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AN OPERATOR $L=aI-D_t^jD_x^{-j-\alpha}-D_t^{-j}D_x^{j+\alpha}$ AND ITS NATURE IN GEVREY FUNCTIONS

By

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1. Introduction.

In this paper, we shall study some spectral properties on the parameter $a\in \boldsymbol{C}$ of the following Goursat problem:

$$(1.1)\qquad \begin{aligned}&\{aD_t^lD_x^\beta-D_t^{l+j}D_x^{\beta-j-\alpha}-D_t^{l-j}D_x^{\beta+j+\alpha}\}u(t,x)=f(t,x),\\ &u(t,x)-w(t,x)=O(t^lx^\beta)\qquad (l\geqq 1,\ \beta\geqq 0),\end{aligned}$$

where $x, t\in \boldsymbol{C}$, $1\leqq j\leqq l$ and $-\beta-j\leqq\alpha\leqq\beta-j$.

In the case $\alpha=0$, the problem (1.1) was studied in an extremely precise form by Leary [1] and Yoshino [5, 6] in the space of holomorphic functions at the origin. Let $a=2\cos\pi\theta\in(-2, 2)$ $(0<\theta<1)$. Leray [1] introduced an auxiliary function $\rho(\theta)$ by

$$(1.2)\qquad \rho(\theta)=\liminf_{N\ni h\to\infty}|\sin(h\pi\theta)|^{1/h}.$$

Here and in what follows, $\boldsymbol{N}$ and $\boldsymbol{Z}$ denotes the set of non negative integers and integers, respectively. They proved that the problem (1.1) is uniquely solvable in the space of holomorphic functions at the origin if and only if

$$(1.3)\qquad a\in \boldsymbol{C}\setminus(-2, 2)\quad\text{or}\quad a=\pm 2\quad\text{or}\quad \rho(\theta)>0.$$

Moreover, Leray-Pisot [2] proved that the set of zero points of $\rho(\theta)$ is uncountable with Lebesgue measure zero.

On the other hand, in the case $\alpha\neq 0$ the problem (1.1) is not solvable in the space of local holomorphic functions, and we have to study the problem in the space of formal or convergent power series with Gevrey estimate for the coefficients according as $\alpha>0$ or $\alpha<0$ (see Miyake-Hashimoto [4, Theorem B]). In this paper, we shall prove that the spectral properties on the parameter a distinguish the case $\alpha+j>0$ from the case $\alpha+j\leqq 0$. In the former case the meaning of the condition (1.3) will be understood clearly.

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In order to proceed our study, we shall study the resolvent sets of the following integro-differential operator in Banach spaces of Gevrey functions,

(1.4) $$A=D_t^j D_x^{-j-\alpha}+D_t^{-j}D_x^{j+\alpha} \qquad (j\geqq 1,\ \alpha\in \boldsymbol{Z}).$$

Because the problem (1.1) is reduced to an integro-differential equation, $(aI-A)U(t, x)=F(t, x)$, by changing the unknown function $u(t, x)$ to $U(t, x)$ by $u=D_t^{-l}D_x^{-\beta}U+w$. Here, $D_t^{-1}U(t, x)$ is defined formally by

$$D_t^{-1}U(t, x):=\sum_{l=1}^{\infty} U_{l-1}(x)t^l/l! \qquad \text{for} \quad U(t, x)=\sum_{l=0}^{\infty} U_l(x)t^l/l!.$$

We shall state main results in § 2 which will be proved in § 4 under some preparative considerations given in § 3. A stability of resolvent sets by small perturbations will be proved in § 5.

The author would like to express his thanks to Mr. Hashimoto for his suggestion of the decomposition of a matrix $C_n(a)$ given in § 3.

2. Statement of results.

Let $U(t, x)=\sum U_{l\beta}t^l x^\beta/l!\beta!$ $(l, \beta\in \boldsymbol{N})$ be a formal power series. Then we define Banach spaces of Gevrey functions as follows:

(i) $U(t, x)\in G^s(R; k)$ $(s, R>0, k\in \boldsymbol{N})$ if

(2.1) $$\|U\|_{R;k}^{(s)}:=\sup_{l,\beta} |U_{l\beta}| \frac{R^{sl+\beta}}{(sl+\beta+k)!}<+\infty.$$

(ii) $U(t, x)\in G_n^s(R; k)$ $(s\leqq 0, R>0, k\in \boldsymbol{N}, n\geqq 1)$ if

(2.2) $$\|U\|_{n;R;k}^{(s)}:=\sup_{l,\beta} |U_{l\beta}| R^{sl+\beta}\frac{\{(n-s)l+n\beta+k\}!}{\{nl+(n+1)\beta+k\}!}<+\infty.$$

The index s is called Gevrey index from the reason below.

Let $U(t, x)\in G^s(R; k)$ or $G_n^s(R; k)$, and put $U_l(x):=\sum U_{l\beta}x^\beta/\beta!\,(l\in \boldsymbol{N})$. Then $U_l(x)$ are holomorphic in a common neighbourhood of $x=0$ for all $l\in \boldsymbol{N}$ and there are positive constants X and T such that

(2.3) $$\max_{|x|\leq X} |U_l(x)|\leqq C\frac{(l!)^s}{T^l}(l\in \boldsymbol{N})$$

holds for some non negative constant C. Precise properties of these spaces are given in Miyake [3] and Miyake-Hashimoto [4].

