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MAXIMAL FUNCTIONS OF PLURISUBHARMONIC FUNCTIONS

By

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Abstract. We show that for nonnegative plurisubharmonic functions on the unit ball of C^n the admissible maximal functions are dominated by the radial maximal functions in L^p-mean. This gives another characterization of the class M^p of holomorphic functions and its invariance under the compositions by automorphisms of the unit ball. As a consequence of the invariance all onto endomorphisms of M^1 $(n=1)$ are characterized.

1. Introduction.

Let B be the unit ball of $C^n(n\geqq 1)$ and let σ denote the Lebesgue measure on $S=\partial B$, normalized so that $\sigma(S)=1$. For a function $u: B\rightarrow C$, the radial maximal function $\mathcal{M}u$ on S is defined by

$$\mathcal{M}u(\eta)=\sup\{|u(r\eta)|: 0\leqq r<1\}, \qquad \eta\in S.$$

For $\alpha>1$ and $\eta\in S$, we let

$$D_\alpha(\eta)=\left\{z\in B: |1-<z, \eta>|<\frac{\alpha}{2}(1-|z|^2)\right\}.$$

The admissible maximal function $\mathcal{M}_\alpha u$ on S is defined by

$$\mathcal{M}_\alpha u(\eta)=\sup\{|u(z)|: z\in D_\alpha(\eta)\}.$$

We prove the following theorem.

THEOREM I. *For $0<p<\infty$, there is a positive constant $C=C(n, p, \alpha)$ such that if $u\geqq 0$ is plurisubharmonic in B then*

$$\int_S \mathcal{M}_\alpha u(\eta)^p d\sigma(\eta)\leqq C\int_S \mathcal{M}u(\eta)^p d\sigma(\eta).$$

For $n=1$, the corresponding theorem for harmonic functions on the upper half plane appears in [3, Theorem 3.6].

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For an application of Theorem I, we consider the class $M^p(B)(0<p<\infty)$ of holomorphic functions f on B for which

$$\int_S (\log^+ \mathcal{M}f(\eta))^p d\sigma(\eta) < \infty .$$

For $n=1$, these classes as topological algebras have been studied in [7, 10] for $p>1$ and in [2, 5, 6] for $p=1$. For $n\geqq 1$, it is shown in [2] that

$$\bigcup_{p>0} H^p \subsetneqq \bigcap_{p>1} M^p \subsetneqq M^1 \subsetneqq N^+ ,$$

where H^p is the usual Hardy space and N^+ is the Smirnov class on B. The main theorem of [2] concerns with the boundary behavior of functions in the class $M^p(p\geqq 1)$, with its application to outer factors of functions in M^1 when $n=1$.

If we take $u=\log^+|f|$ with holomorphic functions f on B in Theorem I, we get the following characterization of M^p immediately.

THEOREM II. *A holomorphic function f on B belongs to M^p if and only if*

$$\int_S (\log^+ \mathcal{M}_\alpha f(\eta))^p d\sigma(\eta) < \infty .$$

Since every automorphism of B maps any radius into a curve which approaches the boundary nontangentially, the following corollary is immediate.

COROLLARY III. *The class $M^p(0<p<\infty)$ is invariant under the compositions of automorphisms of B.*

When $p>1$, this fact is not new because $M^p(p>1)$ can be defined by means of boundary functions. See [2, 7]. As a consequence of this corollary we can characterize all onto algebra endomorphisms of M^1 for the case $n=1$. For the case $p>1$, see [7].

THEOREM IV. *Let $n=1$. Then $\Gamma: M^1 \to M^1$ is an onto algebra endomorphism if and only if*

$$\Gamma(f)=f\circ\varphi , \qquad f\in M^1$$

for some automorphism φ of the unit disc U of C^1. In particular, Γ is invertible in this case and $\Gamma^{-1}(f)=f\circ\varphi^{-1}$, $f\in M^1$.

The proof will be given in the last section. The theorem might be true for $n>1$ but we do not have a proof.

2. An inequality of Hardy and Littlewood.

The following lemma is due to Hardy and Littlewood. It is stated in [3, 4] for $|u|$ with harmonic functions u but the proof is exactly the same for non-negative subharmonic functions.

2.1. LEMMA. *If $u \geqq 0$ is subharmonic on the disc $D(z_0, R)$ with center at z_0 and radius $R>0$ in the complex plane C and if $0<p<\infty$, then*

$$u(z_0) \leqq K\left(\frac{1}{\pi R^2}\iint_{D(z_0, R)} u(z)^p dx dy\right)^{1/p},$$

where $K=K(p)$ is a positive constant independent of u.

