

SOME GENERALIZATIONS OF RAPID ULTRAFILTERS IN TOPOLOGY AND ID-FAN TIGHTNESS

By

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Abstract. In this paper, we introduce the weakly k -rapid points, for $1 \leq k < \omega$, and the rapid points of topological spaces. They extend the concept of rapid ultrafilter. It is evident from the definition that every weak P -point is a rapid point and a weakly k -rapid point for $1 \leq k < \omega$. We show: (a) there is a space containing a rapid, non-weak- P -point \Leftrightarrow there is a rapid ultrafilter on ω ; and (b) there is a space containing a weakly k -rapid, non-weak- P -point, for some $1 \leq k < \omega \Leftrightarrow$ there is a Q -point in $\beta(\omega) \setminus \omega \Leftrightarrow$ for every $1 \leq k < \omega$, there is a space which is weakly $(k+1)$ -rapid and is not weakly k -rapid. Assuming the existence of a Q -point in $\beta(\omega) \setminus \omega$, we give an example of a zero-dimensional homogeneous space without weak P -points such that all its points are rapid. Finally, the concept of Id-fan tightness is introduced as a generalization of countable strong fan tightness.

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1. Preliminaries.

By a space we mean a completely regular Hausdorff space, i. e. Tychonoff space. If X is a space and $x \in X$, then $\mathcal{N}(x)$ denotes the set of all neighborhoods of x . The closure of A in X is denoted by $\text{Cl}_x(A)$ or $\text{Cl}(A)$. For a set X , the set of all finite subsets of X is denoted by $[X]^{<\omega}$ and if $1 \leq m < \omega$, then $[X]^{\leq m} = \{A \subseteq X : |A| \leq m\}$. The Stone-Ćech compactification $\beta(\omega)$ of the natural numbers ω with the discrete topology can be viewed as the set of all ultrafilters on ω , and the remainder $\omega^* = \beta(\omega) \setminus \omega$ consists of all free ultrafilters on ω . For

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$p \in \omega^*$, $\xi(p)$ stands for the subspace $\{p\} \cup \omega$ of $\beta(\omega)$. All functions $f \in {}^\omega \omega$ considered throughout this paper assume only positive values.

G. Mokobodski [Mo] introduced the following class of ultrafilters to respond to a problem in measure theory.

1.1. DEFINITION. $p \in \omega^*$ is *rapid* if

$$\forall h \in {}^\omega \omega \exists A \in p \forall n < \omega (|A \cap h(n)| \leq n).$$

Other two kinds of interesting ultrafilters on ω are:

1.2. DEFINITION. Let $p \in \omega^*$. Then

(1) p is a *Q-point* if for every partition $\{B_n : n < \omega\}$ of ω in finite subsets, there is $A \in p$ such that $|A \cap B_n| \leq 1$ for every $n < \omega$;

(2) p is *semiselective* if $A_n \in p$ for $n < \omega$, then there is $a_n \in A_n$ for each $n < \omega$ such that $\{a_n : n < \omega\} \in p$.

In [CV], the authors say that $p \in \omega^*$ is a *Q-point* if $\forall \{B_n : n < \omega\} \subseteq [\omega]^{<\omega} \exists A \in p \forall n < \omega (|A \cap B_n| \leq 1)$. But, this definition is wrong since none $p \in \omega^*$ satisfies such a condition; indeed, if $p \in \omega^*$ and $B_n = n$ for $n < \omega$, then there is not $A \in p$ such that $|A \cap B_n| \leq 1$ for each $n < \omega$.

We know that every semiselective ultrafilter is rapid and every *Q-point* is rapid. The inclusions among these sorts of ultrafilters on ω are proper: It is shown in [M] that if there is a rapid ultrafilter, then there is also a rapid ultrafilter which is neither *P-point* and nor *Q-point*; (Kunen [K]) $MA \rightarrow \exists p \in \omega^*$ (p is semiselective and not *Q-point*); and Lafflamme [L] proved that $CON(ZFC) \rightarrow CON(ZFC + \exists p \in \omega^*$ (p is *Q-point* and not semiselective)). The existence of these ultrafilters is independent from the axioms of *ZFC*. In fact, Mokobodki [Mo] proved that *CH* implies the existence of rapid ultrafilters on ω ; Miller [M] established that $CON(ZFC) \rightarrow CON(ZFC + \text{there are no rapid ultrafilters})$; Mathias [Ma] and Taylor [T] showed that if there is a dominant family of functions in ${}^\omega \omega$ of cardinality ω_1 , then there exists a *Q-point* in ω^* (for another sufficient condition see [CV]); and the existence of semiselective ultrafilters under *MA* (σ -centered) is shown in [Bo].

In the next theorem, we give four conditions which are equivalent to the rapidness of ultrafilters on ω : clauses (4) and (5) motivated the notions of rapid points and weakly k -rapid points, for $1 \leq k < \omega$, which will be studied in section 2.

1.3. THEOREM. For $p \in \omega^*$, the following are equivalent:

- (1) p is rapid.
- (2) For every sequence $(B_n)_{n < \omega}$ of finite subsets of ω ,

$$\exists A \in p \forall n < \omega (|A \cap B_n| \leq n).$$

(3) There is $h \in {}^\omega \omega$ such that for every sequence $(B_n)_{n < \omega}$ of finite subsets of ω , $\exists A \in p \forall n < \omega (|A \cap B_n| \leq h(n))$.

(4) For every finite-to-one function $f \in {}^\omega \omega$ and every sequence $(B_n)_{n < \omega}$ of finite subsets of ω , $\exists A \in p \forall n < \omega (|A \cap B_n| \leq f(n))$.