In [4], we have proved that the Gevrey index s to the Goursat problem (1.1) is given by

(2.4) $$s=1+\frac{\alpha}{j}\ \left(=\frac{j+\alpha}{j}\right).$$

In what follows we fix s to this number.

Let A be the integro-differential operator given by (1.4). We denote by $\rho_{R;k}(A)$ (resp. $\rho_{n;R;k}(A)$) the resolvent set of the bounded mapping,

$$A: G^s(R;k) \longrightarrow G^s(R;k) \text{ (resp. } A: G_n^s(R;k) \longrightarrow G_n^s(R;k)).$$

Now the main theorem in this paper is the following,

THEOREM A. (i) *Let* $s>0$, *that is,* $j+\alpha>0$. *Then*

$$\rho_{R;k}(A)=\boldsymbol{C}\setminus[-2,2] \quad \text{for any } R>0 \text{ and } k\in\boldsymbol{N}. \tag{2.5}$$

(ii) *Let* $s\leqq 0$. *Let* $C=\max\{e^{s\alpha}, e^{-\alpha}\}$. *Then*

$$\bigcup_{n\geq 1}\rho_{n;R;k}(A)\supset\left\{a\in\boldsymbol{C}\,;\left(\frac{\operatorname{Re} a}{C+1/C}\right)^2+\left(\frac{\operatorname{Im} a}{C-1/C}\right)^2>1\right\} \tag{2.6}$$

for any $R>0$ *and* $k\in\boldsymbol{N}$, *where* $\operatorname{Re} a$ (*resp.* $\operatorname{Im} a$) *denotes the real* (*resp. imaginary*) *part of* $a\in\boldsymbol{C}$.

We remark that in Miyake-Hashimoto [4, Lemma 3.3], we proved the following facts:

(i) In the case $s>0$, the operator norm of A in $G^s(R;k)$ is estimated by $\|A\|\leqq 2$, and hence $\rho_{R;k}(A)\supset\{a\in\boldsymbol{C}\,;\ |a|>2\}$.

(ii) In the case $s\leqq 0$, the operator norm of A in $G_n^s(R;k)$ is estimated by $\|A\|\leqq(1+1/n)^{-j-\alpha}e^{-\alpha}+(1-s/n)^{-j-\alpha}e^{s\alpha}$, and hence $\bigcup_{n\geq 1}\rho_{n;R;k}(A)\supset\{a\in\boldsymbol{C}\,;\ |a|>e^{-\alpha}+e^{s\alpha}\}$.

In the case where $a\in[-2,2]$, the spectral set of the mapping $A: G^s(R;k)\to G^s(R;k)$ $(s>0)$, we can prove the following result corresponding to the result in Introduction.

PROPOSITION B. *Let* $s>0$ *and* $a=2\cos\pi\theta$ $(0\leqq\theta\leqq 1)$. *Then the problem* (1.1) *is uniquely solvable in* $G^s:=\bigcup_{R>0}G^s(R;0)$ *for any* $f(t,x), w(t,x)\in G^s$ *if and only if* $a=\pm 2$ *or* $\rho(\theta)>0$.

3. Preliminary.

We shall study in this section some properties of a special matrix of finite or infinite order of the form:

$$(3.1)\quad C_n(a)=\begin{pmatrix} a & -1 & 0 & & & & \\ -1 & a & -1 & 0 & & \mathbf{0} & \\ 0 & -1 & a & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & 0 \\ & \mathbf{0} & & 0 & -1 & a & -1 \\ & & & & 0 & -1 & a \end{pmatrix},\quad (n+1)^{th}\text{ matrix},$$

where $C_0(a)=a$.

$$(3.2)\quad C_\infty(a)=\begin{pmatrix} a & -1 & 0 & & & \\ -1 & a & -1 & 0 & \mathbf{0} & \\ 0 & -1 & a & -1 & 0 & \\ & 0 & -1 & a & -1 & 0 \\ & \mathbf{0} & & \ddots & \ddots & \ddots \end{pmatrix},\quad \text{infinite matrix.}$$

We shall construct their inverse matrices in exact form.

