 - a_3 - 1$	2
		$a_2 + a_4 - 1 \leq n \leq a_1 - a_3 - 2$	4
	$a_2 > a_3 = \frac{1}{2}$	$n = a_2 + a_4 - 2$ or $a_1 - a_3 - 1$	1
		$a_2 + a_4 - 1 \leq n \leq a_1 - a_3 - 2$	3
	$a_1 > a_2 > a_3 = 0$	$n = a_2 - 1$ or $a_1 - 2$	1
	$a_2 = a_3 > \frac{1}{2}$	$n = a_2 + a_4 - 2$ or $a_1 - a_3 - 1$	1
		$a_2 + a_4 - 1 \leq n \leq a_1 - a_3 - 2$	2
	$a_2 = a_3 = \frac{1}{2}$	$n \leq a_1 - \frac{5}{2}$	1
otherwise			0

4.2 Spectra of the Laplacian. We denote by Δ^p the Laplacian acting on p-forms on the Cayley projective plane. The set of eigenvalues of Δ^p is given as follows (see [3]);

$$\{\langle \lambda + 2\delta, \lambda \rangle | \lambda \in D(G), \dim_{\mathbf{C}} \mathrm{Hom}_K(\textstyle\bigwedge^p T_o(\mathbf{Ca}P^2)^C, V^{F_4}(\lambda)) \neq 0\}$$

By using Theorem 5–11, we can find all of the irreducible F_4-modules $V^{F_4}(\lambda)$ satisfying $\mathrm{Hom}_K(\bigwedge^p T_o(\mathbf{Ca}P^2)^C, V^{F_4}(\lambda)) \neq 0$. We give a list of them in table 2.

Remark. (1) Dimension of the complex irreducible representation $V^{F_4}\left(\sum_{i=1}^4 a_i\varepsilon_i\right)$ of G is calculated by using the Weyl's dimension formula.

$$\begin{aligned}
&\dim\left(V^{F_4}\left(\sum_{i=1}^{4} a_i\varepsilon_i\right)\right) \\
&= \frac{1}{24141680640000}(a_1 + a_2 + 8)(a_1 - a_2 + 3)(a_1 + a_2 + a_3 + a_4 + 10) \\
&\times (a_1 + a_2 + a_3 - a_4 + 9)(a_1 + a_2 - a_3 + a_4 + 7)(a_1 + a_2 - a_3 - a_4 + 6) \\
&\times (a_1 - a_2 + a_3 + a_4 + 5)(a_1 - a_2 + a_3 - a_4 + 4)(a_1 - a_2 - a_3 + a_4 + 2) \\
&\times (a_1 - a_2 - a_3 - a_4 + 1)(a_1 + a_3 + 7)(a_1 - a_3 + 4)(a_1 + a_4 + 6) \\
&\times (a_1 - a_4 + 5)(a_2 + a_3 + 4)(a_2 - a_3 + 1)(a_2 + a_4 + 3)(a_2 - a_4 + 2) \\
&\times (a_3 + a_4 + 2)(a_3 - a_4 + 1)(2a_4 + 1)(2a_3 + 3)(2a_2 + 5)(2a_1 + 11)
\end{aligned}$$

(2) The first eigenvalue of the Laplacian Δ^p acting on p-form $(0 \leq p \leq 5)$ on the Cayley projective plane $\mathbf{Ca}P^2$ is

p	First eigenvalue	mult.
0	0	1
1	12	26
2	18	1053
3	24	4096
4	46	628
5	24	4096

Table 2: Spectra of the Laplacian Δ^p on $\mathbf{Ca}P^2$.

p	highest weight		eigenvalue	mult.
0	$k\lambda_4$	$(k \geq 0)$	$k^2 + 11k$	1
1	$\lambda_1 + k\lambda_4$	$(k \geq 0)$	$k^2 + 13k + 18$	1
	$\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 14k + 24$	1
	$k\lambda_4$	$(k \geq 1)$	$k^2 + 11k$	1
2	$\lambda_1 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 16k + 46$	1
	$\lambda_1 + k\lambda_4$	$(k \geq 0)$	$k^2 + 13k + 18$	$2\ (k \geq 1)$
				$1\ (k = 0)$
	$\lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 15k + 36$	2
	$\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 14k + 24$	2
3	$\lambda_1 + \lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 60$	1
	$\lambda_1 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 16k + 46$	3
	$\lambda_2 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 18k + 68$	1
	$2\lambda_1 + k\lambda_4$	$(k \geq 0)$	$k^2 + 15k + 40$	1
	$2\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 54$	2
	$\lambda_1 + k\lambda_4$	$(k \geq 1)$	$k^2 + 13k + 18$	2
	$\lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 15k + 36$	$3\ (k \geq 1)$
				$2\ (k = 0)$
	$\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 14k + 24$	$3\ (k \geq 1)$
				$1\ (k = 0)$
	$k\lambda_4$	$(k \geq 3)$	$k^2 + 11k$	1