(5) For every finite-to-one function $f \in {}^\omega \omega$ and given $B_n \in [\omega]^{< \omega}$, for $n < \omega$, such that $B_n \cap B_m = \emptyset$ whenever $n < m < \omega$,

$$\exists A \in p \forall n < \omega (|A \cap B_n| \leq f(n)).$$

PROOF. The equivalences (1) \Leftrightarrow (2) and (2) \Leftrightarrow (3) are shown in [M], and the implications (1) \Rightarrow (5), (4) \Rightarrow (3) are evident.

(1) \Rightarrow (4). Let $f \in {}^\omega \omega$ be finite-to-one. Without loss of generality, we may assume that $B_n \subseteq B_{n+1}$ for each $n < \omega$. Define $h \in {}^\omega \omega$ so that $h(m) = \max f^{-1}(m)$ if $f^{-1}(m) \neq \emptyset$, for $m < \omega$, and put $D_m = B_{h(m)}$ for $m < \omega$. By assumption, there is $A \in p$ such that $|A \cap D_m| = |A \cap B_{h(m)}| \leq m$ for all $m < \omega$. If $f(n) = m$ for $n < \omega$, then we have that $n \in f^{-1}(m)$ and $|A \cap B_n| \leq |A \cap B_{h(m)}| \leq m = f(n)$, as desired.

(5) \Rightarrow (3). Let $f \in {}^\omega \omega$ be finite-to-one and define $h: \omega \rightarrow \omega$ by $h(n) = \sum_{i=0}^n f(i)$ for each $n < \omega$. We shall verify that h satisfies our conditions. In fact, let $(B_n)_{n < \omega}$ be a sequence of finite subsets of ω . For $n < \omega$, set $A_n = B_n \setminus \bigcup_{j < n} B_j$. By hypothesis, there is $A \in p$ such that $|A \cap A_n| \leq f(n)$ for all $n < \omega$. Since $B_n \subseteq \bigcup_{j \leq n} A_j$ for each $n < \omega$, we have that $|A \cap B_n| \leq \sum_{i=0}^n f(i) = h(n)$ for each $n < \omega$.

We remark that if a function h satisfies the condition of (3), then h must be finite-to-one. If not, then there is $m < \omega$ such that $h^{-1}(m) = \{m_j: j < \omega\}$, where $m_j < m_{j+1}$ for $j < \omega$, but there is not $A \in p$ such that $|A \cap m_j| \leq h(m_j) = m$ for every $j < \omega$.

Our work in section 3 is based on the following definition.

1.4. DEFINITION. Let X be a space. Then

- (1) [Ar₁] X has *countable tightness* if for each $x \in X$ and $A \subseteq X$ such that $x \in \text{Cl}(A)$ there is a countable subset B of A such that $x \in \text{Cl}(B)$;
- (2) [Ar₂] X has *countable fan tightness* if for every $x \in X$ and every sequence $(A_n)_{n < \omega}$ of subsets of X such that $x \in \bigcap_{n < \omega} \text{Cl}(A_n)$, there exists $F_n \in [A_n]^{< \omega}$ such that $x \in \text{Cl}(\bigcup_{n < \omega} F_n)$;
- (3) [S] X has *countable strong fan tightness* if for every $x \in X$ and every

sequence $(A_n)_{n < \omega}$ of subsets of X such that $x \in \bigcap_{n < \omega} \text{Cl}(A_n)$, there exists $x_n \in A_n$ such that $x \in \text{Cl}(\{x_n : n < \omega\})$.

A natural generalization of countable strong fan tightness is investigated in section 3.

2. Rapid points and weakly k -rapid points.

Clauses (4) and (5) of Theorem 1.3 suggest the following definition.

2.1. DEFINITION. Let $f \in {}^\omega \omega$ and X a space.

(1) A point $x \in X$ is called *f -rapid* if for every sequence $(B_n)_{n < \omega}$ of finite subsets of $X \setminus \{x\}$, $\exists V \in \mathcal{N}(x) \forall n < \omega (|V \cap B_n| \leq f(n))$. X is said to be *f -rapid* if all points of X are f -rapid.

(2) A point $x \in X$ is called *weakly f -rapid* if for every sequence $(B_n)_{n < \omega}$ of finite subsets of $X \setminus \{x\}$ such that $B_n \cap B_m = \emptyset$ whenever $n < m < \omega$, $\exists V \in \mathcal{N}(x) \forall n < \omega (|V \cap B_n| \leq f(n))$. X is said to be *weakly f -rapid* if all points of X are weakly f -rapid.

If f is the identity function from ω to ω , then we simply say rapid (resp. weakly rapid) instead of f -rapid (resp. weakly f -rapid). The meaning of k -rapid and weakly k -rapid should be clear, for $1 \leq k < \omega$. It is evident that $p \in \omega^*$ is a Q -point iff it is weakly k -rapid in $\xi(p)$ for some $1 \leq k < \omega$.

Observe from Theorem 1.3 that $p \in \omega^*$ is a rapid ultrafilter iff p is f -rapid in $\xi(p)$ for each finite-to-one function $f \in {}^\omega \omega$ iff p is weakly f -rapid in $\xi(p)$ for each finite-to-one function $f \in {}^\omega \omega$. The next lemma shows that we cannot withdraw the finite-to-one condition.

2.2. LEMMA. For $p \in \omega^*$ and $f \in {}^\omega \omega$, the following are equivalent.

- (1) p is f -rapid in $\xi(p)$.
- (2) f is finite-to-one and p is a rapid ultrafilter.
- (3) f is finite-to-one and p is weakly f -rapid in $\xi(p)$.

PROOF. The implications (2) \Rightarrow (3) and (3) \Rightarrow (1) are direct consequences of Theorem 1.3.

(1) \Rightarrow (2). According to Theorem 1.3, it is enough to prove that f is finite-to-one. In fact, assume that there is $m < \omega$ such that $f^{-1}(m) = \{m_j : j < \omega\}$, where $m_j < m_{j+1}$ for $j < \omega$. Define $B_n = \{j < \omega : j \leq n\}$ for each $n < \omega$. Then there is $A \in p$ such that $|A \cap B_n| \leq f(n)$ for each $n < \omega$. In particular, $|A \cap B_{m_k}| \leq f(m_k) = m$ for every $k < \omega$. Since A is infinite, there must be $k < \omega$ such that $|A \cap B_{m_k}|$

$> m$, which is a contradiction.

