TSUKUBA J. MATH.
Vol. 18 No. 1 (1994), 135–143

# ON ISOMETRY OF A COMPLETE RIEMANNIAN MANIFOLD TO A SPHERE

By

Yongfan ZHENG

## 0. Introduction

In this paper, we obtain some conditions for a complete Riemannian manifold to be isometric to a sphere. This is to expand the following theorems for a compact Riemannian manifold $M$ into the case where $M$ is complete and not necessarily compact.

THEOREM A (Yano [6]). *If $M$ is a compact orientable Riemannian manifold of dimension $n>2$ with constant scalar curvature and admits a non-isometric conformal vector field $X$: $\mathcal{L}_X g=2\rho g$ such that*

$$\int_M G(d\rho, d\rho)dV \geqq 0, \tag{0.1}$$

*then $M$ is isometric to a sphere.*

As a corollary of this theorem, the condition (0.1) may be replaced by $\mathcal{L}_X|R|^2=0$ or $\mathcal{L}_X|K|^2=0$ (see [3, 6, 8]).

## 1. Notations and Preliminaries

Throughout this paper, by a Riemannian manifold we always mean an $n$-dimensional connected and oriented manifold covered by a system of local coordinates $\{x^i\}$ $(i=1, 2, \cdots, n)$ and furnished with a Riemannian metric tensor $g=g_{ji}dx^j \otimes dx^i$. We use the Einstein summation convention with respect to repeated indices. Furthermore, geometric objects and some functions appeared in this paper are always assumed to be smooth, unless otherwise stated.

Let $M$ be an $n$-dimensional Riemannian manifold with a metric tensor $g$. We use the standard notation for the covariant derivative $\nabla$, the exterior differential $d$, the codifferential $\delta$, the Laplacian $\Delta$ and the volume element $dV$ of $M$. We denote by $\langle\ ,\ \rangle$ and $|\ \ |$ the inner product and the norm induced in

Received July 15, 1992. Revised March 14, 1993.

fibers of various tensor bundles by the metric $g$ of $M$. In this paper, we identify a 1-form with its dual vector field with respect to $g$ and they are represented by the same letter.

By $\mathcal{L}_X$ we mean the operators of Lie derivation with respect to a vector field $X$ on $M$. A vector field (or an infinitesimal transformation) $X$ on $M$ is said to be conformal if it satisfies $\mathcal{L}_X g=2\rho g$ for some function $\rho$ on $M$. In particular, $X$ is isometric if $\rho$ is identically zero.

We denote by $K_{kjih}$ and $R_{ji}$ local components of the curvature tensor $K$ and the Ricci tensor $R$ of $M$ respectively, and by $r$ the scalar curvature of $M$. We put

$$G_{ji}=R_{ji}-(r/n)g_{ji}, \tag{1.1}$$

$$Z_{kjih}=K_{kjih}-\{r/n(n-1)\}(g_{kh}g_{ji}-g_{jh}g_{ki}). \tag{1.2}$$

Then the tensor $G$ measures the deviation of $M$ from an Einstein manifold and the tensor $Z$ that from a manifold of constant curvature.

The following theorem proved by Obata [4] is well known.

THEOREM B. *If a complete Riemannian manifold $M$ of dimension $n \geqq 2$ admits a nonconstant function $\rho$ such that $\nabla\nabla\rho+k^2\rho g=0$, where $k$ is a positive constant, then $M$ is isometric to an n-sphere of radius $1/k$.*

By using this theorem and the above geometric objects, Obata [4], Yano [6, 7, 8], Hsiung [3] and others have obtained some conditions for a compact Riemannain manifold admitting a conformal vector field to be isometric to a sphere. One of these results is Theorem A in the introduction.

The following formulae are well known (see [8]). These were prepared in order to prove Theorem A and others.

$$\langle G, g\rangle=G_{ji}g^{ji}=0, \tag{1.3}$$

where $g^{ji}$ are the contravariant components of $g$ defined by $g^{ji}g_{ik}=\delta_k^j$.

