

A NOTE ON Ω -STABILITY

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

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By Mañé [2] and Palis [3] it was proved that Ω -stability of a C^1 -diffeomorphism of a compact boundaryless manifold implies Axiom A and no cycle conditions. The converse was already proved by Smale [6]. Our aim is to give a simple proof for the following Smale's theorem.

THEOREM. *If a C^1 -diffeomorphism of a compact boundaryless manifold satisfies Axiom A and no cycle conditions, then it is Ω -stable.*

Axiom A is defined by the following (1) the non-wandering set $\Omega(f)$ is hyperbolic (i.e. there exist a Riemannian metric $\langle \cdot, \cdot \rangle$ and a continuous splitting $T_{\Omega(f)}M = E^s \oplus E^u$ such that $Df(E^\sigma) = E^\sigma$, $\sigma = s, u$ and $\|Df|_{E^s}\| \leq \lambda$ and $\|Df^{-1}|_{E^u}\| \leq \lambda$ for some $0 < \lambda < 1$) and (2) the set of all periodic points, $\text{Per}(f)$, is dense in $\Omega(f)$. By Axiom A condition $\Omega(f)$ splits into the finite disjoint union $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_k$ of closed subsets Ω_i , so-called basic sets, such that for $1 \leq i \leq k$ $f(\Omega_i) = \Omega_i$, $\text{Per}(f|_{\Omega_i})$ is dense in Ω_i and $f|_{\Omega_i}$ is topologically transitive. The cycle condition is defined as follows: there exist $n \geq 1$ and $\{\Omega_{i_j}\}_{j=0}^n \subset \{\Omega_i\}$ such that $\Omega_{i_j} \neq \Omega_{i_l}$ ($0 \leq j \neq l \leq n$), $\Omega_{i_{n+1}} = \Omega_{i_0}$ and $W^s(\Omega_{i_j}) \cap W^u(\Omega_{i_{j+1}}) \neq \emptyset$ ($0 \leq j \leq n$). Here $W^s(\Omega_i)$ is denoted by $W^s(\Omega_i) = \{y \in M : d(f^n(y), \Omega_i) \rightarrow 0 \text{ as } n \rightarrow \infty\}$ and also $W^u(\Omega_i) = \{y \in M : d(f^{-n}(y), \Omega_i) \rightarrow 0 \text{ as } n \rightarrow \infty\}$ where d is the metric on M induced by the Riemannian metric. For the space $\text{Diff}^1(M)$ of C^1 -diffeomorphisms with the uniform C^1 -topology, $f \in \text{Diff}^1(M)$ is called Ω -stable if there is a neighborhood $\mathcal{U}(f)$ in $\text{Diff}^1(M)$ such that every $g \in \mathcal{U}(f)$ is Ω -conjugate to f .

If compact f -invariant set A is hyperbolic, then there exists $\varepsilon_0 > 0$ such that $W_\varepsilon^q(x)$ is a C^1 -submanifold, $T_x W_\varepsilon^q(x) = E^q(x)$ and $W_\varepsilon^q(x)$ varies continuously with x ($x \in A$, $0 < \varepsilon \leq \varepsilon_0$ and $\sigma = s, u$), and such that there exists $0 < \lambda_0 < 1$ such that $y \in W_{\varepsilon_0}^s(x)$ ($x \in A$) implies $d(f(x), f(y)) \leq \lambda_0 d(x, y)$ and $y \in W_{\varepsilon_0}^u(x)$ ($x \in A$) implies $d(f^{-1}(x), f^{-1}(y)) \leq \lambda_0 d(x, y)$. Here $W_\varepsilon^s(x)$ is denoted by $W_\varepsilon^s(x) = \{y \in M : d(f^j(x), f^j(y)) \leq \varepsilon, j \geq 0\}$ and also $W_\varepsilon^u(x) = \{x \in M : d(f^{-j}(x), f^{-j}(y)) \leq \varepsilon, j \geq 0\}$.