Let $a\in \boldsymbol{C}\backslash[-2, 2]$ and λ, $\mu\,(=\lambda^{-1})$ $(|\lambda|>1)$ be the roots of the quadratic equation, $z^2-az+1=0$. We define sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ by $b_n=a_n^{-1}$ and the formula,

$$(3.3)\qquad a_0=a,\qquad a_n=a-b_{n-1}\ (n\geqq 1).$$

It is easily proved that

$$(3.4)\qquad a_n=\frac{\lambda^{n+2}-\mu^{n+2}}{\lambda^{n+1}-\mu^{n+1}}\longrightarrow\lambda\ (n\to\infty).$$

The following decomposition of $C_n(a)$ $(1\leqq n\leqq\infty)$ is useful to write down the inverse matrix exactly.

$$C_n(a)=\begin{pmatrix} 1 & & & & & \\ -b_0 & 1 & & & \mathbf{0} & \\ & -b_1 & 1 & & & \\ & & -b_2 & 1 & & \\ & & & \ddots & \ddots & \\ & \mathbf{0} & & & \ddots & \\ & & & & -b_{n-1} & 1 \end{pmatrix}\begin{pmatrix} a_0 & -1 & & & & \\ & a_1 & -1 & & \mathbf{0} & \\ & & a_2 & -1 & & \\ & & & a_3 & -1 & \\ & & & & \ddots & \ddots \\ & \mathbf{0} & & & a_{n-1} & -1 \\ & & & & & a_n \end{pmatrix}.$$

This decomposition implies the following, LEMMA 3.1. *Let a, λ and μ be as above. Then we have:*

(i) *The inverse matrix $C_n^{-1}(a)=(c_{pq})_{0\leq p,q\leq n}$ $(1\leqq n<\infty)$ is given by*

$$c_{pq}=\begin{cases}\dfrac{(\lambda^{q+1}-\mu^{q+1})(\lambda^{n-p+1}-\mu^{n-p+1})}{(\lambda-\mu)(\lambda^{n+2}-\mu^{n+2})}, & 0\leqq q\leqq p\leqq n.\\[2ex] \dfrac{(\lambda^{p+1}-\mu^{p+1})(\lambda^{n-q+1}-\mu^{n-q+1})}{(\lambda-\mu)(\lambda^{n+2}-\mu^{n+2})}, & 0\leqq p<q\leqq n.\end{cases} \tag{3.5}$$

(ii) *Let $C_\infty^{-1}(a)=(c_{pq})_{p,q\geq 0}$. Then*

$$c_{pq}=\begin{cases}\mu^{p+1}\dfrac{\lambda^{q+1}-\mu^{q+1}}{\lambda-\mu}, & 0\leqq q\leqq p.\\[2ex] \mu^{q+1}\dfrac{\lambda^{p+1}-\mu^{p+1}}{\lambda-\mu}, & 0\leqq p<q.\end{cases} \tag{3.6}$$

PROOF. (i) The above decomposition of $C_n(a)$ implies,

$$C_n^{-1}(a)=\begin{pmatrix} b_0 & b_0b_1 & b_0b_1b_2 & \cdots & b_0b_1\cdots b_n\\ & b_1 & b_1b_2 & \cdots & b_1\cdots b_n\\ & & b_2 & \cdots & b_2\cdots b_n\\ & \mathbf{0} & & \ddots & \vdots\\ & & & & b_n\end{pmatrix}\begin{pmatrix}1 & & & & \mathbf{0}\\ b_0 & 1 & & & \\ b_0b_1 & b_1 & \ddots & & \\ b_0b_1b_2 & b_1b_2 & & \ddots & \\ \vdots & \vdots & & & 1 & \\ b_0\cdots b_{n-1} & b_1\cdots b_{n-1} & \cdots & b_{n-1} & 1\end{pmatrix}.$$

Let $0\leqq q\leqq p\leqq n$. Then we have

$$c_{pq}=b_q\cdots b_p\{1+b_pb_{p+1}+b_p(b_{p+1})^2b_{p+2}+\cdots+b_p(b_{p+1}\cdots b_{n-1})^2b_n\}.$$

Since $b_k=(\lambda^{k+1}-\mu^{k+1})/(\lambda^{k+2}-\mu^{k+2})$, we have

$$b_q\cdots b_p=\frac{\lambda^{q+1}-\mu^{q+1}}{\lambda^{p+2}-\mu^{p+2}}.$$

By the relation $b_k^{-1}=a_k=a-b_{k-1}$, we get

$$b_kb_{k+1}=\frac{b_{k+1}-b_k}{b_k-b_{k-1}} \qquad (b_{-1}:=0).$$

Indeed, it is sufficient to notice $b_k^{-1}-b_{k+1}^{-1}=b_k-b_{k-1}$. Hence we have

$$b_p(b_{p+1}\cdots b_{p+l-1})^2b_{p+l}=\frac{b_{p+l}-b_{p+l-1}}{b_p-b_{p-1}}.$$

This implies

$$1+b_pb_{p+1}+b_p(b_{p+1})^2b_{p+2}+\cdots+b_p(b_{p+1}\cdots b_{n-1})^2b_n=\frac{b_n-b_{p-1}}{b_p-b_{p-1}}.$$

By using the relation $\lambda\mu=1$, we have

$$b_n-b_{p-1}=\frac{(\lambda^{n-p+1}-\mu^{n-p+1})(\lambda-\mu)}{(\lambda^{n+2}-\mu^{n+2})(\lambda^{p+1}-\mu^{p+1})}$$

and

$$b_p-b_{p-1}=\frac{(\lambda-\mu)^2}{(\lambda^{p+2}-\mu^{p+2})(\lambda^{p+1}-\mu^{p+1})}.$$

Summing up these results we obtain the expression of c_{pq} for $0\leqq q\leqq p\leqq n$. It is the same in the case $0\leqq p<q\leqq n$.