2.3. LEMMA. *If $f : \omega \rightarrow \omega$ is not finite-to-one, then $p \in \omega^*$ is a Q -point iff p is weakly f -rapid in $\xi(p)$.*

PROOF. Only the sufficiency requires proof. Let $\{B_n : n < \omega\} \subseteq [\omega]^{<\omega}$ be a partition of ω and let $m < \omega$ such that $f^{-1}(m) = \{m_j : j < \omega\}$, where $m_j < m_{j+1}$, for $j < \omega$. Define $\{A_k : k < \omega\}$ by $A_{m_j} = \bigcup_{m_j \leq n < m_{j+1}} B_n$, for each $j < \omega$, and $A_k = \emptyset$ otherwise. By assumption, there is $A \in p$ such that $|A \cap A_k| \leq f(k)$ for all $k < \omega$. Hence, if $m_j \leq n < m_{j+1}$, for some $j < \omega$, then $|A \cap B_n| \leq |A \cap A_{m_j}| \leq f(m_j) = m$. We may write $A = \bigcup_{i \leq m} A_i$ so that $|A_i \cap B_n| \leq 1$ for each $i \leq m$ and each $n < \omega$. Since $A \in p$, there is $i \leq m$ such that $A_i \in p$ and then $|A_i \cap B_n| \leq 1$ for every $n < \omega$. Therefore, p is a Q -point.

We omit the proof of the next theorem since it is completely similar to that of Theorem 1.3.

2.4. THEOREM. *For a finite-to-one function $f \in {}^\omega\omega$ and $x \in X$, the following are equivalent.*

- (1) x is rapid in X .
- (2) x is f -rapid in X .
- (3) x is weakly f -rapid in X .

The relationship between weakly k -rapid points, for $1 \leq k < \omega$, and rapid points is established in the next corollary.

2.5. COROLLARY. *For $1 \leq k < \omega$, every weakly k -rapid point is rapid.*

PROOF. Let $1 \leq k < \omega$ and $x \in X$. Suppose that x is weakly k -rapid in X . Let $(B_n)_{n < \omega}$ be a sequence of finite subsets of $X \setminus \{x\}$. For $n < \omega$, set $A_n = B_n \setminus \bigcup_{j < n} B_j$. By assumption, there is $V \in \mathcal{N}(x)$ such that $|V \cap A_n| \leq k$ for each $n < \omega$. Hence, $|V \cap B_n| \leq \sum_{j \leq n} |V \cap A_j| \leq (n+1)k$, since $B_n \subseteq \bigcup_{j \leq n} A_j$, for each $n < \omega$. Thus, x is f -rapid, where $f(n) = (n+1)k$ for every $n < \omega$. The conclusion now follows from 2.4.

Next, we shall show that if $f \in {}^\omega\omega$ is not finite-to-one, then there is $k < \omega$ such that weak f -rapidness agrees with weak k -rapidness. It will be shown in 2.11 that for every $1 \leq k < \omega$ there is a space which is weakly $(k+1)$ -rapid and is not weakly k -rapid.

2.6. THEOREM. *Let $f \in {}^\omega\omega$ be non-finite-to-one and X a space. If $k =$*

$\min\{m < \omega : f^{-1}(m) \text{ is infinite}\}$, then $x \in X$ is weakly k -rapid iff it is weakly f -rapid.

PROOF. First, assume that $x \in X$ is weakly k -rapid. Let $(B_n)_{n < \omega}$ be a sequence in $[X \setminus \{x\}]^{< \omega}$ such that $B_i \cap B_j = \emptyset$ whenever $i < j < \omega$. Choose $r < \omega$ such that $f^{-1}(m) \subseteq r$ for each $m < k$. Then, we may find $V \in \mathcal{N}(x)$ such that $|V \cap B_n| \leq k$, for each $n < \omega$, and $V \cap B_n = \emptyset$ for every $n < r$. Hence, if $f(n) < k$, then $|V \cap B_n| = 0 \leq f(n)$. Thus, $|V \cap B_n| \leq f(n)$ for all $n < \omega$.

Now suppose that $x \in X$ is weakly f -rapid and let $(B_n)_{n < \omega}$ be a sequence in $[X \setminus \{x\}]^{< \omega}$ such that $B_i \cap B_j = \emptyset$ whenever $i < j < \omega$. Enumerate $f^{-1}(k)$ by $\{k_n : n < \omega\}$, where $k_n < k_{n+1}$ for $n < \omega$. For every $n < \omega$, set $D_{k_n} = B_n$ and $D_m = \emptyset$ otherwise. Then, there is $V \in \mathcal{N}(x)$ such that $|V \cap D_m| \leq f(m)$ for each $m < \omega$. Hence, $|V \cap B_n| = |V \cap D_{k_n}| \leq f(k_n) = k$ for $n < \omega$. This shows that x is weakly k -rapid.

The weakly f -rapid points, for $f \in {}^\omega \omega$, satisfy the following property.

2.7. THEOREM. *If $x \in X$ is a weakly f -rapid point for $f \in {}^\omega \omega$, then no non-trivial sequence converges to x .*

PROOF. Assume that $\{x_n\}_{n < \omega}$ is a non-trivial sequence converging to a weakly f -rapid point x of a space X . We may assume that $x \neq x_n$ for all $n < \omega$ and $x_n \neq x_m$ for $n < m < \omega$. Define, for each $n < \omega$, $B_n = \{x_m : n + \sum_{i=0}^{n-1} f(i) \leq m < n + 1 + \sum_{i=0}^n f(i)\}$. Notice that $|B_n| = f(n) + 1$ for each $n < \omega$. By assumption there exists $V \in \mathcal{N}(x)$ such that $|V \cap B_n| \leq f(n)$ for each $n < \omega$. So we may pick $y_n \in (X \setminus V) \cap B_n$ for each $n < \omega$; that is, $B_n \setminus V \neq \emptyset$ for each $n < \omega$. This implies that $(x_n)_{n < \omega}$ does not converge to x , which is a contradiction.