Table 2 (continued)

p	highest weight		eigenvalue	mult.
4	$2\lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 19k + 84$	1
	$\lambda_1 + 2\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 19k + 80$	1
	$\lambda_1 + \lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 60$	2
	$2\lambda_1 + k\lambda_4$	$(k \geq 0)$	$k^2 + 15k + 40$	3 $(k \geq 1)$
				2 $(k = 0)$
	$2\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 54$	6 $(k \geq 1)$
				5 $(k = 0)$
	$\lambda_2 + k\lambda_4$	$(k \geq 1)$	$k^2 + 15k + 36$	2
	$\lambda_1 + k\lambda_4$	$(k \geq 1)$	$k^2 + 13k + 18$	3 $(k \geq 3)$
				2 $(k = 1, 2)$
	$k\lambda_4$	$(k \geq 2)$	$k^2 + 11k$	3 $(k \geq 5)$
				2 $(k = 3, 4)$
	$2\lambda_1 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 18k + 72$	1
	$3\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 20k + 90$	1
	$\lambda_2 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 18k + 68$	3
	$\lambda_1 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 16k + 46$	5 $(k \geq 1)$
				4 $(k = 0)$
	$\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 14k + 24$	5 $(k \geq 2)$
				4 $(k = 1)$

Table 2 (continued)

p	highest weight		eigenvalue	mult.
5	$3\lambda_1 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 66$	1
	$\lambda_2 + 2\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 21k + 106$	1
	$2\lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 19k + 84$	1
	$\lambda_1 + 2\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 19k + 80$	4
	$\lambda_1 + \lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 60$	3 $(k \geq 1)$
				2 $(k = 0)$
	$2\lambda_1 + k\lambda_4$	$(k \geq 0)$	$k^2 + 15k + 40$	3 $(k \geq 1)$
				1 $(k = 0)$
	$2\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 17k + 54$	7 $(k \geq 1)$
				4 $(k = 0)$
	$\lambda_2 + k\lambda_4$	$(k \geq 0)$	$k^2 + 15k + 36$	6 $(k \geq 2)$
				5 $(k = 1)$
				2 $(k = 0)$
	$\lambda_1 + k\lambda_4$	$(k \geq 1)$	$k^2 + 13k + 18$	5 $(k \geq 3)$
				2 $(k = 1, 2)$
	$k\lambda_4$	$(k \geq 2)$	$k^2 + 11k$	3 $(k \geq 5)$
				1 $(k = 3, 4)$
	$\lambda_1 + \lambda_2 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 20k + 96$	1
	$2\lambda_1 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 18k + 72$	2
	$3\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 20k + 90$	3
	$\lambda_2 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 18k + 68$	5 $(k \geq 1)$
				4 $(k = 0)$
	$\lambda_1 + \lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 16k + 46$	8 $(k \geq 1)$
				4 $(k = 0)$
	$\lambda_3 + k\lambda_4$	$(k \geq 0)$	$k^2 + 14k + 24$	7 $(k \geq 3)$
				5 $(k = 2)$
				4 $(k = 1)$
				1 $(k = 0)$

References

[1] M. R. Bremner, R. V. Moody and J. Patera, *Dominant weight multiplicities for representations of simple Lie algebras*, Pure and Applied Math. Ser. 90, Marcel Dekker, New York and Basel (1985).

[2] P. Cariter, *On H. Weyl's character formula*, Bull. Amer. Math. Soc. 67 (1961), 228–230

[3] A. Ikeda and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbf{C})$*, Osaka J. Math. **15** (1978), 515–546.

[4] E. Kaneda, *The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces*, J. Math. Kyoto Univ. **23** (1983), 369–395 and **24** (1984), 141–162.

[5] J. Lepowsky, *Representations of semisimple Lie groups and an enveloping algebra decomposition*, Thesis, Massachusetts Institute of Technology (1970).

[6] ———, *Multiplicity formulas for certain semisimple Lie groups*, Bull. Amer. Math. Soc. **77** (1971), 601–605.

[7] H. Strese, *Zur harmonischen Analyse*, Math. Nachr. **98** (1980), 61–73.

[8] ———, *Spectren symmetrischen Räume*, ibid, 757-82.

[9] C. Tsukamoto, *The spectra of the Laplace-Beltrami operator on $SO(n+2)/SO(2) \times SO(n)$ and $Sp(n+1)/Sp(1) \times Sp(n)$*, Osaka J. Math. **18** (1981), 407–426.

[10] N. Wallach, *Harmonic analysis on homogeneous spaces*, Pure and Applied Math. Ser. 19, Marcel-Dekker, New-York (1973).

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