(ii) The expression (3.6) is obtained by letting $n\to\infty$ in the expression (3.5). This is only an explanation, but we can prove it precisely, so we omit it. □

We consider that C^{n+1} is a vector space equipped with the norm by the maximum of absolute values of components. Let $a\in C\setminus[-2, 2]$ and take $c>1$ such that

(3.7) $$\left(\frac{\operatorname{Re} a}{c+1/c}\right)^2+\left(\frac{\operatorname{Im} a}{c-1/c}\right)^2=1.$$

Then $|\lambda|=c$ and $|\mu|=|\lambda|^{-1}=c^{-1}$. By this choice of c, we can prove the following,

LEMMA 3.2. *The operator norm of* $C_n^{-1}(a)$ $(a\in C\setminus[-2, 2])$ *in* C^{n+1} *is estimated by*

(3.8) $$\|C_n^{-1}(a)\|\leqq\frac{c(c^2+1)}{(c-1)^3(c+1)}.$$

PROOF. By the definition, $\|C_n^{-1}(a)\|=\max_p\sum_{q=0}^n|c_{pq}|$. We fix $p\in\{0, 1, \cdots, n\}$. Considering (3.5), we set

$$\sum_{q=0}^n|c_{pq}|=\left\{\sum_{q=0}^p+\sum_{q=p+1}^n\right\}|c_{pq}|\underset{\text{put}}{=\!=}I+II.$$

First, we estimate the part I.

$$I=|\mu|^{p+2}\frac{|1-\mu^{2(n-p+1)}|}{|1-\mu^2||1-\mu^{2(n+2)}|}\sum_{q=0}^q|\lambda|^{q+1}|1-\mu^{2(q+1)}|$$

(since $1-c^{-2}\leqq|1-\mu^{2(k+1)}|\leqq 1+c^{-2}$, $|1-\mu^{2(n+2)}|\geqq 1-c^{-4}$)

$$\leqq\frac{(1+c^{-2})^2}{c^{p+2}(1-c^{-2})(1-c^{-4})}\sum_{q=0}^p c^{q+1}$$

(since $\sum_{q=0}^p c^{q+1}<c^{p+2}/(c-1)$)

$$<\frac{(1+c^{-2})^2}{(c-1)(1-c^{-2})(1-c^{-4})}=\frac{c^2(c^2+1)}{(c-1)^3(c+1)^2}.$$

By the same way we can estimate the part II as follows.

$$II=|\mu|^{n-p+2}\frac{|1-\mu^{2(p+1)}|}{|1-\mu^2||1-\mu^{2(n+2)}|}\sum_{q=p+1}^{n}|\lambda|^{n-q+1}|1-\mu^{2(n-q+1)}|$$
$$<\frac{c(c^2+1)}{(c-1)^3(c+1)^2}.$$

These estimates imply (3.8) immediately. □

REMARK. Yoshino proved the similar but somewhat different estimate to (3.8) by more complicated calculation [5, Lemma 3].

By the same way as above, we can prove the following,

LEMMA 3.3. *Let l^∞ be the Banach space of bounded sequences with supremum norm. Then $C_\infty(a)$ has a bounded inverse operator in l^∞ if and only if $a\in \boldsymbol{C}\setminus[-2, 2]$ and the operator norm of $C_\infty^{-1}(a)$ is estimated by*

$$\|C_\infty^{-1}(a)\|\leqq\frac{c^2+1}{c(c-1)^2}, \tag{3.9}$$

where $c>1$ is the same constant as above.

PROOF. Let $a\in(-2, 2)$ and put $a=2\cos\pi\theta$ $(0<\theta<1)$. Then it is easy to check that a is an eigen value with an eigen function $(\sin\pi\theta, \sin 2\pi\theta, \sin 3\pi\theta, \cdots)\in l^\infty$. The inequality (3.9) is proved by the same way as above, so we omit the proof. □

Next, we consider the case $a\in[-2, 2]$. Let $a=2\cos\pi\theta$ $(0\leqq\theta\leqq1)$. In this case, $\{\lambda, \mu\}=\{e^{\pm\sqrt{-1}\pi\theta}\}$ and

$$\det C_n(a)=\begin{cases}\dfrac{\lambda^{n+2}-\mu^{n+2}}{\lambda-\mu}=\dfrac{\sin(n+2)\pi\theta}{\sin\pi\theta}, & \theta\neq0, 1.\\ n+2, & \theta=0.\\ (-1)^{n+1}(n+2), & \theta=1.\end{cases} \tag{3.10}$$

Therefore, $C_n(a)$ is invertible for every $n\in\boldsymbol{N}$ if and only if $a=\pm2$ or θ is an irrational. Let θ be an irrational and put $C_n^{-1}(a)=(c_{pq})_{0\leq p,q\leq n}$. Then the expression (3.5) implies

$$c_{pq}=\begin{cases}\dfrac{\sin(q+1)\pi\theta\sin(n-p+1)\pi\theta}{\sin\pi\theta\sin(n+2)\pi\theta}, & 0\leqq q\leqq p\leqq n.\\ \dfrac{\sin(p+1)\pi\theta\sin(n-q+1)\pi\theta}{\sin\pi\theta\sin(n+2)\pi\theta}, & 0\leqq p<q\leqq n.\end{cases} \tag{3.11}$$

The inverse matrices in cases $\theta=0, 1$ are obtained by letting $\theta\downarrow 0$ and $\theta\uparrow 1$ in this expression.