Observe from 2.7 that every non-isolated, weakly f -rapid point of a space has uncountable character.

It is evident that every weak P -point is an f -rapid point for each $f \in {}^\omega \omega$. For the converse, we have the following two results. First, we state a definition.

2.8. DEFINITION (Bernstein [B]). Let $p \in \omega^*$ and X a space. We say that $x \in X$ is the p -limit of a sequence $(x_n)_{n < \omega}$, we write $x = p\text{-lim } x_n$, if for every $V \in \mathcal{N}(x)$, $\{n < \omega : x_n \in V\} \in p$.

2.9. THEOREM. *Let $f \in {}^\omega \omega$. There is a space X containing an f -rapid, non-weak- P -point iff f is finite-to-one and there is a rapid ultrafilter on ω .*

PROOF. Necessity. Let X be a space and $x \in X$ a f -rapid, non-weak- P -

point. Then there exists $\{x_j: j < \omega\} \subseteq X \setminus \{x\}$ such that $x \in \text{Cl}_X \{x_j: j < \omega\}$. It is not hard to prove (see [GS, Lemma 2.2]) that there is $p \in \omega^*$ such that $x = p\text{-lim } x_j$. We shall verify that p is a rapid ultrafilter on ω . Indeed, let $\{B_n: n < \omega\} \subseteq [\omega]^{<\omega}$ and define $D_n = \{x_j: j \in B_n\}$ for $n < \omega$. By assumption, we can find $V \in \mathcal{N}(x)$ such that $|V \cap D_n| \leq f(n)$ for each $n < \omega$. Since $x = p\text{-lim } x_j$, $A = \{j < \omega: x_j \in V\} \in p$. If $j \in A \cap B_n$, then $x_j \in V \cap D_n$. Thus, $A \in p$ and $|A \cap B_n| \leq f(n)$ for each $n < \omega$. The conclusion now follows from Lemma 2.2.

Sufficiency. If $p \in \omega^*$ is a rapid ultrafilter and f is finite-to-one, by Lemma 2.2, then p is an f -rapid, non-weak- P -point of $\xi(p)$.

As an immediate consequence of the previous theorem we have:

2.10. COROLLARY. *If $f \in {}^\omega\omega$ is not finite-to-one, then the concepts of weak P -point and f -rapid point coincide.*

We remark that if M is a model of ZFC in which there are not rapid ultrafilters on ω (see [M]), then $M \models$ If X is a space, then $x \in X$ is a weak P -point iff x is f -rapid in X for every $f \in {}^\omega\omega$.

2.11. THEOREM. *The following statements are equivalent.*

- (1) *There is a space X containing a non-weak- P -point, weakly k -rapid for some $1 \leq k < \omega$.*
- (2) *There is a Q -point $p \in \omega^*$.*
- (3) *For every $1 \leq k < \omega$, there is a space which is weakly $(k+1)$ -rapid and is not weakly- k -rapid.*

PROOF. To prove (1) \Rightarrow (2) we apply the same reasoning used in the proof of Theorem 2.9 and Lemma 2.3, and (1) is the particular case of (3) when $k=1$.

(2) \Rightarrow (3). Fix $1 \leq k < \omega$ and let $p \in \omega^*$. We define a topology on $\mathcal{E}(p, k) = \{p\} \cup \{(j, n): j \leq k, n < \omega\}$ as follows: $\{(j, n)\}$ is open for all $j \leq k$ and $n < \omega$. $V \subseteq \mathcal{E}(p, k)$ is a neighborhood of p if $p \in V$ and $\{n < \omega: (j, n) \in V\} \in p$ for each $j \leq k$. Assume that p is a Q -point. First, we show that $\mathcal{E}(p, k)$ is weakly $(k+1)$ -rapid. Let $(B_m)_{m < \omega}$ be a sequence in $[\mathcal{E}(p, k) \setminus \{p\}]^{<\omega}$. For each $j \leq k$, put $B_{j,m} = B_m \cap \{(j, n): n < \omega\}$. Since p is a Q -point there is $A_j \in p$ such that $|A_j \cap B_{j,m}| \leq 1$ for $m < \omega$. Then $V = \{p\} \cup \bigcup_{j \leq k} \{(j, n): n \in A_j\} \in \mathcal{N}(p)$ and it is evident that $|V \cap B_m| \leq k+1$ for each $m < \omega$. Thus, $\mathcal{E}(p, k)$ is weakly k -rapid. Now, define $B_m = \{(j, m): j \leq k\}$, for each $m < \omega$, and suppose that $\mathcal{E}(p, k)$ is weakly- k -rapid. So there is $W \in \mathcal{N}(p)$ such that $|W \cap B_m| \leq k$ for each $m < \omega$. Set $A_j = \{n < \omega: (j, n) \in W\}$ for $j \leq k$. We have that $A = \bigcap_{j \leq k} A_j \in p$. If $m \in A$, then $(j, m) \in W \cap B_m$ for each $j \leq k$ and so $|W \cap B_m| = k+1$, which is a contradiction.

For $1 \leq k < \omega$, it is not hard to show that if X_i is a weakly k -rapid (resp. rapid) space with more than two points, for $i \in I$, and I is infinite, then $\prod_{i \in I} X_i$ has no weakly k -rapid (resp. rapid) points. For finite products, we have that (p, p) is not weakly $(k+1)$ -rapid in $\mathcal{E}(p, k) \times \mathcal{E}(p, k)$, and if x is rapid in X and y is rapid in Y , then (x, y) is rapid in $X \times Y$.