The next lemma will be a polydisc version of the above inequality. Its statement is suitably adapted for the proof of Theorem I.

Let $z=r\zeta \in B$ and $R>0$. Let $\zeta_2, \cdots, \zeta_n \in S$ be such that $\zeta, \zeta_2, \cdots, \zeta_n$ form an orthonomal basis for C^n. Define a polydisc $\Delta(z, R)$ with respect to the basis $\zeta, \zeta_2, \cdots, \zeta_n$ at z as follows:

$$\begin{aligned}\Delta(z, R) &\equiv \Delta(z, R; \zeta, \zeta_2, \cdots, \zeta_n)\\ &=\left\{w=z+\lambda\zeta+\sum_2^n \lambda_j\zeta_j : |\lambda|<R, |\lambda_j|<R^{1/2}, 2\leqq j\leqq n\right\}.\end{aligned}$$

2.2. LEMMA. *Let $\Delta=\Delta(z, R) \subset B$. If $u \geqq 0$ is plurisubharmonic in B and $0<p<\infty$, then*

$$u(z)^p \leqq K\frac{1}{m_n(\Delta)}\int_\Delta u(w)^p dm_n(w),$$

where $K=K(n, p)$ is a positive constant independent of u and dm_n is the Lebesgue measure on C^n.

PROOF. We define

$$v(\lambda, \lambda_2, \cdots, \lambda_n)=u(z+\lambda\zeta+\lambda_2\zeta_2+\cdots+\lambda_n\zeta_n).$$

Since u is plurisubharmonic in B, v is an n-subharmonic function for $|\lambda|<R$, $|\lambda_j|<R^{1/2}(2\leqq j\leqq n)$. We now apply Lemma 2.1 n times to v. The positive constants K's in the following are not the same in each occurence but are independent of v.

$$\begin{aligned}v(0, \cdots, 0)^p &\leqq K\frac{1}{R}\int_{|\lambda_n|<R^{1/2}} v(0, \cdots, 0, \lambda_n) dm_1(\lambda_n)\\ &\leqq \cdots\\ &\leqq K\frac{1}{R^{n-1}}\int\cdots\int_{|\lambda_j|<R^{1/2}(2\leqq j\leqq n)} v(0, \lambda_2, \cdots, \lambda_n)^p dm_1(\lambda_2)\cdots dm_1(\lambda_n)\end{aligned}$$

$$\leqq K\frac{1}{R^{n+1}}\int\cdots\int_{|\lambda_1|<R,\,|\lambda_j|<R^{1/2}(2\leqq j\leqq n)} v(\lambda_1,\cdots,\lambda_n)^p dm_1(\lambda_1)\cdots dm_1(\lambda_n).$$

Therefore, we have

$$u(z)^p\leqq K\frac{1}{m_n(\Delta)}\int_\Delta u(w)^p dm_n(w). \qquad \text{Q. E. D.}$$

3. Geometric lemmas.

3.1. LEMMA. *Let $z=r\zeta\in B$ and let $\Delta(z,\varepsilon(1-r^2))\subset B$ for a choice of $\zeta_2,\cdots,\zeta_n\in S$ and $\varepsilon>0$. If $r>1/2$ and $w\in\Delta(z,\varepsilon(1-r^2))$ then*

$$r-\delta(1-r^2)<|w|<r+\delta(1-r^2)$$

for some choice of a positive constant $\delta=\delta(n,\varepsilon)$ independent of z and ζ's.

PROOF. Suppose $w=z+\lambda\zeta+\sum_2^n\lambda_j\zeta_j\in\Delta(z;\varepsilon(1-r^2))$. Then

$$\begin{aligned}|w|^2&=|r+\lambda|^2+\sum_2^n|\lambda_j|^2\leqq r^2+|\lambda|^2+2|\lambda|+(n-1)\varepsilon(1-r^2)\\&\leqq r^2+(n+2)\varepsilon(1-r^2).\end{aligned}$$

Also,

$$\begin{aligned}|w|^2&\geqq(r-|\lambda|)^2=r^2-2r|\lambda|+|\lambda|^2\\&\geqq r^2-2|\lambda|\geqq r^2-2\varepsilon(1-r^2).\end{aligned}$$

If $r>1/2$ then

$$||w|-r|\leqq 2||w|^2-r^2|\leqq 2(n+2)\varepsilon(1-r^2).$$

So we can take $\delta=2(n+2)\varepsilon$. Q. E. D.

The following lemma appears in [1] but its proof is included for the sake of completeness.