4. Proofs of results.

Let $U(t, x)=\sum U_{l\beta}t^l x^\beta/l!\beta!$ and $F(t, x)=\sum F_{l\beta}t^l x^\beta/l!\beta!$. Then the equation, $(aI-A)U(t, x)=F(t, x)$, implies the following relations,

$$aU_{l\beta}-U_{l+j,\beta-j-\alpha}-U_{l-j,\beta+j+\alpha}=F_{l\beta}\quad (l, \beta\in \boldsymbol{N}). \tag{4.1}$$

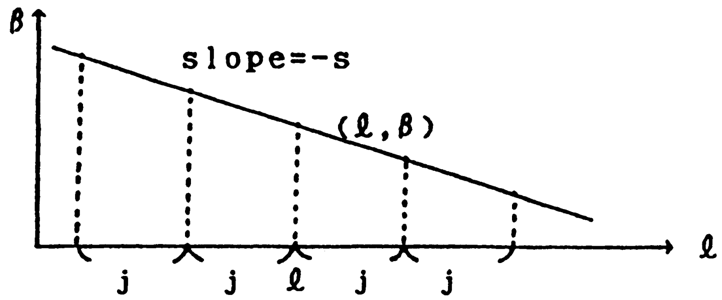
PROOF OF THEOREM A. (i) Let $s>0$, *i.e.*, $j+\alpha>0$. For $(l, \beta)\in \boldsymbol{N}^2$, we put

$$d(l, \beta)=[l/j]+[\beta/(j+\alpha)], \tag{4.2}$$

where $[l/j]$ denotes the integral part of l/j. We define a vector $\mathcal{U}_{l\beta}\in \boldsymbol{C}^{d(l+\beta)+1}$ by

$$\mathcal{U}_{l\beta}={}^t(\cdots, U_{l-j,\beta+j+\alpha}, U_{l\beta}, U_{l+j,\beta-j-\alpha}, \cdots), \tag{4.3}$$

which is defined from the coefficients of $U(t, x)\in G^s(R, k)$.



The relation (4.1) induces the following equation

$$C_{d(l,\beta)}(a)\mathcal{U}_{l\beta}=\mathcal{F}_{l\beta}\in \boldsymbol{C}^{d(l,\beta)+1}, \tag{4.4}$$

where $\mathcal{F}_{l\beta}$ is defined from $F(t, x)$ similarly to $\mathcal{U}_{l\beta}$. Since $|U_{l+pj,\beta-p(j+\alpha)}|\leqq \|U\|_{R;k}^{(s)}(sl+\beta+k)!/R^{sl+\beta}$ $(-[l/j]\leqq p\leqq[\beta/(j+\alpha)])$, (4.4) is an equation in $\boldsymbol{C}^{d(l,\beta)+1}$ with the usual norm as in Lemma 3.2. Hence by Lemma 3.2, we have $\rho_{R;k}(A)\supset \boldsymbol{C}\setminus[-2, 2]$ and the operator norm of $(\boldsymbol{aI}\text{-}\boldsymbol{A})^{-1}$ is estimated by

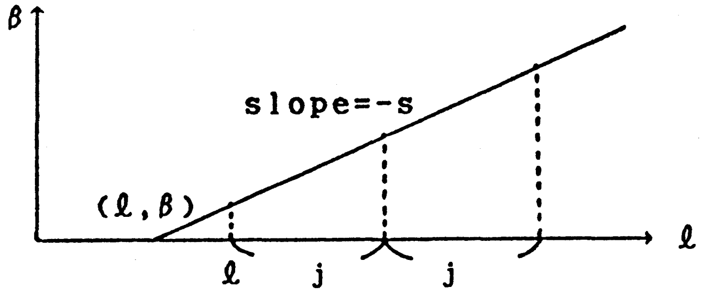
$$\|(aI-A)^{-1}\|\leqq\frac{c(c^2+1)}{(c-1)^3(c+1)}. \tag{4.5}$$

Let $a=2\cos\pi\theta\in(-2, 2)$ $(0<\theta<1)$. Then a is an eigen value if θ is a rational number, because an equation $(aI-A)U(t, x)=0$ has a non trivial polynomial solution since $\det C_n(a)=0$ for some $n\in \boldsymbol{N}$. This proves that $[-2, 2]$ is the spectral set of A. This completes the proof of (i).