$$Z_{kjih}g^{kh}=G_{ji}, \tag{1.4}$$

$$|G|^2=|R|^2-(1/n)r^2, \tag{1.5}$$

$$|Z|^2=|K|^2-\{2/n(n-1)\}r^2, \tag{1.6}$$

$$\delta G=-g^{kj}\nabla_k G_{ji}dx^i=-\{(n-2)/2n\}\,dr. \tag{1.7}$$

Let $X$ be a conformal vector field on $M$, that is, it satisfies

$$\mathcal{L}_X g_{ji}=\nabla_j X_i+\nabla_i X_j=2\rho g_{ji}, \tag{1.8}$$

where $\rho$ is a function, and then we have

$$\rho=-(1/n)\delta X=(1/n)\nabla_i X^i, \tag{1.9}$$

$$\mathcal{L}_X r=2(n-1)\Delta\rho-2\rho r, \tag{1.10}$$

$$\mathcal{L}_X |G|^2=-2(n-2)\langle\nabla\nabla\rho, G\rangle-4\rho|G|^2, \tag{1.11}$$

$$\mathcal{L}_X |Z|^2=-8\langle\nabla\nabla\rho, G\rangle-4\rho|Z|^2. \tag{1.12}$$

Now, we assume that $M$ is complete. Let $f$ be the geodesic distance function from a fixed point on $M$ and $B(t)$ the geodesic ball of radius $t$, i.e.,

$$B(t)=\{x\in M \mid f(x)\leqq t\} \tag{1.13}$$

for $t>0$. Then there exists a Lipschitz continuous function $w_t$ on $M$ satisfying the following properties:

$$0\leqq w_t(x)\leqq 1, \qquad x\in M, \tag{1.14}$$

$$w_t(x)=1, \qquad x\in B(t), \tag{1.15}$$

$$\operatorname{supp} w_t\subset B(2t), \tag{1.16}$$

$$w_t \longrightarrow 1 \ (t\to\infty), \tag{1.17}$$

$$|dw_t|\leqq C/t \qquad \text{almost everywhere on } M, \tag{1.18}$$

where $C$ is a positive constant independent of $t$ (see [1, 2, 9]).

## 2. Main Results

THEOREM 1. *Let $M$ be a complete Riemannian manifold of dimension $n\geqq 2$, and admit a non-constant function $\rho$ such that $\Delta\rho=nk\rho$ for some non-zero constant $k$. If $\rho$ satisfies that*

$$\liminf_{t\to\infty}\int_M\langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV\geqq 0, \tag{2.1}$$

*and has first derivatives in $L^2(M)$, then $M$ is isometric to a sphere.*

Especially, if $R(d\rho, d\rho)\geqq(n-1)k|d\rho|^2$, then we get the condition (2.1) in Theorem 1. Thus we obtain the following

COROLLARY. *Let $M$ be a complete Riemannian manifold of dimension $n\geqq 2$, aud admit a non-constant function $\rho$ such that $\Delta\rho=nk\rho$ for some non-zero constant $k$. If the Ricci curvature of $M$ in the direction $d\rho$ is not less than $(n-1)$ $k$ and $\rho$ has first derivatives in $L^2(M)$, then $M$ is isometric to a sphere.*

REMARK 1. In Theorem 1, if $M$ is compact, then automatically the first derivatives of $\rho$ are in $L^2(M)$ and $\liminf_{t\to\infty}\int_M\langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV=\int_M\langle R-(n-1)kg,\ d\rho\otimes d\rho\rangle dV$. From the proof of Theorem 1 it follows that the assumption $\int_M |d\rho|^2 dV<+\infty$ may be replaced by

$$\lim_{t\to\infty}(1/t^2)\int_{B(2t)}|d\rho|^2 dV=0 \tag{2.2}$$

as in the case of [5].

As a special case of Theorem 1, we assert the following

THEOREM 2. *Let $M$ be a complete Riemannain manifold of dimension $n\geqq 2$ with non-zero constant scalar curvature, and admit a non-isometric comformal vector field $X$: $\mathcal{L}_X g=2\rho g$. If $\rho$ satisfies that*

$$\liminf_{t\to\infty}\int_M\langle w_t^2 G,\ d\rho\otimes d\rho\rangle dV\geqq 0, \tag{2.3}$$

*and has first derivatives in $L^2(M)$, then $M$ is isometric to a sphere.*

PROOF OF THEOREM 2. It follows from (1.10) that $\Delta\rho=nk\rho$, $k$ being the nonzero conatant $r/n(n-1)$. Then we have completed the proof of Theorem 2 as an application of Theorem 1. □

REMARK 2. From the comment in Remark 1 we can consider that Theorem 2 is a generalization of Theorem A.

