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ON THE CAUCHY PROBLEM FOR ANALYTIC SEMIGROUPS WITH WEAK SINGULALITY

By

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I. Introduction and Results

Let X be a Banach space with norm $\|\cdot\|$ and $\mathfrak{A}$ a linear operator defined in X. We consider the following initial-value problem: Given an element $u_0 \in X$ and an X-vauled function f defined on an interval $I=[0, T]$, find an X-valued function u defined on I such that

$$(*)\qquad \begin{cases} \dfrac{du}{dt}(t)=\mathfrak{A}u(t)+f(t), & 0<t\leqq T, \\ u(0)=u_0. \end{cases}$$

In this paper, under the condition that the operator $\mathfrak{A}$ generates an analytic semigroup *with weak singularity*, we give sufficient conditions on the function f for the existence and uniqueness of solutions of the problem $(*)$.

We say that a function $u(t)$ is a *strict solution* or simply a *solution* of the problem $(*)$ if it satisfies the following three conditions:

$$(1.1)\qquad u\in C([0, T]; X)\cap C^1((0, T]; X).$$

(1.2) $u(t)$ is in the domain $\mathcal{D}(\mathfrak{A})$ of the operator $\mathfrak{A}$ for $0<t\leqq T$.

$$(1.3)\qquad u(0)=u_0 \quad\text{and}\quad \frac{du}{dt}(t)=\mathfrak{A}u(t)+f(t), \qquad 0<t\leqq T.$$

Here $C([0, T]; X)$ denotes the space of continuous functions on $[0, T]$ taking values in X, and $C^1((0, T]; X)$ denotes the space of continuously differentiable functions on $(0, T]$ taking values in X, respectively.

We recall the following fundamental result in the theory of analytic semigroups (cf. Pazy [2]; Tanabe [4]):

THEOREM 1.0. *Assume that the following three assumptions are satisfied*:

(A.1) *The operator $\mathfrak{A}$ is a densely defined, closed linear operator in X.*

(A.2) *There exist constants $0<\omega<\pi/2$ and $\lambda_0<0$ such that the resolvent set of $\mathfrak{A}$ contains the region $\Sigma(\omega)=\{\lambda\in \boldsymbol{C};\ |\arg(\lambda-\lambda_0)|<\pi/2+\omega\}$.*

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(A.3) *If* $0<\varepsilon<\omega$, *then there exists a constant* $C(\varepsilon)>0$ *such that the resolvent* $(\mathfrak{A}-\lambda)^{-1}$ *satisfies the estimate*:

$$\|(\mathfrak{A}-\lambda)^{-1}\|\leqq\frac{C(\varepsilon)}{1+|\lambda|}, \qquad \lambda\in\Sigma(\varepsilon).$$

Then the operator $\mathfrak{A}$ *generates a semigroup* $e^{z\mathfrak{A}}$ *in* X *which is* analytic *in the sector* $\Delta(\omega)=\{z=t+is\in\boldsymbol{C};\ z\neq 0,\ |\arg z|<\omega\}$.

If $0<\gamma<1$, we let

$C^{\gamma}([0, T]; X)$=the space of X-valued, continuous functions $f(t)$ on $[0, T]$ such that we have $\|f(t)-f(s)\|\leqq M|t-s|^{\gamma}$, $t, s\in[0, T]$ for some constant $M>0$.

Now it is known (cf. Pazy [2], Theorem 3.2) that the following theorem holds.

THEOREM 1.1. *Assume that the operator* $\mathfrak{A}$ *satisfies Assumptions* (A.1), (A.2) *and* (A.3). *If* $f\in C^{\gamma}([0, T]; X)$ *with* $0<\gamma\leqq 1$, *then, for any* $u_0\in X$, *the problem* (*) *has a unique solution which takes the following form*:

$$u(t)=e^{t\mathfrak{A}}u_0+\int_0^t e^{(t-s)\mathfrak{A}}f(s)ds. \tag{1.4}$$

The next Besov space version of Theorem 1.1 is due to Muramatu [1] (see [1], Theorem B).

THEOREM 1.2. *Assume that the operator* $\mathfrak{A}$ *satisfies Assumptions* (A.1), (A.2) *and* (A.3). *If* f *belongs to the Besov space* $B^0_{\infty,1}((0, T); X)$, *then, for any* $u_0\in X$, *the problem* (*) *has a unique solution which takes the form of* (1.4).