(ii) Let $s\leqq 0$, *i.e.*, $j+\alpha\leqq 0$. In this case, the situation is more complicated, since the relation (4.1) implies infinite systems of equations. We pick up such $(l, \beta)\in \boldsymbol{N}^2$ that $l-j<0$ or $\beta+j+\alpha=\beta+sj<0$ $(s=(j+\alpha)/j\leqq 0)$. For $U(t, x)\in G_n^s(R; k)$, we define $\mathcal{U}_{l\beta}\in \boldsymbol{C}^{+\infty}$ by

$$\text{(4.6)}\qquad \mathcal{U}_{l\beta}={}^t(U_{l\beta}, U_{l+j,\beta-sj}, \cdots, U_{l+pj,\beta-p(sj)}, \cdots)$$

$$\underset{\text{put}}{=\!=}{}^t(U_0^{(l,\beta)}, U_1^{(l,\beta)}, U_2^{(l,\beta)}, \cdots).$$



By the definition of the norm of $G_n^s(R; k)$, we have

$$\text{(4.7)}\qquad |U_p^{(l,\beta)}|\leqq\frac{\|U\|}{R^{sl+\beta}}\frac{\{nl+(n+1)\beta+k+(n-(n+1)s)jp\}\,!}{\{(n-s)l+n\beta+k+(n-(n+1)s)jp\}\,!},$$

for every $p\in\boldsymbol{N}$. Now we define a weight function by

$$\text{(4.8)}\qquad w_{l,\beta;n}(p)=\frac{\{nl+(n+1)\beta+k+(n-(n+1)s)jp\}\,!}{\{(n-s)l+n\beta+k+(n-(n+1)s)jp\}\,!}\qquad (p\in\boldsymbol{N}).$$

Then $\{U_p^{(l,\beta)}w_{l,\beta;n}(p)^{-1}\}_{p=0}^{\infty}\in l^{\infty}$.

Let $a\in\boldsymbol{C}\setminus[-2, 2]$ and put $C_{\infty}^{-1}(a)=(c_{pq})_{p,q\geqq 0}$. Then $a\in\rho_{n;R;k}(A)$ if

$$\text{(4.9)}\qquad \sup_{l,\beta}\sup_{p\geqq 0}\sum_{q=0}^{\infty}|c_{pq}|\,w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1}<\infty.$$

Let us prove the following,

LEMMA 4.1. (i) *Let* $0\leqq q\leqq p$. *Then we have*

$$\text{(4.10)}\qquad w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1}\leqq e^{-sj(p-q)\{1-s(1+1/n)\}}.$$

(ii) *Let* $0\leqq p<q$. *Then we have*

$$\text{(4.11)}\qquad w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1}\leqq e^{j(q-p)\{1-s(1+1/n)\}}.$$

PROOF. (i) Let $p\geqq q\geqq 0$. Then we have

$$\begin{aligned} &w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1} \\ &=\prod_{r=1}^{(n-(n+1)s)j(p-q)}\frac{(n-s)l+n\beta+k+(n-(n+1)s)jq+r}{nl+(n+1)\beta+k+(n-(n+1)s)jq+r} \\ &\leqq\{(n-s)/n\}^{(n-(n+1)s)j(p-q)}\leqq e^{-s\{1-s(1+1/n)\}j(p-q)}. \end{aligned}$$

(ii) Let $q>p\geqq 0$. Then we have

$$\begin{aligned} &w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1} \\ &=\prod_{r=1}^{(n-(n+1)s)j(q-p)}\frac{nl+(n+1)\beta+k+(n-(n+1)s)jp+r}{(n-s)l+n\beta+k+(n-(n+1)s)jp+r} \\ &\leqq\{(n+1)/n\}^{(n-(n+1)s)j(q-p)}\leqq e^{\{1-s(1+1/n)\}j(q-p)}. \quad \square \end{aligned}$$

Next we shall estimate each component of $C_\infty^{-1}(a)$. Let $a\in \boldsymbol{C}\backslash[-2, 2]$ and $C_\infty^{-1}(a)=(c_{pq})_{p,q\geq 0}$. Take $c>1$ as in (3.7). Then by Lemma 3.1, we have:

$$|c_{pq}|\leqq c^{q-p-1}\frac{c^2+1}{c^2-1}, \qquad p\geqq q\geqq 0. \tag{4.12}$$

$$|c_{pq}|\leqq c^{p-q-1}\frac{c^2+1}{c^2-1}, \qquad q>p\geqq 0. \tag{4.13}$$

We are now in the final stage of the proof. We put

$$\begin{aligned} &\sum_{q=0}^{\infty}|c_{pq}|\, w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1} \\ &\quad =\Big(\sum_{q=0}^{p}+\sum_{q=p+1}^{\infty}\Big)|c_{pq}|\, w_{l,\beta;n}(q)\cdot w_{l,\beta;n}(p)^{-1}=I(n)+II(n). \end{aligned}$$

Let $C=\max\{e^{s\alpha}, e^{-\alpha}\}$ (>1) and $a\in \boldsymbol{C}$ satisfy

$$\Big(\frac{\operatorname{Re} a}{C+1/C}\Big)^2+\Big(\frac{\operatorname{Im} a}{C-1/C}\Big)^2>1.$$

Then, $c>\max\{e^{s\alpha}, e^{-\alpha}\}$ in the above inequalities. First, we examine $I(n)$.

$$I(n)\leqq\frac{c^2+1}{c^2-1}\sum_{q=0}^{p}c^{q-p-1}e^{-sj\{1-s(1+1/n)\}(p-q)}.$$

We notice that $-sj(1-s)=s\alpha$, since $s=1+(\alpha/j)$. Hence, $c>e^{s\alpha}$ implies that $I(n)$ is bounded for sufficiently large n.