THEOREM 3. *Let $M$ be a complete Riemannain manifold of dimension $n>2$ with non-zero constant scalar curvature, and admit a non-isometric comformal vector field $X$: $\mathcal{L}_X g=2\rho g$. If $\mathcal{L}_X|R|^2=0$ (or $\mathcal{L}_X|K|^2=0$) and $\rho$ has first derivative in $L^2(M)$, then $M$ is isometric to a sphere.*

REMARK 3. Here we remark the following fact concerning constant scalar curvatures.

PROPOSITION. *Let $M$ be a complete Riemannian manifold $M$ with constant scalar curvature $r$, and admit a non-isometric conformal vector field $X$: $\mathcal{L}_X g=2\rho g$. If $\rho$ has first derivatives in $L^2(M)$, then $r$ is non-negative.*

## 3. Proof of Theorems

In this section, we give the proofs of the theorems mentioned in § 2. We need the lemma below.

LEMMA. *Let $M$ be a complete Riemannian manifold, and admit a non-trivial solution $\rho$ of the partial differential equation $\Delta\rho=k\rho$ for some constant $k$. If $\rho$ has first derivatives in $L^2(M)$, then $k$ is non-negative.* (see also [10]).

PROOF. We can easily find that

$$\delta(w_t^2\rho d\rho)=-w_t^2|d\rho|^2+w_t^2\rho\Delta\rho-\langle w_t\rho d\rho, 2dw_t\rangle \quad \text{a.e. on } M. \tag{3.1}$$

We integrate the both sides of (3.1) over $B(2t)$. Since Stokes' theorem holds for Lipschitz differential forms and $w_t=0$ on the boundary $\partial B(2t)$ of $B(2t)$, the left hand becomes zero:

$$\int_{B(2t)}\delta(w_t^2\rho d\rho)dV=-\int_{\partial B(2t)}w_t^2\langle\rho d\rho, N\rangle dS=0,$$

where $N$ and $dS$ are the unit normal to $\partial B(2t)$ and the volume element of $\partial B(2t)$ respectively. Then we see

$$\int_{B(2t)}w_t^2|d\rho|^2dV-k\int_{B(2t)}w_t^2\rho^2dV+\int_{B(2t)}\langle w_t\rho d\rho, 2dw_t\rangle dV=0. \tag{3.2}$$

From Schwartz's inequality and (1.18), we have

$$\begin{aligned}&\left|\int_{B(2t)}\langle w_t\rho d\rho, 2dw_t\rangle dV\right|\\ &\leqq\left[\int_{B(2t)}(w_t\rho)^2dV\right]^{1/2}\left[\int_{B(2t)}\langle d\rho, 2dw_t\rangle^2dV\right]^{1/2}\\ &\leqq\left[\int_{B(2t)}(w_t\rho)^2dV\right]^{1/2}\left[\int_{B(2t)}4|d\rho|^2|dw_t|^2dV\right]^{1/2}\\ &\leqq\left[\int_{B(2t)}(w_t\rho)^2dV\right]^{1/2}\cdot\frac{2C}{t}\left[\int_{B(2t)}|d\rho|^2dV\right]^{1/2}.\end{aligned}$$

Now we suppose that $k$ is negative. Using the fundamental inequality

$$2ab=-\left(\sqrt{-k}\,a-\frac{1}{\sqrt{-k}}b\right)^2-ka^2-\frac{1}{k}b^2\leqq-ka^2-\frac{1}{k}b^2,$$

we get the following:

$$\begin{aligned}&\left|\int_{B(2t)}\langle w_t\rho d\rho, 2dw_t\rangle dV\right|\\ &\qquad\leqq\frac{1}{2}\left[-k\int_{B(2t)}(w_t\rho)^2dV-\frac{4C^2}{kt^2}\int_{B(2t)}|d\rho|^2dV\right].\end{aligned} \tag{3.3}$$

Then it follows from (3.2) combined with (3.3) that

$$-\frac{2C^2}{kt^2}\int_{B(2t)}|d\rho|^2dV\geqq\int_{B(2t)}w_t^2|d\rho|^2dV-\frac{1}{2}k\int_{B(2t)}w_t^2\rho^2dV\geqq0.$$

Furthermore, from (1.16), we also have

$$-\frac{2C^2}{kt^2}\int_M |d\rho|^2 dV \geqq \int_M w_t^2 |d\rho|^2 dV - \frac{1}{2}k\int_M w_t^2 \rho^2 dV \geqq 0. \tag{3.4}$$

Since $\int_M |d\rho|^2 dV < \infty$, letting $t \to \infty$ in (3.4), we have

$$\begin{aligned} 0 &\geqq \liminf_{t\to\infty} \int_M w_t^2 |d\rho|^2 dV - \frac{1}{2}k \liminf_{t\to\infty} \int_M w_t^2 \rho^2 dV \\ &\geqq \int_M |d\rho|^2 dV - \frac{1}{2}k\int_M \rho^2 dV \geqq 0. \end{aligned}$$

Then we see that $\rho=0$ on $M$. This contradicts the hypothesis. Therefore $k$ must be nonnegative. □

The previous proposition is immediately proved by (1.10) and this lemma.