REMARK 1.1. Theorem 1.2 is a generalization of Theorem 1.1. In fact, the following inclusion holds:

$$\bigcup_{0<\gamma\leqq 1} C^{\gamma}([0, T]; X)\subsetneqq B^0_{\infty,1}((0, T); X).$$

EXAMPLE 1.1. The following function f belongs to the space $B^0_{\infty,1}((0, T); \boldsymbol{R})$, but does not belong to the spaces $C^{\gamma}([0, T]; \boldsymbol{R})$ for any $0<\gamma\leqq 1$.

$$f(t)=\begin{cases}\dfrac{1}{\log t} & \text{if } 0<t\leqq T,\\ 0 & \text{if } t=0.\end{cases}$$

For the precise definition of the Besov space $B^0_{\infty,1}((0, T); X)$, we refer to Section 2.

We say that the operator $\mathfrak{A}$ satisfies Assumption $(\mathrm{AS})_\theta$ with $0<\theta<1$ if it satisfies Assumptions (A.1) and (A.2) and the following weaker assumption than (A.3):

$(\mathrm{A.3})_\theta$ If $0<\varepsilon<\omega$, then there exists a constant $C(\varepsilon)>0$ such that the resolvent $(\mathfrak{A}-\lambda)^{-1}$ satisfies the estimate:

$$\|(\mathfrak{A}-\lambda)^{-1}\| \leqq \frac{C(\varepsilon)}{(1+|\lambda|)^\theta}, \qquad \lambda \in \Sigma(\varepsilon).$$

By Theorem 5.3 of Taira [3], we know that the operator $\mathfrak{A}$ which satisfies Assumption $(\mathrm{AS})_\theta$ with $0<\theta<1$ generates an analytic semigroup $e^{z\mathfrak{A}}$ such that

$$\|e^{z\mathfrak{A}}\| \leqq \frac{M_0}{|z|^{1-\theta}}, \qquad z \in \Delta(\omega).$$

Thus, such an analytic semigroup as $e^{z\mathfrak{A}}$ may be called an *analytic semigroup with weak singularity*. We remark that Assumption $(\mathrm{A.3})_1$ is nothing but Assumption (A.3).

A concrete example of $\mathfrak{A}$ which satisfies Assumption $(\mathrm{AS})_\theta$ is given by Taira [3]. Furthermore, Taira [3] has demonstrated that the operator $\mathfrak{A}$ generates an analytic semigroup $e^{t\mathfrak{A}}$ which does not necessarily have the following property:

$$\lim_{\substack{|t|\to 0\\ t\in\Delta(\omega)}} e^{t\mathfrak{A}}u_0 = u_0 \qquad \text{for } \textit{all} \quad u_0 \in X.$$

Here $\Delta(\omega)=\{\lambda\in \boldsymbol{C}\,;\ |\arg\lambda|<\omega\}$. More precisely, using fractional powers of the operator $\mathfrak{A}$, Taira [3] has proved that if Assumption $(\mathrm{AS})_\theta$ is satisfied, then the operator $\mathfrak{A}$ generates an analytic semigroup $e^{t\mathfrak{A}}$ which has the property

$$\lim_{\substack{|t|\to 0\\ t\in\Delta(\omega)}} e^{t\mathfrak{A}}u_0 = u_0$$

for all $u_0 \in \mathcal{D}((-\mathfrak{A})^\alpha)$ with $1-\theta<\alpha<1$. Here if the operator $\mathfrak{A}$ satisfies Assumptions (A.1), (A.2) and $(\mathrm{A.3})_\theta$, we can define the fractional powers $(-\mathfrak{A})^{-\alpha}$ of $\mathfrak{A}$ for $1-\theta<\alpha<1$ by

$$(-\mathfrak{A})^{-\alpha} = \frac{\sin \alpha\pi}{\pi}\int_0^\infty t^{-\alpha}(t-\mathfrak{A})^{-1}dt,$$

and also define the fractional powers $(-\mathfrak{A})^\alpha$ by

$$(-\mathfrak{A})^\alpha = \text{the inverse of } (-\mathfrak{A})^{-\alpha}.$$

By the definition of $(-\mathfrak{A})^\alpha$, we have the following:

$$\mathcal{D}(\mathfrak{A}) \subset \mathcal{D}((-\mathfrak{A})^\alpha) \subset X, \qquad 1-\theta<\alpha<\theta,$$

$$\mathcal{D}((-\mathfrak{A})^0) = X.$$

The following theorem is due to Taira [3] (cf. [3], Theorem 8.2). In the case $\theta=1$, the theorem coincides with Theorem 1.1.

