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A CLASS OF MULTIVALENT FUNCTIONS

By

Mamoru NUNOKAWA

1. Introduction.

Let $A(p)$ be the class of functions of the form

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n} \qquad (p\in N=1, 2, 3, \cdots)$$

which are analytic in $U=\{z \mid |z|<1\}$.

A function $f(z)\in A(p)$ is said to be p-valently starlike iff

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \qquad \text{in } U .$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p-valently starlike in U. Further, a function in $A(p)$ is said to be p-valently convex iff

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}>0 \qquad \text{in } U .$$

Also we denote by $C(p)$ the subclass of $A(p)$ consisting of all p-valently convex functions in U.

MacGregor [2] investigated the class of functions which are analytic in U, $f(0)=f'(0)-1=0$ and satisfy the condition

$$|f'(z)-1|<1 \qquad \text{in } U .$$

Let F denote the class of functions which satisfy the above conditions.

MacGregor [2, Theorem 6] obtained the following result:

THEOREM A. *If $f(z)\in F$, then $f(z)$ is starlike in $|z|<\sqrt{4/4}\doteqdot 0.894$.*

Nunokawa [4] and Nunokawa, Fukui, Owa, Saitoh and Sekine [6] improved Theorem A. Mocanu [3] showed that there is a function $f(z)\in A(1)$ which is a member of F but not starlike in $|z|<1$.

THEOREM B. *If $f(z)\in F$, then $f(z)$ is starlike in $|z|<r_{1}<1$, where r_{1} is the*

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root of the equation

$$\log(9-4r^2+4r^3-r^4)-\log 9(1-r^2)+\sin^{-1}r=\pi\,,$$

that is $r_1 \doteqdot 0.934$.

A proof of Theorem B can be found in [6, Corollary].

2. Main theorem.

In this paper, we need the following lemmata.

LEMMA 1. *Let* $w(z)$ *be analytic in the unit disk* U, *with* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z|=r$ *at a point* z_0, *then we can write*

$$z_0 w'(z_0)=k w(z_0)$$

where k *is a real number and* $k \geqq 1$.

A proof can be found in [1].

Applying Lemma 1, we can obtain the following lemma.

LEMMA 2. *Let* $p(z)$ *be analytic in* U, $p(0)=1$ *and suppose that*

(1) $$|p(z)+zp'(z)-1|<\sqrt{2} \quad \text{in } U\,.$$

Then we have

$$|p(z)-1|<\frac{\sqrt{2}}{2} \quad \text{in } U$$

and

$$|\arg p(z)|<\frac{\pi}{4} \quad \text{in } U\,.$$

PROOF. Putting

$$p(z)=1+\frac{\sqrt{2}}{2}w(z)\,,$$

then $w(z)$ is analytic in U and $w(0)=0$. If there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|\geqq 1\,,$$

then from Lemma 1, we nave

$$z_0 w'(z_0)=k w(z_0)\,, \qquad (k\geqq 1)\,.$$

Then w have

$$|p(z_0)+z_0p'(z_0)-1|=\frac{\sqrt{2}}{2}|w(z_0)+kw(z_0)|$$

$$=\frac{\sqrt{2}}{2}|w(z_0)|(1+k)\geqq\frac{\sqrt{2}}{2}(1+k)\geqq\sqrt{2}\ .$$

This contradicts (1). Therefore we have

$$|w(z)|<1 \qquad \text{in } U\ .$$

This shows that

$$|p(z)-1|<\frac{\sqrt{2}}{2} \qquad \text{in } U\ .$$

and therefore we have

$$|\arg p(z)|<\frac{\pi}{4} \qquad \text{in } U\ .$$

This completes our proof.

Applying the same method as in the proof of [5, Lemma 6 and Theorem 5], we can easily obtain the following lemma.

LEMMA 3. *Let $p\geqq 2$. If $f(z)\in A(p)$ satisfies the condition*

$$\operatorname{Re}\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}>0 \qquad \textit{in } U\ ,$$

then we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0 \qquad \textit{in } U\ .$$

MAIN THEOREM. Let $p\geqq 2$. If $f(z)\in A(p)$ satisfies the condition

(2) $$|f^{(p)}(z)-p\,!|<\sqrt{2}(p\,!) \qquad \text{in } U\ ,$$

then $f(z)$ is p-valently starlike in U.

PROOF. Let us put

(3) $$p(z)=\frac{f^{(p-1)}(z)}{p\,!\,z}\ , \qquad (\ p(0)=1\ ).$$

Then we have

$$p(z)+zp'(z)-1=\frac{f^{(p)}(z)}{p\,!}-1\ ,$$

and from the hypothesis (2), we have

$$|p(z)+zp'(z)-1|<\sqrt{2} \qquad \text{in } U\ .$$

Then, from Lemma 2 and (3), we have

$$|\arg p(z)|=\left|\arg\frac{f^{(p-1)}(z)}{p!\,z}\right|=\left|\arg\frac{f^{(p-1)}(z)}{z}\right|<\frac{\pi}{4}\qquad\text{in } U\,. \tag{4}$$

Applying the same idea as in the proof of [7, Theorem 1] and integrating on the line segment from 0 to z, we have

$$\begin{aligned}\frac{f^{(p-2)}(z)}{z^2}&=\frac{1}{z^2}\int_0^z f^{(p-1)}(t)dt\\&=\frac{1}{r^2}\int_0^r\frac{f^{(p-1)}(t)}{t}\rho d\rho\end{aligned} \tag{5}$$

where $z=re^{i\theta}$, $0<r<1$, $t=\rho e^{i\theta}$ and $0\leqq\rho\leqq r$.