Using Axiom A condition, we can check that there exist neighborhoods V of $\Omega(f)$, $\mathcal{CV}(f)$ of f and $c > 0$ such that for $g \in \mathcal{CV}(f)$ and $x, y \in \bigcap_{i \in \mathbb{Z}} g^i(V)$ if $d(g^i(x), g^i(y)) \leq c$ for $i \in \mathbb{Z}$ then $x = y$ (this remark will be discussed in the last part). For $\varepsilon > 0$ small enough there exists $0 < \gamma < \varepsilon$ such that $d(x, y) < \gamma$ ($x, y \in \Omega(f)$) implies that $W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ is one point set and its intersection is transversal. By λ -lemma we have $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \in \Omega(f)$ which ensures that there exist a neighborhood $U \subset V$ and $\delta > 0$ satisfying the following: for every δ -pseudo orbit $\{x_i\} \subset U$ (i.e. $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$) there is a unique point $x \in M$ such that $d(f^i(x), x_i) < \varepsilon$ ($i \in \mathbb{Z}$). In fact notice that $x \in \Omega(f)$. Therefore $\bigcap_{i \in \mathbb{Z}} f^i(U) = \Omega(f)$ holds and f has a filtration for $\Omega(f)$ (i.e. a sequence $\phi = M_0 \subset M_1 \subset \dots \subset M_k = M$ of smooth, compact codimension 0 submanifolds with boundary of M such that $f(M_i) \subset \text{int } M_i$ and $\bigcap_{n \in \mathbb{Z}} f^n(M_i - M_{i-1}) = \Omega_i$) (see [6]). Thus $\Omega(f)$ coincides with the chain recurrent set $R(f)$ (which means the set of all points x such that for $\alpha > 0$ there is a finite α -pseudo orbit $\{x_0, x_1, \dots, x_n\}$ with $x_0 = x_n = x$).

Choose a neighborhood $\mathcal{U}(f) \subset \mathcal{CV}(f)$ of f in $\text{Diff}^1(M)$ such that $\max_{x \in M} d(f(x), g(x)) < \delta$ and $R(g) \subset U$ for every $g \in \mathcal{U}(f)$. Then we have $R(g) \subset \bigcap_{i \in \mathbb{Z}} g^i(U)$ because $R(g)$ is g -invariant set. Let $x \in \bigcap_{i \in \mathbb{Z}} g^i(U)$, then $\{g^i(x)\}$ is a δ -pseudo orbit of f . Thus there is a unique point $h(x) \in R(f) = \Omega(f)$ such that $d(f^i(h(x)), g^i(x)) < \varepsilon$ for $i \in \mathbb{Z}$, and then $h: \bigcap_{i \in \mathbb{Z}} g^i(U) \rightarrow R(f)$ is injective and $h \circ g = f \circ h$ holds. It is not difficult to see that h is continuous. By Hartman's theorem we have $\#\{x \in M: f^m(x) = x\} = \#\{x \in M: g^m(x) = x\} < \infty$ for $m > 0$ and $g \in \mathcal{U}(f)$. Then $R(f) = h(\bigcap_{i \in \mathbb{Z}} g^i(U))$. Since h is a homeomorphism and $\text{Per}(f)$ is dense in $\Omega(f)$, we have $\Omega(g) = R(g) = \bigcap_{i \in \mathbb{Z}} g^i(U)$.

Finally we explain the above remark. Since f satisfies Axiom A, there is $m > 0$ such that for $x \in \Omega(f)$ and $v \in T_x M$, $\|Df^n(v)\| \geq 6\|v\|$ for $n = m$ or $n = -m$. And there exists an extended continuous splitting $T_v M = \tilde{E}^s \oplus \tilde{E}^u$. Then we can find a neighborhood $\mathcal{CV}(f)$ such that for $g \in \mathcal{CV}(f)$ and $x \in \bigcap_{i \in \mathbb{Z}} g^i(V)$

$$D_x g^m = \begin{pmatrix} A^{ss} & A^{us} \\ A^{su} & A^{uu} \end{pmatrix}: \tilde{E}^s(x) \oplus \tilde{E}^u(x) \longrightarrow \tilde{E}^s(g^m(x)) \oplus \tilde{E}^u(g^m(x))$$

$$D_x g^{-m} = \begin{pmatrix} B^{ss} & B^{us} \\ B^{su} & B^{uu} \end{pmatrix}: \tilde{E}^s(x) \oplus \tilde{E}^u(x) \longrightarrow \tilde{E}^s(g^{-m}(x)) \oplus \tilde{E}^u(g^{-m}(x))$$