THEOREM 1.3. *Assume that the operator* $\mathfrak{A}$ *satisfies Assumption* $(AS)_\theta$ *with* $1/2<\theta<1$. *If* $f\in C^\gamma([0, T]; X)$ *with* $1-\theta<\gamma\leqq 1$, *then, for any* $u_0\in\mathcal{D}((-\mathfrak{A})^\alpha)$ *with* $1-\theta<\alpha<\theta$, *the problem* (*) *has a unique solution which takes the form of* (1.4).

In this paper, using Besov space theory, we prove the following result:

THEOREM 1.4. *Assume that the operator* $\mathfrak{A}$ *satisfies Assumption* $(AS)_\theta$ *with* $1/2<\theta<1$. *If* f *belongs to the Besov space* $B^{1-\theta}_{\infty,1}((0, T); X)$, *then, for any* $u_0\in\mathcal{D}((-\mathfrak{A})^\alpha)$ *with* $1-\theta<\alpha<\theta$, *the problem* (*) *has a unique solution which takes the form of* (1.4).

REMARK 1.2. Theorem 1.4 is a generalization of Theorem 1.3 and Theorem 1.2. In fact, the following inclusion holds (cf. Corollary 2.1 and Remark 2.2):

$$\bigcup_{1-\theta<\gamma\leq 1} C^\gamma([0, T]; X)\subsetneqq B^{1-\theta}_{\infty,1}((0, T); X).$$

EXAMPLE 1.2. The following function f belongs to the space $B^{1-\theta}_{\infty,1}((0, T); \boldsymbol{R})$, but does not belong to the spaces $C^\gamma([0, T]; \boldsymbol{R})$ for any $1-\theta<\gamma\leqq 1$.

$$f(t)=\begin{cases} \dfrac{t^{1-\theta}}{\log t} & \text{if } 0<t\leqq T, \\ 0 & \text{if } t=0. \end{cases}$$

The rest of this paper is organized as follows:

In Section 2 we state the basic definition and properties of Besov spaces that will be used in the sequel.

In Section 3 we present a brief description of the analytic semigroups with weak singularity generated by the operator $\mathfrak{A}$ which satisfies Assumption $(\mathrm{AS})_\theta$ with $0<\theta<1$.

Section 4 is devoted to the proof of our main Theorem 1.4 by following the argument in the proof of Theorem B of Muramatu [1].

2. Besov spaces

This section is devoted to a description of the definition and properties of Besov spaces (for the details, see Muramatu [1]). We define Besov spaces on an open set Ω in $\boldsymbol{R}^N$, but, in this paper, only use the case when Ω is an open *interval* $I(N=1)$.

Let Ω be an open set in $\boldsymbol{R}^N$, X a Banach space with norm $\|\cdot\|$, $1\leqq p\leqq\infty$ and m a non-negative integer. For an X-valued function f on Ω, we define

$$\|f\|_{L^p(\Omega;\, X)}=\begin{cases}\left(\int_\Omega\|f(x)\|^p dx\right)^{1/p} & \text{if } 1\leqq p<\infty,\\ \operatorname*{ess\,sup}_{x\in\Omega}\|f(x)\| & \text{if } p=\infty,\end{cases}$$

$$\|f\|_{L^p_*(\Omega;\, X)}=\begin{cases}\left(\int_\Omega\|f(x)\|^p|x|^{-N} dx\right)^{1/p} & \text{if } 1\leqq p<\infty,\\ \operatorname*{ess\,sup}_{x\in\Omega}\|f(x)\| & \text{if } p=\infty,\end{cases}$$

$$\|f\|_{H^{m,p}(\Omega;\, X)}=\sum_{|\alpha|\leqq m}\|\partial^\alpha f\|_{L^p(\Omega;\, X)}.$$

Here all the derivatives $\partial^\alpha f$ are taken in the sense of distributions. If $X=\boldsymbol{R}$, we simply write $\|\cdot\|_{L^p(\Omega;\, X)}$, $\|\cdot\|_{L^p_*(\Omega;\, X)}$ and $\|\cdot\|_{H^{m,p}(\Omega;\, X)}$ as $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{L^p_*(\Omega)}$ and $\|\cdot\|_{H^{m,p}(\Omega)}$ respectively.