Next, we give an example, assuming the existence of a rapid ultrafilter on ω , of a rapid homogeneous space without weak P -points.

2.12. EXAMPLE. In [AF], the authors defined the homogeneous zero-dimensional space S_ω . In a similar way, for every $p \in \omega^*$, we may define the space $S_\omega(p)$ by replacing convergence sequences by p -limits in the construction (for a similar procedure see [G-F]). $S_\omega(p)$ is also a homogeneous, zero-dimensional space without weak P -points. For $p \in \omega^*$, set $S_\omega(p) = \{x\} \cup \{x_{n_1, \dots, n_r} : n_j < \omega \text{ for } 1 \leq j \leq r < \omega\}$. Then, we have that $x = p\text{-lim } x_n$ and $x_{n_1, \dots, n_r} = p\text{-lim } x_{n_1, \dots, n_r, n}$, for every $n_1, \dots, n_r < \omega$. To describe a neighborhood of x in $S_\omega(p)$, we put $S(A) = \{x_n : n \in A\}$ and $S(x_{n_1, \dots, n_r}, A) = \{x_{n_1, \dots, n_r, n} : n \in A\}$ for $x_{n_1, \dots, n_r} \in S_\omega(p)$ and for $A \subseteq \omega$. If $\{A\} \cup \bigcup_{1 \leq r < \omega} \{A_{n_1, \dots, n_r} : n_j < \omega \text{ for } 1 \leq j \leq r\}$ are elements of p , then the set $\{x\} \cup S(A) \cup \bigcup_{1 \leq r < \omega} (\bigcup_{n_1 \in A} \bigcup_{n_2 \in A_{n_1}} \dots \bigcup_{n_r \in A_{n_1, \dots, n_{r-1}}} S(x_{n_1, \dots, n_r}, A_{n_1, \dots, n_r}))$ is a basic neighborhood of x in $S_\omega(p)$. It is shown in the proof of 2.11 ((2) \Rightarrow (3)), the condition of Q -point is not essential, that the space $\mathcal{E}(p, k)$ is not weakly k -rapid for each $1 \leq k < \omega$ and for each $p \in \omega^*$. Since $\mathcal{E}(p, k)$ is homeomorphic to the subspace $\{x\} \cup \{x_{j, n} : j \leq k, n < \omega\}$ of $S_\omega(p)$ for each $1 \leq k < \omega$, $S_\omega(p)$ is not weakly k -rapid for all $1 \leq k < \omega$. Now suppose that p is a rapid ultrafilter on ω . We shall show that $S_\omega(p)$ is a rapid space. It is enough to prove that x is a rapid point of $S_\omega(p)$. In fact, let $(B_m)_{m < \omega}$ be a sequence of finite subsets of $S_\omega(p) \setminus \{x\}$ and let $\sigma : \omega \rightarrow \bigcup_{1 \leq r < \omega} \{(n_1, \dots, n_r) : n_j < \omega \text{ for } 1 \leq j \leq r\}$ be a bijection. Since p is a rapid ultrafilter, we may find $A \in p$ such that $|B_m \cap S(A)| \leq m$ for each $m < \omega$. By induction, for each $x_{n_1, \dots, n_r} \in S_\omega(p)$ we define $A_{n_1, \dots, n_r} \in p$ such that

- (1) $|B_m \cap S(x_{n_1, \dots, n_r}, A_{n_1, \dots, n_r})| \leq m$ for each $m < \omega$; and
- (2) $B_m \cap S(x_{n_1, \dots, n_r}, A_{n_1, \dots, n_r}) = \emptyset$ for every $m \leq \sigma^{-1}((n_1, \dots, n_r))$.

Define

$$V = \{x\} \cup S(A) \cup \bigcup_{1 \leq r < \omega} (\bigcup_{n_1 \in A} \bigcup_{n_2 \in A_{n_1}} \dots \bigcup_{n_r \in A_{n_1, \dots, n_{r-1}}} S(x_{n_1, \dots, n_r}, A_{n_1, \dots, n_r})).$$

For every $m < \omega$, let $z(m) = |V \cap B_m|$. Fix an arbitrary $m < \omega$ and put $V \cap B_m = \{x_{n_1^s, \dots, n_{r_s}^s} : 1 \leq s \leq z(m)\}$. Then $x_{n_1^s, \dots, n_{r_s}^s} \in S(x_{n_1^s, \dots, n_{r_s^s-1}^s}, A_{n_1^s, \dots, n_{r_s^s}^s}) \cap B_m$ for each $1 \leq s \leq z(m)$. From (1) and (2) it follows that $\sigma^{-1}((n_1^s, \dots, n_{r_s^s-1}^s)) < m$, for each $1 \leq s \leq z(m)$, and $|\{1 \leq t \leq z(m) : (n_1^s, \dots, n_{r_s^s-1}^s) = (n_1^t, \dots, n_{r_t^t-1}^t), n_{r_s^s}^s \neq n_{r_t^t}^t\}| \leq m$, for

each $1 \leq s \leq z(m)$. So $z(m) \leq m^2$. Thus, $|V \cap B_m| \leq m^2$ for every $m < \omega$. Theorem 2.4 implies that x is rapid in $S_\omega(p)$.

Finally, we state some problems.

2.13. QUESTION. Assume the existence of a Q -point $p \in \omega^*$.

(1) Is there a compact weakly k -rapid (resp. rapid) space without weak P -points, for each $1 \leq k < \omega$?

(2) Is there a weakly k -rapid (resp. rapid) topological group without weak P -points, for each $1 \leq k < \omega$?

(3) For every $1 \leq k < \omega$, is there a weakly $(k+1)$ -rapid homogeneous space which is not weakly k -rapid?

3. On Id-fan tightness.

We begin with a definition that generalizes countable strong fan tightness (1.4 (3)).