3.2. LEMMA. *Let $\beta>\alpha>1$ and $z=r\zeta\in D_\alpha(\eta)$. Then there is a positive constant $\varepsilon=\varepsilon(n,\alpha,\beta)$ such that*

$$\Delta(z,\varepsilon(1-r^2))\subset D_\beta(\eta)$$

for any choice of $\zeta_2,\cdots,\zeta_n\in S$.

PROOF. Suppose $w=z+\lambda\zeta+\sum_2^n\lambda_j\zeta_j\in\Delta(z,\varepsilon(1-r^2))$. Then $|\lambda|<\varepsilon(1-r^2)$ and and $|\lambda_j|<\{\varepsilon(1-r^2)\}^{1/2}$. By the orthogonality of ζ and ζ_j, the Schwarz lemma and the hypothesis $z\in D_\alpha(\eta)$, we have

$$\begin{aligned}|\langle\zeta_j,\eta\rangle|&=|\langle\zeta_j,\eta-r\zeta\rangle|\\&\leqq|\eta-r\zeta|\end{aligned}$$

$$\leqq \sqrt{2}\,|1-\langle r\zeta,\ \eta\rangle|^{1/2}$$
$$\leqq \{\alpha(1-r^2)\}^{1/2}.$$

We compute

$$|1-\langle w,\ \eta\rangle| = \Big|1-\Big(\langle r\zeta,\ \eta\rangle+\lambda\langle\zeta,\ \eta\rangle+\sum_2^n \lambda_j\langle\zeta_j,\ \eta\rangle\Big)\Big|$$
$$\leqq \frac{\alpha}{2}(1-r^2)+\varepsilon(1-r^2)+\sum_2^n \{\varepsilon(1-r^2)\}^{1/2}|\langle\zeta_j,\ \eta\rangle|$$
$$\leqq \Big\{\frac{\alpha}{2}+\varepsilon+(n-1)\varepsilon^{1/2}\alpha^{1/2}\Big\}(1-r^2)$$

On the other hand, from the proof of Lemma 3.1, we have

$$1-|w|^2 \geqq \{1-(n+2)\varepsilon\}(1-r^2).$$

Therefore we can choose $\varepsilon=\varepsilon(n,\ \alpha,\ \beta)>0$ so small that

$$|1-\langle w,\ \eta\rangle| < \frac{\beta}{2}(1-|w|^2),$$

for any $w\in\Delta(z,\ \varepsilon(1-r^2))$. Therefore $\Delta(z,\ \varepsilon(1-r^2))\subset D_\beta(\eta)$. Q.E.D.

We define the radial projection π from $B\setminus\{0\}$ onto S as

$$\pi(w)=w/|w|, \qquad w\in B\setminus\{0\}.$$

For $\eta\in S$ and $\delta>0$,

$$Q(\eta,\ \delta)=\{\zeta\in S:\ |1-\langle\zeta,\ \eta\rangle|<\delta\}$$

is the nonisotropic "ball" of radius $\delta^{1/2}$ around η. The volume $\sigma(Q(\eta,\ \delta))$ is roughly propotional to δ^n, i.e., $\sigma(Q(\eta,\ \delta))\approx\delta^n$. See [9, Proposition 5.1.4].

33. LEMMA. *Let $z=r\zeta\in D_\alpha(\eta)$, $r>0$ and $\beta>\alpha>1$. Then there is a positive constant $\varepsilon=\varepsilon(n,\ \alpha,\ \beta)$ so small that*

$$\pi(\Delta(z,\ \varepsilon(1-r^2))\subset Q\Big(\eta,\ \Big(\frac{\beta}{2}+1\Big)(1-r^2)\Big)$$

for any choice of $\zeta_2,\ \cdots,\ \zeta_n$.

PROOF. Chooce β' so that $\beta>\beta'>\alpha$. Let $w=\rho w\in\Delta(z,\ \varepsilon(1-r^2))$. Then

$$|1-\langle\omega,\ \eta\rangle| = |1-\langle\rho\omega,\ \eta\rangle-(1-\rho)\langle\omega,\ \eta\rangle|$$
$$\leqq |1-\langle\omega,\ \eta\rangle|+(1-\rho^2).$$

By Lemma 3.2, we can choose $\varepsilon>0$ so small that

$$|1-\langle w,\ \eta\rangle| < \frac{\beta'}{2}(1-r^2).$$

From the proof of Lemma 3.1, we have

$$1-\rho^2 \leqq (1+2\varepsilon)(1-r^2).$$

Therefore we have

$$|1-\langle\omega, \eta\rangle| < \left(\frac{\beta'}{2}+1+2\varepsilon\right)(1-r^2).$$

If we choose $\varepsilon=\varepsilon(n, \alpha, \beta)>0$ even smaller so that $\beta'/2+1+2\varepsilon<\beta/2+1$, we have

$$|1-\langle\omega, \eta\rangle| < \left(\frac{\beta}{2}+1\right)(1-r^2);$$

so that $\omega \in Q(\eta, (\beta/2+1)(1-r^2))$. Q.E.D.