We introduce function spaces as follows:

$L^p(\Omega;\, X)$=the space of X-valued functions such that $\|f\|_{L^p(\Omega;\, X)}$ is finite.
$L^p_*(\Omega;\, X)$=the space of X-valued functions such that $\|f\|_{L^p_*(\Omega;\, X)}$ is finite.
$H^{m,p}(\Omega;\, X)$=the space of functions $f\in L^p(\Omega;\, X)$ whose derivatives $\partial^\alpha f$, $|\alpha|\leqq m$, in the sense of distributions, belong to $L^p(\Omega;\, X)$.

The spaces $L^p(\Omega;\, X)$ and $H^{m,p}(\Omega;\, X)$ are Banach spaces with the norms $\|\cdot\|_{L^p(\Omega;\, X)}$ and $\|\cdot\|_{H^{m,p}(\Omega;\, X)}$, respectively.

Definition of Besov spaces. Let X be a Banach space with norm $\|\cdot\|$, Ω an open set in $\boldsymbol{R}^N$, $1\leqq p,\, q\leqq\infty$ and σ a real number such that $\sigma=m+\theta$ with an integer m and $0<\theta\leqq 1$.

(a) The case $m\geqq 0$ and $0<\theta<1$: The Besov space $B^\sigma_{p,q}(\Omega;\, X)$ is the set of all functions $f\in H^{m,p}(\Omega;\, X)$ such that the seminorm

$$\begin{aligned}|f|_{B^\sigma_{p,q}(\Omega;\, X)}&=\sum_{|\alpha|=m}\||y|^{-\theta}\|\partial^\alpha f(x+y)-\partial^\alpha f(x)\|_{L^p(\Omega_{1,y};\, X)}\|_{L^q_*(\boldsymbol{R}^N)}\\ &=\sum_{|\alpha|=m}\left(\int_{\boldsymbol{R}^N}\left(\int_{\Omega_{1,y}}\|\partial^\alpha f(x+y)-\partial^\alpha f(x)\|^p dx\right)^{q/p}\frac{dy}{|y|^{q\theta+N}}\right)^{1/q}\end{aligned}$$

is finite. Here $\Omega_{k,y}=\bigcap_{j=0}^{k}\Omega-jy$ and $\Omega-jy=\{z-jy;\ z\in\Omega\}$.

(b) The case $m\geqq 0$ and $\theta=1$: The Besov space $B^\sigma_{p,q}(\Omega;\, X)$ consists of all functions $f\in H^{m,p}(\Omega;\, X)$ such that the seminorm

$$\begin{aligned}&|f|_{B^\sigma_{p,q}(\Omega;\, X)}\\ &=\sum_{|\alpha|=m}\||y|^{-1}\|\partial^\alpha f(x+2y)-2\partial^\alpha f(x+y)+\partial^\alpha f(x)\|_{L^p(\Omega_{2,y};\, X)}\|_{L^q_*(\boldsymbol{R}^N)}\end{aligned}$$

is finite.

The space $B^{\sigma}_{p,q}(\Omega; X)$ is a Banach space with the norm

$$\|f\|_{B^{\sigma}_{p,q}(\Omega; X)}=\|f\|_{H^{m,p}(\Omega; X)}+|f|_{B^{\sigma}_{p,q}(\Omega; X)}.$$

(c) The case $m<0$: The Besov space $B^{\sigma}_{p,q}(\Omega; X)$ is the set of all distributions f of the form

$$f=\sum_{|\alpha|\leq -m}\partial^{\alpha}f_{\alpha}, \qquad f_{\alpha}\in B^{\theta}_{p,q}(\Omega; X). \tag{2.1}$$

The space $B^{\sigma}_{p,q}(\Omega; X)$ is a Banach space with the norm

$$\|f\|_{B^{\sigma}_{p,q}(\Omega; X)}=\inf\sum_{|\alpha|\leq -m}\|f_{\alpha}\|_{B^{\theta}_{p,q}(\Omega; X)},$$

where the infimum is taken over all expressions of the form (2.1).