3.1. DEFINITION. Let $h \in {}^\omega \omega$. A space X has h -fan tightness if for every $x \in X$ and for every sequence $(A_n)_{n < \omega}$ of subsets of X such that $x \in \bigcap_{n < \omega} \text{Cl}(A_n)$, there is $F_n \in [A_n]^{< h(n)}$, for every $n < \omega$, such that $x \in \text{Cl}(\bigcup_{n < \omega} F_n)$.

If $h \in {}^\omega \omega$ is the constant function of value k for $1 \leq k < \omega$, then k -fan tightness stands for h -fan tightness. Henceforth, $\text{Id}: \omega \rightarrow \omega$ will denote the identity map on ω . It is evident that countable strong fan tightness \Leftrightarrow 1-fan tightness \Rightarrow h -fan tightness for each $h \in {}^\omega \omega \Rightarrow$ countable fan tightness \Rightarrow countable tightness. There is an easy example of a space with countable tightness which does not have countable fan tightness. In fact, for $p \in \omega^*$, we define a topology on $\mathcal{E}(p, \omega) = \{p\} \cup \omega \times \omega$ as follows: the singleton $\{(n, m)\}$ is open for every $(n, m) \in \omega \times \omega$, and $V \in \mathcal{N}(p)$ provided that $p \in V$ and $\{m < \omega : (n, m) \in V\} \in p$ for each $n < \omega$ (see the proof 2.11). It is not hard to show that $\mathcal{E}(p, \omega)$ has countable tightness and does not have countable fan tightness for every $h \in {}^\omega \omega$. Example 3.7 has Id-fan tightness and does not have countable strong fan tightness, and Example 3.8 has countable fan tightness and does not have h -fan tightness.

Next, we shall show that if $h \in {}^\omega \omega$, then h -fan tightness coincides with either 1-fan tightness (=countable strong fan tightness) or Id-fan tightness. First, we give some preliminary results.

3.2. LEMMA. Let $h \in {}^\omega \omega$ and let $f \in {}^\omega \omega$ be non-bounded. Then every space with h -fan tightness has f -fan tightness.

PROOF. Let X be a space with h -fan tightness, $x \in X$ and $(A_n)_{n < \omega}$ a sequence of subsets of X such that $x \in \bigcap_{n < \omega} \text{Cl}(A_n)$. Since f is not bounded, we may choose positive integers $n_0 < n_1 < \dots < n_k < \dots$ such that $h(k) \leq f(n_k)$ for each $k < \omega$. Define $B_k = A_{n_k}$ for each $k < \omega$. Then, for every $k < \omega$ there is $E_k \in [B_k]^{\leq h(k)}$ such that $x \in \text{Cl}(\bigcup_{k < \omega} E_k)$. For $n < \omega$, put $F_n = E_k$ if $n = n_k$ and $F_n = \emptyset$ otherwise. Thus, we have that $\bigcup_{n < \omega} F_n = \bigcup_{k < \omega} E_k$ and $F_{n_k} = E_k \in [A_{n_k}]^{h(k) \leq f(n_k)}$ for each $k < \omega$. Therefore, $x \in \bigcup_{n < \omega} F_n$ and $F_n \in [A_n]^{\leq f(n)}$ for all $n < \omega$.

The following two corollaries are direct consequences of 3.2.

3.3. COROLLARY. *If $h, f \in {}^\omega \omega$ are non-bounded, then the notions of h -fan tightness and f -fan tightness are the same.*

3.4. COROLLARY. *If $h \in {}^\omega \omega$, then every space with h -fan tightness has Id-fan tightness.*

3.5. LEMMA. *If $h \in {}^\omega \omega$ is bounded then h -fan tightness agrees with countable strong fan tightness.*

PROOF. Assume that $h \in {}^\omega \omega$ is bounded by the integer $k < \omega$. Let X be a space with h -fan tightness, $x \in X$ and $(A_n)_{n < \omega}$ a sequence of subsets of X such that $x \in \bigcap_{n < \omega} \text{Cl}(A_n)$. By assumption, for each $n < \omega$ there is $F_n \in [A_n]^{\leq k}$ such that $x \in \text{Cl}(\bigcup_{n < \omega} F_n)$. We may suppose that $|F_n| = k$ for all $n < \omega$. Enumerate each F_n by $\{x_1^n, \dots, x_k^n\}$ and set $B_j = \{x_j^n; n < \omega\}$ for each $1 \leq j \leq k$. Since $x \in \text{Cl}(\bigcup_{n < \omega} F_n) = \text{Cl}(B_1 \cup \dots \cup B_k) = \text{Cl}(B_1) \cup \dots \cup \text{Cl}(B_k)$, there is $1 \leq j \leq k$ such that $x \in \text{Cl}(B_j)$. Thus, $x_j^n \in A_n$ for each $n < \omega$ and $x \in \text{Cl}(\{x_j^n; n < \omega\})$.

We turn now to the principal result of this section.

3.6. THEOREM. *If $h \in {}^\omega \omega$, then h -fan tightness coincides with either 1-fan tightness or Id-fan tightness.*

The next two examples show that Id-fan tightness is a new concept.

3.7. EXAMPLE. Let $x \notin \omega \times \omega$. We consider the following topology on $X = \{x\} \cup (\omega \setminus \{0\}) \times \omega$: the set $(\omega \setminus \{0\}) \times \omega$ has the discrete topology and a neighborhood of x consists of a finite intersection of the sets $V_f = \{x\} \cup \{(n, m) \in (\omega \setminus \{0\}) \times \omega : (n, m) \neq (n, f(n))\}$ for $f \in {}^\omega \omega$. Notice that X is a zero-dimensional space. We shall verify that X with this topology has Id-fan tightness and does not have strong fan tightness. Indeed, for $1 \leq n < \omega$, we put $A_n = \{(n, m) : m < \omega\}$. In order to show that X has Id-fan tightness we note that $x \in \text{Cl}(B) \setminus B$, for $B \subseteq X$, whenever for every $1 \leq n < \omega$ there is $k_n < \omega$ such that $|B \cap A_{k_n}| > n$.