4. Proof of Theorem I.

It suffices to prove the theorem for a modified admissible maximal function (with the same notation) as

$$\mathcal{M}_\alpha u(\eta)=\sup\left\{|u(z)| : |z| \geqq \frac{1}{2}, z \in D_\alpha(\eta)\right\}.$$

Let $z=r\zeta \in D_\alpha(\eta)$, $r \geqq 1/2$ and $\beta>\alpha$. By Lemmas 3.1, 3.2 and 3.3, we can choose positive constants $\varepsilon=\varepsilon(n, \alpha, \beta)$ and $\delta=\delta(n, \varepsilon)=\delta(n, \alpha, \beta)$ so that

(i) $\Delta=\Delta(z, \varepsilon(1-r^2)) \subset D_\beta(\eta)$ for a choice of $\zeta_2, \cdots, \zeta_n$,

(ii) $\pi(\Delta) \subset Q(\eta, (\beta/2+1)(1-r^2))$,

(iii) $r-\delta(1-r^2)<|w|<r+\delta(1-r^2)$ if $w \in \Delta$.

Using Lemma 2.2, we have the following computation in which the constants $K=K(n, p, \delta)$ are not the same in each occurrence, but are independent of u.

$$\begin{aligned}
u(z)^{p/2} &\leqq K\frac{1}{(1-r^2)^{n+1}}\int_\Delta u(w)^{p/2}dm_n(w) \\
&\leqq K\frac{1}{(1-r^2)^{n+1}}\int_{r-\delta(1-r^2)}^{r+\delta(1-r^2)}\rho^{2n-1}d\rho\int_{Q(\eta,(\beta/2+1)(1-r^2))}\mathcal{M}u(\omega)^{p/2}d\sigma(\omega) \\
&\leqq K\frac{1}{(1-r^2)^n}\int_Q \mathcal{M}u(\omega)^{p/2}d\sigma(\omega) \\
&\leqq K\frac{1}{\sigma(Q)}\int_Q \mathcal{M}u(\omega)^{p/2}d\sigma(\omega) \\
&\leqq KM\{(\mathcal{M}u)^{p/2}\}(\eta),
\end{aligned}$$

where M is the Hardy-Littlewood maximal function operator on S. Therefore we have

$$\{\mathcal{M}_\alpha u(\eta)\}^{p/2} \leqq KM\{(\mathcal{M}u)^{p/2}\}(\eta).$$

We note that the constan K is eventally dependent on n, p, α from the choice of β and δ. By the Hardy-Littlewood maximal theorem [9, Theorem 5.2.6], we have

$$\int_S \mathcal{M}_\alpha u(\eta)^p d\sigma(\eta) \leqq C\int_S \mathcal{M}u(\eta)^p d\sigma(\eta).$$

for some positive constant $C=C(n, p, \alpha)$ indpendent of u. Q.E.D.

5. Proof of Theorem IV.

By corollary III, every automorphism φ of U defines an algebra isomorphism $\Gamma(f)=f\circ\varphi$, $\varphi\in M^1$. Conversely, let Γ be any onto endomorphism of M^1. We will follow the corresponding proof for the case N^+ [8]. Let $\varphi=\Gamma(z)$ and let $\lambda=U$. (z denotes the identity function on U.) Define $\gamma(f)=\Gamma(f)(\lambda)$, $f\in M^1$. Since γ is a multiplicative linear functional on M^1, γ corresponds to the point evaluation at some $\beta\in U$ by Theorem 6.4 of [6]. Thus $\beta=\gamma(z)=\Gamma(z)(\lambda)=\varphi(\lambda)$. Hence $\varphi(\lambda)\in U$ for all $\lambda\in U$ and $\Gamma(f)(\lambda)=f(\varphi(\lambda))$, $f\in M^1$, $\lambda\in U$. Since Γ is onto, φ is not constant. Thus $\varphi(U)$ is open in U. Therefore Γ is one-to-one (and onto). Thus Γ^{-1} is also an onto endomorphism, so $\Gamma^{-1}(f)=f\circ\psi$, $f\in M^1$, for some holomorphic self-map ψ of U. But then $z=\Gamma\Gamma^{-1}(z)=\Gamma(\psi)=\psi\circ\varphi$ and $\varphi\circ\psi=z$. Therefore φ is an automorphism of U. Q.E.D.

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