Let us prove Theorem 1.

PROOF OF THEOREM 1. Let $i_\zeta$ be the inner product operator with respect to a vector field $\zeta$ on $M$, that is, operating it to a (0, 2)-type tensor $T$, then we get the 1-form $i_\zeta T = \zeta^j T_{ji} dx^i$.

The second equality can be shown by direct computation:

$$\begin{aligned} &\delta\left\{w_t^2 i_\zeta\left(\mathcal{L}_\zeta g + \frac{2}{n}\delta\zeta\cdot g\right)\right\} \\ &= \left\langle \Delta\zeta + \frac{n-2}{n} d\delta\zeta,\ w_t^2\zeta\right\rangle - \langle 2R,\ w_t^2 \zeta\otimes\zeta\rangle \\ &\quad -\frac{1}{2}\left|w_t\left(\mathcal{L}_\zeta g + \frac{2}{n}\delta\zeta\cdot g\right)\right|^2 - \left\langle \mathcal{L}_\zeta g + \frac{2}{n}\delta\zeta\cdot g,\ 2w_t dw_t \otimes\zeta\right\rangle \\ &\qquad\qquad \text{a.e. on } M, \end{aligned} \tag{3.5}$$

for any vector field $\zeta$ on $M$. Integrating the both sides of (3.5) over $B(2t)$ and applying Stokes' theorem, we have

$$\begin{aligned} 0 &= \int_{B(2t)} \left\langle \Delta\zeta + \frac{n-2}{n} d\delta\zeta,\ w_t^2\zeta\right\rangle dV - \int_{B(2t)} \langle 2R,\ w_t^2\zeta\otimes\zeta\rangle dV \\ &\quad -\frac{1}{2}\int_{B(2t)} \left|w_t\left(\mathcal{L}_\zeta g + \frac{2}{n}\delta\zeta\cdot g\right)\right|^2 dV - \int_{B(2t)} \left\langle \mathcal{L}_\zeta g + \frac{2}{n}\delta\zeta\cdot g,\ 2w_t dw_t\otimes\zeta\right\rangle dV. \end{aligned} \tag{3.6}$$

Putting $\zeta = d\rho$ in (3.6) and using $\Delta d\rho = d\Delta\rho = nkd\rho$, it follows that

$$\begin{aligned} 0 &= \int_{B(2t)} \langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV \\ &\quad + \int_{B(2t)} w_t^2 |\nabla\nabla\rho + k\rho g|^2 dV + \int_{B(2t)} \langle \nabla\nabla\rho + k\rho g,\ 2w_t dw_t \otimes d\rho\rangle dV. \end{aligned} \tag{3.7}$$

From Schwartz's inequality and (1.18), we have

$$(3.8)\qquad \left|\int_{B(2t)}\langle\nabla\nabla\rho+k\rho g,\ 2w_t dw_t\otimes d\rho\rangle dV\right|$$

$$\leqq\int_{B(2t)}|w_t(\nabla\nabla\rho+k\rho g)|\cdot 2|dw_t\otimes d\rho|dV$$

$$\leqq\left[\int_{B(2t)}|w_t(\nabla\nabla\rho+k\rho g)|^2 dV\right]^{1/2}\left[4\int_{B(2t)}|dw_t\otimes d\rho|^2 dV\right]^{1/2}$$

$$\leqq\frac{1}{2}\left[\int_{B(2t)}|w_t(\nabla\nabla\rho+k\rho g)|^2 dV+4\int_{B(2t)}|dw_t\otimes d\rho|^2 dV\right]$$

$$\leqq\frac{1}{2}\left[\int_{B(2t)}|w_t(\nabla\nabla\rho+k\rho g)|^2 dV+\frac{4C^2}{t^2}\int_{B(2t)}|d\rho|^2 dV\right]$$