For each $1 \leq n < \omega$, let $B_n \subseteq X$ such that $x \in \bigcap_{1 \leq n < \omega} \text{Cl}(B_n)$ and $x \notin B_n$, for each $n < \omega$. Then for each $1 \leq n < \omega$ there is $k_n < \omega$ such that $|B_n \cap A_{k_n}| > n$. For every $1 \leq n < \omega$, choose $F_n \subseteq B_n \cap A_{k_n}$ such that $|F_n| = n$. Let $V = \bigcap_{j \leq s} V_{f_j} \in \mathcal{N}(x)$, where $f_j \in {}^\omega \omega$ for $j \leq s < \omega$. Since $|F_{2s} \cap \{(k_{2s}, f_j(k_{2s})) : j \leq s\}| \leq s+1$ and $|F_{2s}| = 2s$, we obtain that $F_{2s} \cap V \neq \emptyset$ and hence $V \cap \bigcup_{1 \leq n < \omega} F_n \neq \emptyset$. Thus, $x \in \text{Cl}(\bigcup_{1 \leq n < \omega} F_n)$. Suppose that X has countable strong fan tightness. Then for every $1 \leq n < \omega$ there is $t_n < \omega$ such that $x \in \text{Cl}(\{(n, t_n) : 1 \leq n < \omega\})$. Let $f \in {}^\omega \omega$ be defined by $f(n) = t_n$ for each $1 \leq n < \omega$. Then $V_f \cap \{(n, t_n) : 1 \leq n < \omega\} = \emptyset$, which is a contradiction.

3.8. EXAMPLE. Let $Y = \{y\} \cup (\omega \setminus \{0\}) \times \omega$, where $y \notin \omega \times \omega$. We equip $(\omega \setminus \{0\}) \times \omega$ with the discrete topology and let $\mathcal{N}(y)$ be the set of all finite intersections of the sets W_s , where $W_s = \{y\} \cup \{(n, m) : m \notin S_n, 1 \leq n < \omega\}$ and $S = (S_n)_{1 \leq n < \omega}$ is a sequence of subsets of ω such that $|S_n| \leq n$ for each $1 \leq n < \omega$. We claim that Y is a zero-dimensional space which has countable fan tightness and does not have Id-fan tightness. It is evident that Y is zero-dimensional and does not have Id-fan tightness. We claim that Y does not have countable fan tightness. First, observe that $y \in \text{Cl}(B) \setminus B$ if and only if for every $1 \leq n < \omega$ there is $k_n < \omega$ such that $|B \cap A_n| > nk_n$, where $A_n = \{(n, m) : m < \omega\}$ for $1 \leq n < \omega$. Assume that $y \in \bigcap_{1 \leq n < \omega} \text{Cl}(B_n)$ and $y \notin B_n$ for each $1 \leq n < \omega$. Then, for each $1 \leq n < \omega$ there is $k_n < \omega$ such that $|B_n \cap A_{k_n}| > nk_n$. For each $1 \leq n < \omega$, choose $F_n \subseteq B_n \cap A_{k_n}$ with $|F_n| > nk_n$. Let $W = \bigcap_{j \leq r} W_{s_j} \in \mathcal{N}(y)$, where $S_j = (S_n^j)_{n < \omega}$ for $j \leq r < \omega$. Since $|F_r \cap \{(k_r, m) : m \notin S_{k_r}^j, j \leq r\}| \leq rk_r$ and $|F_r| > rk_r$, we have that $W \cap F_r \neq \emptyset$ and hence $W \cap (\bigcup_{1 \leq n < \omega} F_n) \neq \emptyset$. Thus, $y \in \text{Cl}(\bigcup_{1 \leq n < \omega} F_n)$.

Certain ultrafilters on ω can be characterized in terms of countable fan tightness and Id-fan tightness.

3.9. THEOREM. *An ultrafilter p on ω is a P -point iff $\xi(p)$ has countable fan tightness.*

3.10. THEOREM. *For $p \in \omega^*$, the following statements are equivalent.*

- (1) p is semiselective;
- (2) $\xi(p)$ has countable strong fan tightness;
- (3) $\xi(p)$ has Id-fan tightness;
- (4) there is $k \in {}^\omega \omega$ such that given $A_n \in p$ for $n < \omega$, there exists $F_n \in [A_n]^{\leq h(n)}$ such that $\bigcup_{n < \omega} F_n \in p$.

PROOF. The proofs of (1) \Leftrightarrow (2), (2) \Rightarrow (3) and (3) \Rightarrow (4) are direct from the definitions, and (4) \Rightarrow (3) follows from 3.2. Only the implication (3) \Rightarrow (1) requires

proof. Assume that $\xi(p)$ has Id-fan tightness. Let $(A_n)_{n<\omega}$ be a sequence of elements of p . Without loss of generality, we may suppose that $A_{n+1} \subseteq A_n$ for $n<\omega$. Define $B_n = A_{n(n+1)/2}$ for each $n<\omega$. By hypothesis, for each $n<\omega$ there is $F_n \in [B_n]^{<n}$ such that $A = \bigcup_{n<\omega} F_n \in p$. By adding integers if it necessary and by induction, we may assume that $|F_n| = n$, for each $n<\omega$, and $F_n \cap F_m = \emptyset$ whenever $n<m<\omega$. Enumerate successively the F_n 's by $\{a_j : j<\omega\}$. Then we have that $A = \{a_j : j<\omega\} \in p$. Fix $1<j<\omega$ and let $1 \leq n<\omega$ be such that $a_j \in F_n$. It then follows that $j \leq n(n+1)/2$ and hence $a_j \in F_n \subseteq A_{n(n+1)/2} \subseteq A_j$, as desired.