From (4), we have

$$\left|\arg\frac{f^{(p-1)}(t)}{t}\rho\right|=\left|\arg\frac{f^{(p-1)}(t)}{t}\right|<\frac{\pi}{4}\qquad\text{in } U\,. \tag{6}$$

Applying the same idea as in the proof of [8, Lemma 1] and since $s=f^{(p-1)}(t)/t$ lies in the convex sector $\{|\arg s|<\pi/4\}$, then from (5) and (6), the same is true of its integral mean value of (5).

Therefore, we have

$$\begin{aligned}\left|\arg\frac{f^{(p-2)}(z)}{z^2}\right|&=\left|\arg\frac{1}{r^2}\int_0^r\frac{f^{(p-1)}(t)}{t}\rho d\rho\right|\\&=\left|\arg\int_0^r\frac{f^{(p-1)}(t)}{t}\rho d\rho\right|<\frac{\pi}{4}\qquad\text{in } U\,.\end{aligned} \tag{7}$$

From (4) and (7), we have

$$\begin{aligned}\left|\arg\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right|&=\left|\arg\frac{f^{(p-1)}(z)}{z}\frac{z^2}{f^{(p-2)}(z)}\right|\\&\leqq\left|\arg\frac{f^{(p-1)}(z)}{z}\right|+\left|\arg\frac{f^{(p-1)}(z)}{z^2}\right|<\frac{\pi}{2}\qquad\text{in } U\,.\end{aligned}$$

This shows that

$$\mathrm{Re}\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}>0\qquad\text{in } U\,. \tag{8}$$

From Lemma 3 and (8), we have

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0\qquad\text{in } U\,.$$

This completes our proof.

From the main theorem, we easily have the following corollary.

COROLLARY 1. *Let $p\geqq 2$. If $f(z)\in A(p)$ satisfies the condition*

(9) $$|zf^{(p+1)}(z)+pf^{(p)}(z)-p(p!)|<\sqrt{2}p(p!) \qquad in\ U\,,$$

then $f(z)$ is p-valently convex in U.

PROOF. Putting

$$g(z)=\frac{zf'(z)}{p}\,,$$

then $g(z)$ is a function of $A(p)$.

From the hypothesis (9) and the main theorem, we have that $g(z)$ is p-valently starlike in U. Therefore, $f(z)$ is p-valently convex in U.

This completes our proof.

COROLLARY 2. *Let $p\geqq 2$. If $f(z)\in A(p)$ satisfies the condition*

(10) $$|f^{(p+1)}(z)|<\sqrt{2}(p!) \qquad in\ U\,,$$

then, $f(z)$ is p-valently starlike in U.

PROOF. By an easy calculation and from (10), we have

$$|f^{(p)}(z)-p!|=\left|\int_0^z f^{(p+1)}(t)dt\right|$$

$$\leqq\int_0^r |f^{(p+1)}(t)|d\rho<\sqrt{2}(p!)|z|<\sqrt{2}(p!)$$

where $|z|=r<1$ and $0\leqq|t|=\rho<r$.

From the main theorem, $f(z)$ is p-valently starlike in U.

This completes our proof.

REMARK 1. It is easily confirmed that the function

$$f(z)=z^p+\frac{p!e^{-\alpha}}{\alpha^p}\sqrt{2}\left\{e^{\alpha z}-\sum_{k=0}^{p}\frac{(\alpha z)^k}{k!}\right\}$$

satisfies the conditions (2) and (10), therefore $f(z)$ is p-valently starlike in U. On the other hand, the function

$$g(z)=z^p+\int_0^z\frac{p!e^{-\alpha}}{t\alpha^p}\left(e^{\alpha t}-\sum_{k=0}^{p}\frac{(\alpha t)^k}{k!}\right)dt$$

satisfies the condition (9), therefore $g(z)$ is p-valently convex in U.

REMARK 2. To prove the main theorem, we have to obtain

$$\mathrm{Re}\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}>0 \qquad \text{in } U\,.$$

Therefore we have to suppose $p \geqq 2$.

On the other hand, it is easily confirmed that the function

$$f(z)=z+\frac{\sqrt{2}}{2}z^2 \in A(1)$$

satisfies the condition

$$|f'(z)-1|<\sqrt{2} \qquad \text{in } U\,,$$

but $f(z)$ is not starlike in U.

This shows that the main theorem does not hold good for the case $p=1$.

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Department of Mathematics
University of Gunma
Aramaki, Maebashi 371
Japan