Then it follows from (3.7) combined with (3.8) that

$$\frac{2C^2}{t^2}\int_{B(2t)}|d\rho|^2 dV\geqq\int_{B(2t)}\langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV$$
$$+\frac{1}{2}\int_{B(2t)}w_t^2|\nabla\nabla\rho+k\rho g|^2 dV.$$

Furthermore, from (1.16), we also have

$$(3.9)\quad \frac{2C^2}{t^2}\int_M|d\rho|^2 dV\geqq\int_M\langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV+\frac{1}{2}\int_M w_t^2|\nabla\nabla\rho+k\rho g|^2 dV.$$

Since $\int_M|d\rho|^2 dV<\infty$ and $\liminf_{t\to\infty}\int_M\langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV\geqq 0$, letting $t\to\infty$ in (3.9), we see

$$(3.10)\quad 0\geqq\liminf_{t\to\infty}\int_M\langle R-(n-1)kg,\ w_t^2 d\rho\otimes d\rho\rangle dV+\frac{1}{2}\liminf_{t\to\infty}\int_M w_t^2|\nabla\nabla\rho+k\rho g|^2 dV$$
$$\geqq\frac{1}{2}\int_M|\nabla\nabla\rho+k\rho g|^2 dV\geqq 0.$$

Hence we have

$$(3.11)\qquad \nabla\nabla\rho+k\rho g=0 \quad \text{on } M.$$

This combined with Theorem B and Lemma completes the proof of Theorem 1. □

Theorem 3 is proved below as an application of Theorem 2.

PROOF OF THEOREM 3. Since the scalar curvature $r$ is constant, we first note that $\mathcal{L}_X|G|^2=0$ [resp. $\mathcal{L}_X|Z|^2=0$] is equivalent to $\mathcal{L}_X|R|^2=0$ [resp. $\mathcal{L}_X|K|^2=0$].

The next equality can be shown by direct computation:

$$-\delta(w_t^2\rho i_{d\rho}G)=\langle w_t\rho G,\ 2dw_t\otimes d\rho\rangle+\langle w_t^2G,\ d\rho\otimes d\rho\rangle \tag{3.12}$$
$$+\langle w_t^2\rho G,\ \nabla\nabla\rho\rangle \quad \text{a.e. on } M.$$

Integrating the both sides of (3.12) over $B(2t)$, applying Stokes' theorem, and using (1.11) and the condition $\mathcal{L}_X|R|^2=0$, we have

$$0=\int_{B(2t)}\langle w_t\rho G,\ 2dw_t\otimes d\rho\rangle dV+\int_{B(2t)}\langle w_t^2G,\ d\rho\otimes d\rho\rangle dV \tag{3.13}$$
$$-\frac{2}{n-2}\int_{B(2t)} w_t^2\rho^2|G|^2dV.$$

Hence we know the inequality

$$|\langle w_t\rho G,\ 2dw_t\otimes d\rho\rangle|\leqq\frac{1}{n-2}|w_t\rho G|^2+(n-2)|2dw_t\otimes d\rho|^2 \tag{3.14}$$
$$\leqq\frac{1}{n-2}w_t^2\rho^2|G|^2+\frac{4(n-2)C^2}{t^2}|d\rho|^2.$$

Then it follows from (3.13) combined with (3.14) and also (1.16) that

$$\int_M\langle w_t^2G,\ d\rho\otimes d\rho\rangle dV\geqq\frac{1}{n-2}\int_M w_t^2\rho^2|G|^2dV-\frac{4(n-2)C^2}{t^2}\int_M|d\rho|^2dV. \tag{3.15}$$

Letting $t\to\infty$ in (3.15), we see

$$\liminf_{t\to\infty}\int_M\langle w_t^2G,\ d\rho\otimes d\rho\rangle dV\geqq\frac{1}{n-2}\liminf_{t\to\infty}\int_M w_t^2\rho^2|G|^2dV\geqq 0. \tag{3.16}$$

Then we get the condition (2.3) in Theorem 2.

Similarly, using (1.12) and the condition $\mathcal{L}_X|K|^2=0$ in place of (1.11) and $\mathcal{L}_X|R|^2=0$, we can obtain the condition (2.3). Thus we can apply Theorem 2, thereby completing the proof of Theorem 3. □

**Acknowledgement.** The author wishes to express his hearty thanks to the referee whose valuable suggestions improved very much the present paper.

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Department of Mathematics,
Liaoning University,
Shenyang 110036,
China