3.11. QUESTION. Is there a topological group G such that G has Id-fan tightness (resp. countable fan tightness) and does not have countable strong fan tightness (resp. Id-fan tightness)?

For a space X we denote by $C_\pi(X)$ the function space on X with the topology of pointwise convergence. In the next theorem, we shall show that the concepts of countable strong fan tightness and Id-fan tightness coincide on the class of spaces of the form $C_\pi(X)$. Recall that X has property C'' if for every sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X there is $G_n \in \mathcal{G}_n$, for each $n<\omega$, such that $X = \bigcup_{n<\omega} G_n$. The following lemma is needed.

3.12. LEMMA. *For a space X , the following are equivalent.*

- (1) X has property C'' ;
- (2) for every sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X , for each $n<\omega$ there is $\mathcal{D}_n \in [\mathcal{G}_n]^{<n}$ such that $X = \bigcup_{n<\omega} \bigcup \mathcal{D}_n$;
- (3) there is $h \in {}^\omega \omega$ such that for every sequence $(\mathcal{G}_n)_{n<\omega}$ of open covers of X there is $\mathcal{D}_n \in [\mathcal{G}_n]^{<h(n)}$, for each $n<\omega$, for which $X = \bigcup_{n<\omega} \bigcup \mathcal{D}_n$.

PROOF. Only (3) \Rightarrow (1) requires proof. Let $h \in {}^\omega \omega$ satisfy the conditions of clause (3) and let $(\mathcal{G}_n)_{n<\omega}$ be a sequence of open covers of X . Without loss of generality we may suppose that h is strictly increasing. Put $\mathcal{H}_0 = \mathcal{G}_0 \wedge \dots \wedge \mathcal{G}_{h(0)-1}$ and for $n<\omega$, we define $\mathcal{H}_n = \mathcal{G}_{h(n)} \wedge \dots \wedge \mathcal{G}_{h(n+1)-1}$, where $\mathcal{G} \wedge \mathcal{H} = \{G \cap H : G \in \mathcal{G} \text{ and } H \in \mathcal{H}\}$ for \mathcal{G} and \mathcal{H} covers of X . Then for each $n<\omega$ there is $\mathcal{D}_n \in [\mathcal{H}_n]^{<h(n)}$ such that $X = \bigcup_{n<\omega} \bigcup \mathcal{D}_n$. We may assume that $\mathcal{D}_0 = \{H_j : j<h(0)\}$ and $\mathcal{D}_n = \{H_{h(n)+j} : j<h(n+1)-h(n)\}$ for every $1 \leq n<\omega$. Now, we have that if $n<\omega$ and $j<h(n+1)-h(n)$ (resp. if $j<h(0)$), then there is $G_{h(n)+j} \in \mathcal{G}_{h(n)+j}$ (resp. $G_j \in \mathcal{G}_j$) such that $H_{h(n)+j} \subseteq G_{h(n)+j}$ (resp. $H_j \subseteq G_j$). It then follows that $X = \bigcup_{m<\omega} G_m$ and $G_m \in \mathcal{G}_m$ for each $m<\omega$.

3.13. THEOREM. *For a space X , the following are equivalent.*

- (1) $C_\pi(X)$ has countable strong fan tightness;
- (2) each finite product of X has property C'' ;
- (3) $C_\pi(X)$ has Id-fan tightness.

PROOF. The equivalence (1) \Leftrightarrow (2) is shown in [S] and by a slight modification of Sakai's argument we can prove that $C_\pi(X)$ has Id-fan tightness iff each finite product of X satisfies the property of clause (2) of 3.12. Thus, (2) \Leftrightarrow (3) follows from 3.12.

References

- [Ar₁] A. V. Arhangel'skii, On bcompacta hereditarily satisfying Suslin's condition, tightness and free sequences, Soviet Math. Dokl. 12 (1971), 1253-1257.
- [Ar₂] A. V. Arhangel'skii, Hurewicz spaces, analytic sets, and fan tightness of function spaces, Soviet Math. Dokl. 33 (1986), 396-399.
- [AF] A. V. Arhangel'skii and S. P. Franklin, Ordinal invariants for topological spaces, Michigan Math. J. 15 (1968), 313-320.
- [B] A. R. Bernstein, A new kind of compactness for topological spaces, Fund. Math. 66 (1970), 185-193.
- [Bo] D. D. Booth, Ultrafilters on a countable set, Ann. Math. Logic 2 (1970), 1-24.
- [CV] E. Copláková and P. Vojtaš, A new sufficient condition for the existence of Q -points in $\beta(\omega)\setminus\omega$, Colloq. Math. Soc. János Bolyai, 41 (1983), 199-208.
- [G-F] S. Garcia-Ferreira, On $FU(p)$ -spaces and p -sequential spaces, Comment. Math. Univ. Carolinae 32 (1991), 161-171.
- [GS] J. Ginsburg and V. Saks, Some applications of ultrafilters in topology, Pacific J. Math. 57 (1975), 403-418.
- [K] K. Kunen, Some points in βN , Proc. Cambridge Philos. Soc. 80 (1976), 358-398.
- [L] C. Laflamme, Forcing with filters and complete combinatorics, Ann. Math. Logic, 42 (1989), 125-163.
- [Ma] A. R. D. Mathias, O^* and the P -point problem, Higher Set Theory, Proc. Oberwolfach 1977, Lecture Notes Math. 669 (1978), 375-384.
- [M] A. W. Miller, There are no Q -points in Laver's model for the Borel conjecture, Proc. Amer. Math. Soc. 78 (1980), 103-106.
- [Mo] G. Mokobodski, Ultrafilters rapides sur N , Séminaire BreLOT-Choquet-Deny. Théorie de Potentiel, 12-ème année, 1967/68, No. 12.
- [S] M. Sakai, Property C'' and function spaces, Proc. Amer. Math. Soc. 104 (1988), 917-919.
- [T] A. D. Taylor, A note on the existence of P -points and Q -points, unpublished.

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