Next, we examine $II(n)$.

$$II(n)\leqq\frac{c^2+1}{c^2-1}\sum_{q=p+1}^{\infty}c^{p-q-1}e^{j\{1-s(1+1/n)\}(q-p)}.$$

The codition $c>e^{-\alpha}$ implies that $II(n)$ is bounded for sufficiently large n, since $j(1-s)=-\alpha$. These prove that $a\in\rho_{n;R;k}(A)$ if n is sufficiently large.

This completes the proof of (ii). □

REMARK. In the case $s=0$, suppose that we take the norm of $G^0(R\,;\,k)$ by

$$\|U\|_{R;k}^{(0)}=\sup_{\beta}|U_{l\beta}|\frac{R^{\beta}}{(\beta+k)!}<+\infty$$

instead of (2.2). Then we can prove that $\rho_{R;k}(A)=C\setminus[-2,2]$. The proof is just the same as Lemma 3.3, but the stability of the resolvent set by small perturbations, which will be proved in the next section, does not hold by this choice of the norm.

PROOF OF PROPOSITION B. Instead of the problem (1.1), let us consider an equation,

$$(aI-A)U(t,x)=F(t,x)\in G^s(R_0\,;\,0)\qquad (s>0). \tag{4.14}$$

First, suppose $a\in(-2,2)$ and $\rho(\theta)>0$. By (4.4), the uniquenss of the solution $U(t,x)\in G^s:=\cup_{R>0}G^s(R\,;\,0)$ is trivial, since $C_{d(l,\beta)}(a)$ is invertible because θ is an irrational in this case. And also by (4.4), we see that the equation (4.14) has a solution in G^s if the operator norm of $C_n^{-1}(a)=(c_{pq})$ in $\boldsymbol{C}^{n+1}$ is estimated by

$$\|C_n^{-1}(a)\|\leqq Cr^n \tag{4.15}$$

for some positive constants C and r. Indeed, it is sufficient to notice the existence of positive constants c_0 and c_1 such that $c_0(sl+\beta)\leqq d(l,\beta)+1\leqq c_1(sl+\beta)$.

By (3.11), we have

$$\sum_{q=0}^{n}|c_{pq}|\leqq(n+1)|\sin\pi\theta|^{-1}|\sin(n+2)\pi\theta|^{-1}.$$

This, together with $\rho(\theta)>0$, implies (4.15) by taking $r>1/\rho(\theta)$.

The unique solvability of the equation (4.14) in the case $a=\pm 2$ is also easily proved.

Next consider the case $\rho(\theta)=0$. In the case where θ is a rational number, the uniqueness of solutions of the equation (4.14) does not hold trivially. In the case where θ is an irrational, it is easy to give $F(t,x)\in G^s(R_0\,;\,0)$ such that the formal power series solution $U(t,x)$ (which exists uniquely) does not belong to G^s. Indeed, it is sufficient to notice the existence of a sequence $\{\varepsilon_r\}_{r=1}^{\infty}$ and $1\leqq h(1)\leqq h(2)\leqq\cdots\to\infty$ such that

$$\varepsilon_1>\varepsilon_2>\varepsilon_3\cdots\longrightarrow 0,$$

and

$$|\sin(h(r)+2)\pi\theta|<(\varepsilon_r)^{h(r)},\qquad r=1,2,\cdots.$$

For the simplicity, we consider the case $s=1$. Let $F(t, x)=\sum_{r=1}^{\infty} t^{h(r)} \in G^1(1; 0)$, *i.e.*, $F_{h(r),0}=h(r)!$ and $F_{l\beta}=0$ otherwise. Then by (3.11) and (4.4), we have

$$U_{0,h(r)}=h(r)!\frac{\sin\pi\theta}{\sin(h(r)+2)\pi\theta},$$

and this shows that $U(t, x)$ does not belong to $G^1(R; 0)$ for any $R>0$. This completes the proof of Proposition B. $\square$

5. Stability of resolvent sets.

We shall prove that the resolvent sets given in Theorem A are stable by small perturbations defined below.

Let

$$B=\sum_{\sigma,l,\beta}^{\text{finite}} b_{\sigma l\beta}(x)t^{\sigma}D_t^l D_x^{\beta} \qquad ((\sigma, l, \beta)\in \boldsymbol{N}\times \boldsymbol{Z}\times \boldsymbol{Z}), \tag{5.1}$$

be an integro-differential operator with holomorphic coefficients in a neighbourhood of $x=0$. We say that the operator B is a "small perturbation" with respect to A if

$$(1-s)\sigma+sl+\beta<0 \qquad \text{for any } (\sigma, l, \beta) \text{ with } b_{\sigma l\beta}(x)\not\equiv 0. \tag{5.2}$$

The meaning of this condition is easily understood by the Newton polygon of the operator $A+B$ (see Miyake-Hashimoto [4]). We remark that in the case $s\geqq 1$ we can replace $b_{\sigma l\beta}(x)$ by $b_{\sigma l\beta}(t, x)\in G^s:=\cup_{R>0}G^s(R; 0)$ (see [3], [4]).