In the rest of this section we describe a characterization theorem of Besov spaces. In the following we denote the interval $(0, T)$ by I.

We introduce two function spaces.

(i) $\mathcal{K}_0(I)$ is the set of all functions $\phi\in C^{\infty}(\boldsymbol{R}^2)$ which satisfy the following conditions:

(2.2) For any $t\in\boldsymbol{R}$, there exists a compact set K_t in $\boldsymbol{R}$ such that K_t contains the support of $\phi(t, \cdot)$.

(2.3) For any compact set K in I, there is a compact set $K_1\subset I$ such that $\operatorname{supp}\phi(t, (t-\cdot)/\tau)\subset K_1$ for $t\in K$ and $0<\tau\leqq 1$.

(ii) $\mathcal{K}_m(I)$ is the set of m-th derivatives $\partial_s^m\phi(t, s)$ of the functions in $\mathcal{K}_0(I)$.

Let ϕ_0 be a function in $C_0^{\infty}(\boldsymbol{R})$ which satisfies the conditions:

$$\operatorname{supp}\phi_0\subset I, \int_{\boldsymbol{R}}\phi_0(t)dt=1.$$

If $0<c\leqq 1$, we define ϕ, e_m, e_m^* as follows:

$$\phi(t, s)=\frac{m}{m!}s^m\phi_0(t-s), \tag{2.4}$$

$$e_m(t, s)=\sum_{k=0}^{m-1}\partial_s^k\left\{\frac{1}{k!}s^k\phi_0(t-s)\right\}, \qquad m=1, 2, \cdots,$$

$$e_m^*(t, s)=2e_m(t, s)-\int e_m(t, r)e_m(t-cr, s-r)dr, \qquad m=1, 2, \cdots. \tag{2.5}$$

Then we have the following results:

LEMMA 2.1. *The functions ϕ, e_m and e_m^* introduced above belong to the space $\mathcal{K}_0(I)$. Further ϕ, e_m and e_m^* belong to the space $\mathcal{K}_0(J)$ for any open interval $J\supset I$.*

LEMMA 2.2 (*Integral representation of distributions*). *Let* $0<c\leqq 1$ *and* $m=l+h$ *where* l *and* h *are non negative integers. Let* ϕ, e_m^* *be the functions as above. If* f *is an* X*-valued distribution on* I, *then it can be represented as follows*:

$$f(t)=\int_0^c\left\langle\frac{1}{\tau}\phi_{0,h}\left(t,\ \frac{t-s}{\tau}\right),\ u_l(\tau,\ s)\right\rangle_s\frac{d\tau}{\tau}$$
$$+\sum_{j=0}^h\int_0^c\left\langle\frac{1}{\tau}\phi_{0,m+j}\left(t,\ \frac{t-s}{\tau}\right),\ u_{jh}(\tau,\ s)\right\rangle_s\frac{d\tau}{\tau}$$
$$+\frac{1}{c}\left\langle e_m^*\left(t,\ \frac{t-s}{c}\right),\ f(s)\right\rangle_s$$

where $\langle\ ,\ \rangle_s$ *denotes the pairing of* $\mathcal{D}(\boldsymbol{R})\times\mathcal{D}'(\boldsymbol{R};\ X)$ *and*

$$\phi_{i,j}(t,\ s)=\partial_t^i\partial_s^j\phi(t,\ s),$$

$$u_l(\tau,\ t)=\int_\tau^c\left(\frac{\tau}{\tau'}\right)^l\sum_{k=0}^l\binom{l}{k}\tau'^k\left\langle\frac{1}{\tau'}\phi_{k,m+l-k}\left(t,\ \frac{t-s}{\tau'}\right),\ f(s)\right\rangle_s\frac{d\tau'}{\tau'},$$

$$u_{jh}(\tau,\ s)=(-\tau)^{h-j}\binom{h}{j}\int_0^\tau\left(\frac{\tau'}{\tau}\right)^h\left\langle\frac{1}{\tau'}\phi_{h-j,l}\left(t,\ \frac{t-s}{\tau'}\right),\ f(s)\right\rangle_s\frac{d\tau'}{\tau'}.$$