$$\|A^{ss}\| \leq 1/5, \quad |A^{uu}| \geq 5, \quad \|B^{uu}\| \leq 1/5, \quad |B^{ss}| \geq 5$$

$$\max\{\|A^{us}\|, \|A^{su}\|, \|B^{us}\|, \|B^{su}\|\} \leq 1/5$$

where $|E| = \|E^{-1}\|^{-1}$. Since $g^i(x) \in V$ for $|i| \leq m$, when $\|v^u\| \geq \|v^s\|$ for $v = v^s + v^u \in \tilde{E}^s(x) \oplus \tilde{E}^u(x)$ we have $\|v\| \leq 2\|v^u\|$ and then

$$\begin{aligned} \|Dg^m(v)\| &= \|A^{ss}(v^s) + A^{su}(v^s) + A^{us}(v^u) + A^{uu}(v^u)\| \\ &\geq 5\|v^u\| - (2\|v^s\|/5 + \|v^u\|/5) \\ &\geq 4\|v^u\| \geq 2\|v\|. \end{aligned}$$

For the case when $\|v^s\| \geq \|v^u\|$ we have also $\|Dg^{-m}(v)\| \geq 2\|v\|$.

Take and fix $\varepsilon > 0$ such that $\varepsilon(1 + K + \dots + K^{m-1}) < 1/2$ where $K = \sup\{\|D_x g\| : x \in M, g \in \mathcal{CV}(f)\}$. Then there is $c > 0$ such that for $g \in \mathcal{CV}(f)$

$$\|\exp_{g^\sigma(x)}^{-1} \circ g^\sigma \circ \exp_x(v) - D_x g^\sigma(v)\| \leq \varepsilon \|v\| \quad (x \in M)$$

whenever $\|v\| \leq c$ ($\sigma = 1, -1$). Let $g \in \mathcal{CV}(f)$ and $x, y \in \bigcap_{i \in \mathbb{Z}} g^i(V)$. Then we assume $d(g^i(x), g^i(y)) \leq c$ ($i \in \mathbb{Z}$) does not ensure $x = y$. Take δ with $0 < \delta \leq c_1/4$ where $c_1 = \sup\{d(g^i(x), g^i(y)) : i \in \mathbb{Z}\}$. Obviously $c_1 - \delta < d(g^k(x), g^k(y)) \leq c_1$ for some $k \in \mathbb{Z}$. Put $u = g^k(x)$, $v = g^k(y)$ and $w = \exp_u^{-1}(v)$. Then $c_1 - \delta < \|w\| = d(u, v)$ and $\|Dg^n(w)\| \geq 2\|w\|$ for some $|n| = m$. We deal with only the case $\|Dg^m(w)\| \geq 2\|w\|$ (because the case $\|Dg^{-m}(w)\| \geq 2\|w\|$ follows from a similar way). Since $\|w\| = d(u, v) \leq c$ we have $\|\exp_{g^2(u)}^{-1} \circ g^2 \circ \exp_u(w) - D_u g(w)\| \leq \varepsilon \|w\|$, from which $\|D_u g(w)\| \leq c_1(1 + \varepsilon)$, and $\|\exp_{g^2(u)}^{-1} \circ g^2 \circ \exp_u(w) - D_u g^2(w)\| \leq c_1\varepsilon + Kc_1\varepsilon = c_1\varepsilon(1 + K)$. Since $\|\exp_{g^2(u)}^{-1} \circ g^2 \circ \exp_u(w)\| = d(g^2(u), g^2(v)) \leq c_1$, we have $\|D_u g^2(w)\| \leq c_1(1 + \varepsilon(1 + K))$ and by induction we have $2\|w\| \leq \|D_u g^m(w)\| \leq c_1(1 + \varepsilon(1 + K + \dots + K^{m-1}))$. Consequently $c_1 - \delta < \|w\| \leq 3c_1/4$, i.e. $c_1/4 < \delta$ which is a contradiction.

REMARK. By the same method of the above proof it is easily proved that Anosov diffeomorphisms are structurally stable. In fact, an Anosov diffeomorphism f has the pseudo orbit tracing property and every diffeomorphism in some neighborhood of f has the same expansive constant.

References

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