Now we can prove the following,

THEOREM C. *Suppose that B is a small perturbation with respect to A. Then there is a positive constant R_0 such that:*

(i) $\bigcup_{k\in N}\rho_{R_0;k}(A+B)\supset \boldsymbol{C}\setminus[-2, 2]$, $\bigcup_{R>0}\rho_{R;k}(A+B)\supset \boldsymbol{C}\setminus[-2, 2]$.

(ii) $\bigcup_{n\geq 1}\bigcup_{k\in N}\rho_{n;R_0;k}(A+B)\supset \left\{a\in \boldsymbol{C};\ \left(\frac{\operatorname{Re} a}{C+1/C}\right)^2+\left(\frac{\operatorname{Im} a}{C-1/C}\right)^2>1\right\}$.

Here C is the same positive constant as in Theorem A.

PROOF OF THEOREM C. We denote by $\mathcal{O}(|x|\leqq X)$ $(X>0)$ the set of holomorphic functions in $|x|<X$ and continuous on $|x|\leqq X$. For $b(x)\in\mathcal{O}(|x|\leqq X)$ we define

$$\|b\|_X=\max_{|x|\leq X}|b(x)|. \tag{5.3}$$

Then we can prove the following lemmas.

LEMMA 5.1 ([4, Lemma 3.1]). (i) *Let* $s>0$ *and* $b(x)\in\mathcal{O}(|x|\leqq\rho R)$ $(\rho>1)$. *Then* $b(x)$ *defines a bounded operator of multiplier in* $G^s(R;k)$ *with norm estimated by*

$$\|b\|\leqq C\frac{\rho}{\rho-1}\|b\|_{\rho R}, \tag{5.4}$$

for some positive constant C.

(ii) *Let* $s\leqq 0$ *and* $b(x)\in\mathcal{O}(|x|\leqq\rho R)$. *Then if* $\rho>e^{-s}/(n+1)$, $b(x)$ *defines a bounded operator of multiplier in* $G_n^s(R;k)$ $(k\geqq n-1)$ *with norm estimated by*

$$\|b\|\leqq C\frac{(n+1)\rho}{(n+1)\rho-e^{-s}}\|b\|_{\rho R}, \tag{5.5}$$

for some positive constant C.

LEMMA 5.2 ([4, Lemma 3.3]). *Let* $(\sigma, l, \beta)\in\boldsymbol{N}\times\boldsymbol{Z}\times\boldsymbol{Z}$ *satisfiy*

$$(1-s)\sigma+sl+\beta \underset{\text{put}}{=\!=} -\delta\leqq 0.$$

Then $t^\sigma D_t^l D_x^\beta$ *defines a bounded operator in* $G^s(R;k)$ $(s>0)$ *and also in* $G_n^s(R;k)$ $(s\leqq 0)$ *with norm estimated by*

$$\|t^\sigma D_t^l D_x^\beta\|\leqq C(\sigma, l, \beta, s, n)R^{\sigma+\delta}k^{-\delta}, \tag{5.6}$$

where $C(\sigma, l, \beta, s, n)$ *does not depend on* R *and* k.

Theorem C is now obvious from these lemmas. Indeed, let B be a small perturbation with respect to A.

(i) Let $s>0$. Then there is a positive constant R_0 such that B is a bounded operator in $G^s(R;k)$ for every $0<R\leqq R_0$ and $k\in\boldsymbol{N}$. By the definition of small perturbation, there is a positive constant δ such that the operator norm of B is estimated by

$$\|B\|\leqq CR^\delta k^{-\delta}, \tag{5.7}$$

for some positive constant C. This implies (i) immediately.

(ii) Let $s\leqq 0$. Then there is a positive constant R_0 such that B is a bounded operator in $G_n^s(R;k)$ for every $0<R\leqq R_0$, $n\in\boldsymbol{N}$ and $k\geqq n-1$, and the operator norm is estimated by

$$\|B\|\leqq CR^\delta k^{-\delta} \tag{5.8}$$

for some positive constants C and δ. This implies (ii).

This completes the proof. □

REMARK. In the case $s \leqq 0$, if the coefficients of the small perturbation B are all constants, then we have

$$\bigcup_{n\geq 1}\bigcup_{R>0}\rho_{n;R;k}(A+B)\supset\left\{a\in \boldsymbol{C}\,;\,\left(\frac{\mathrm{Re}\,a}{C+1/C}\right)^2+\left(\frac{\mathrm{Im}\,a}{C-1/C}\right)^2>1\right\},$$

for any $k\in \boldsymbol{N}$.

Added in proof. After this paper was accepted for its publication, the author and M. Yoshino generalized the results in this paper in more precise form for general operators in the following preprint by employing the Toeplitz operator method:

M. Miyake and M. Yoshino, Wiener-Hopf equation and Fredholm property of the Goursat problem in Gevrey space, Preprint series, No. 13 (1992), College of General Education, Nagoya University.

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