THEOREM 2.1 (*Characterization of Besov spaces*). *Let* $1\leqq p,\ q\leqq\infty$, $\sigma\in\boldsymbol{R}$ *and* m *a non-negative integer such that* $m>\sigma$, *and* $0<c\leqq 1$. *An* X*-valued distribution* f *on* I *belongs to the space* $B_{p,q}^\sigma(I;\ X)$ *if and only if the following conditions are satisfied*:

$$\left\langle\phi\left(t,\ \frac{t-s}{c}\right),\ f(s)\right\rangle_s\in L^p(I;\ X)\qquad \textit{for any}\quad \phi\in\mathcal{K}_0(I),$$

$$\tau^{-\sigma}\left\langle\phi\left(t,\ \frac{t-s}{\tau}\right),\ f(s)\right\rangle_s\in L_*^q((0,\ c);\ L^p(I;\ X))\qquad \textit{for any}\quad \phi\in\mathcal{K}_m(I).$$

REMARK 2.1. (A) Let m, h and l be integers such that $-h<\sigma<l$, $m=l+h$. Set

$$\psi_k(t,\ s)=\partial_t^k\partial_s^{l-k}e_m^*(t,\ s),\qquad k=0,\ \cdots,\ l.$$

Then $f\in B_{p,q}^\sigma(I;\ X)$ if the following conditions are satisfied:

$$\tau^{-\sigma}\left\langle\frac{1}{\tau}\phi_{k,m+l-k}\left(s,\ \frac{s-r}{\tau}\right),\ f(r)\right\rangle_r\in L_*^q((0,\ c);\ L^p(I;\ X))\quad \text{for}\quad k=0,\ \cdots,\ l,$$

$$\tau^{-\sigma}\left\langle\frac{1}{\tau}\phi_{h-j,l}\left(s,\ \frac{s-r}{\tau}\right),\ f(r)\right\rangle_r\in L_*^q((0,\ c);\ L^p(I;\ X))\quad \text{for}\quad j=0,\ \cdots,\ h,$$

$$\left\langle\psi_k\left(t,\ \frac{t-s}{c}\right),\ f(s)\right\rangle_s\in L^p(I;\ X)\qquad \text{for}\quad k=0,\ \cdots,\ l.$$

(B) Furthermore, the norm of f in $B_{p,q}^{\sigma}(I, X)$ is equivalent with the sum of the corresponding norms of the above functions.

COROLLARY 2.1. *We have the following inclusions*:

(2.6) $B_{\infty,q_1}^{\sigma_1}(I; X) \subset B_{\infty,q_2}^{\sigma_2}(I; X)$ *for* $1 \leqq q_1, q_2 \leqq \infty, \sigma_2 < \sigma_1$.

(2.7) $B_{\infty,q_1}^{\sigma}(I; X) \subset B_{\infty,q_2}^{\sigma}(I; X)$ *for* $1 \leqq q_1 \leqq q_2 \leqq \infty, \sigma \in \boldsymbol{R}$.

(2.8) $B_{\infty,1}^{0}(I; X) \subset L^{\infty}(I; X)$.

(2.9) $B_{\infty,1}^{m}(I; X) \subset C^{m}([0, T]; X)$ *if m is a non negative integer.*

(2.10) $B_{\infty,\infty}^{\theta}(I; X) = C^{\theta}([0, T]; X)$ *for* $0 < \theta < 1$.

Further the inclusions (2.6), (2.7) *and* (2.8) *are continuous.*

REMARK 2.2. From the inclusions (2.6) and (2.10), it follows that

$$C^{\gamma}([0, T]; X) \subset B_{\infty,1}^{1-\theta}(I; X) \quad \text{for} \quad 1-\theta < \gamma \leqq 1.$$

THEOREM 2.2. *Let* $1 \leqq p, q \leqq \infty$ *and* $\sigma \in \boldsymbol{R}$. *If* $g \in B_{p,q}^{\sigma}(I; X)$, *then there exists a sequence* $\{g_n\}_{n=1}^{\infty}$ *such that*

$$g_n \in B_{p,q}^{\sigma}(I; X) \cap C^1([0, T]; X),$$

$$g_n \longrightarrow g \quad in \quad B_{p,q}^{\sigma}(I; X) \cap L^1(I; X) \quad as \quad n \longrightarrow \infty.$$

3. Analytic semigroups with weak singularity

In this section we briefly state properties of analytic semigroups with weak singularity which will be used in the following section.

THEOREM 3.1. *Assume that a linear operator* $\mathfrak{A}$ *satisfies conditions* (A.1), (A.2) *and* (A.3)$_{\theta}$ *for* $0 < \theta < 1$. *Then we have the following*:

(3.1) *The operator* $\mathfrak{A}$ *generates a semigroup* $e^{z\mathfrak{A}}$ *on* X *which is analytic in the sector* $\Delta(\omega)$.

(3.2) *The operators* $\mathfrak{A}^m e^{z\mathfrak{A}}$ *and* $(d^m/dz^m)e^{z\mathfrak{A}}$ *are bounded operators on* X *for any non-negative integer* m *and* $z \in \Delta(\omega)$, *and satisfy the following relation and estimate.*

$$\frac{d^m}{dz^m}(e^{z\mathfrak{A}}) = \mathfrak{A}^m e^{z\mathfrak{A}}, \qquad z \in \Delta(\omega).$$

$$\|\mathfrak{A}^m e^{z\mathfrak{A}}\| \leqq M_m |z|^{\theta-1-m}, \qquad z \in \Delta(\omega).$$

Here the letter M_m *is a constant depending on* m *and* ω.

PROOF. We can define the semigroup $e^{z\mathfrak{A}}$ for any $0<\varepsilon<\omega$ as follows:

$$e^{z\mathfrak{A}}=-\frac{1}{2\pi i}\int_{\Gamma}e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}d\lambda.$$

Here Γ is a path in the set $\Sigma(\varepsilon)$ such that $\Gamma=-\Gamma_1+\Gamma_2$ where

$$\Gamma_1=\{re^{-i(\pi/2+\varepsilon)};\ 0\leqq r<\infty\}.$$

$$\Gamma_2=\{re^{i(\pi/2+\varepsilon)};\ 0\leqq r<\infty\}.$$

Then, according to Theorem 5.3 of Taira [3], we have the conditions (3.1) and (3.2) for $m=0, 1$. In the following we show the condition (3.2) for general $m\geqq 2$.

First we show the following formula:

$$\frac{d^m}{dz^m}(e^{z\mathfrak{A}})=-\frac{1}{2\pi i}\int_{\Gamma}\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}d\lambda,\qquad m\geqq 1,\ z\in\Delta(\varepsilon). \tag{3.3}$$

For $z\in\Delta(\varepsilon)$ and $\lambda\in\Gamma_1$, we set

$$z=|z|e^{i\alpha},\qquad 0\leqq\alpha<\varepsilon,$$

$$\lambda=re^{-i(\pi/2+\varepsilon)},\qquad 0\leqq r<\infty.$$

Then we have

$$|e^{z\lambda}|=|e^{|z|r(\cos(\alpha-\pi/2-\varepsilon)+i\sin(\alpha-\pi/2-\varepsilon))}|$$

$$=e^{-|z|r\cdot\sin(\varepsilon-\alpha)}.$$

Hence it follows that for $z\in\Delta(\varepsilon)$ and $\lambda\in\Gamma_1$

$$\|\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}\|\leqq r^m e^{-|z|r\cdot\sin(\varepsilon-\alpha)}\frac{C(\varepsilon)}{(1+r)^{\theta}}. \tag{3.4}$$

Similarly, for $z\in\Delta(\varepsilon)$ and $\lambda\in\Gamma_2$, we let

$$z=|z|e^{i\alpha},\qquad 0\leqq\alpha<\varepsilon,$$

$$\lambda=re^{i(\pi/2+\varepsilon)},\qquad 0\leqq r<\infty.$$

Then we have

$$|e^{z\lambda}|=|e^{|z|r(\cos(\alpha+\pi/2+\varepsilon)+i\sin(\alpha+\pi/2+\varepsilon))}|$$

$$=e^{-|z|r\cdot\sin(\varepsilon+\alpha)}.$$

Hence it follows that for $z\in\Delta(\varepsilon)$ and $\lambda\in\Gamma_2$

$$\|\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}\|\leqq r^m e^{-|z|r\cdot\sin(\varepsilon+\alpha)}\frac{C(\varepsilon)}{(1+r)^{\theta}}. \tag{3.5}$$

If $z\in\Delta(\varepsilon)$, we have by the estimates (3.4) and (3.5)

$$\int_{\Gamma}\|\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}\|d\lambda$$
$$\leqq \sum_{i=1}^{2}\int_{\Gamma_i}\|\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}\|d\lambda$$
$$\leqq C(\varepsilon)\int_0^\infty \frac{r^m}{(1+r)^\theta}(e^{-|z|r\cdot\sin(\varepsilon-\alpha)}+e^{-|z|r\cdot\sin(\varepsilon+\alpha)})dr.$$

Let $\rho=|z|r$. By interchanging the integral order, we have

$$\int_0^\infty \frac{r^m}{(1+r)^\theta}(e^{-|z|r\cdot\sin(\varepsilon-\alpha)}+e^{-|z|r\cdot\sin(\varepsilon+\alpha)})dr$$
$$=\int_0^\infty (\rho/|z|)^m\Big(\frac{1}{1+\rho/|z|}\Big)^\theta(e^{-\rho\cdot\sin(\varepsilon-\alpha)}+e^{-\rho\cdot\sin(\varepsilon+\alpha)})\frac{d\rho}{|z|}$$
$$\leqq |z|^{\theta-1-m}\int_0^\infty \rho^{m-\theta}(e^{-\rho\cdot\sin(\varepsilon-\alpha)}+e^{-\rho\cdot\sin(\varepsilon+\alpha)})d\rho.$$

Since $\sin(\varepsilon-\alpha)>0$ and $\sin(\varepsilon+\alpha)>0$, we obtain that

$$\int_0^\infty \rho^{m-\theta}(e^{-\rho\cdot\sin(\varepsilon-\alpha)}+e^{-\rho\cdot\sin(\varepsilon+\alpha)})d\rho<\infty.$$

This implies that the operator $\int_{\Gamma}\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}d\lambda$ is bounded on X for $z\in\Delta(\varepsilon)$. Further we have

$$\frac{d^m}{dz^m}(e^{z\mathfrak{A}})=-\frac{1}{2\pi i}\int_{\Gamma}\lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}d\lambda,\qquad z\in\Delta(\varepsilon) \tag{3.6}$$

and

$$\Big\|\frac{d^m}{dz^m}(e^{z\mathfrak{A}})\Big\|\leqq C|z|^{\theta-1-m},\qquad z\in\Delta(\varepsilon). \tag{3.7}$$

Here the letter C is a constant depending on m and ω.

Next, using induction on m, we show that

$$\frac{d^m}{dz^m}(e^{z\mathfrak{A}})=\mathfrak{A}^m e^{z\mathfrak{A}},\qquad z\in\Delta(\varepsilon). \tag{3.8}$$

By Theorem 5.3 of [3], we have the equality (3.8) for $m=1$. We assume that the equality (3.8) holds for $m\geqq 1$. Then it follows from (3.6) that

$$\frac{d^{m+1}}{dz^{m+1}}(e^{z\mathfrak{A}})=-\frac{1}{2\pi i}\int_{\Gamma}\lambda^{m+1}e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}d\lambda$$
$$=-\frac{1}{2\pi i}\int_{\Gamma}\lambda^m e^{z\lambda}\lambda(\mathfrak{A}-\lambda)^{-1}d\lambda.$$

By Remarking that $\mathfrak{A}(\mathfrak{A}-\lambda)^{-1}=1+\lambda(\mathfrak{A}-\lambda)^{-1}$, it follows that

$$\frac{d^{m+1}}{dz^{m+1}}(e^{z\mathfrak{A}})=-\frac{1}{2\pi i}\int_\Gamma \lambda^m e^{z\lambda}\mathfrak{A}(\mathfrak{A}-\lambda)^{-1}d\lambda-\frac{1}{2\pi i}\int_\Gamma \lambda^m e^{z\lambda}d\lambda.$$

The closedness of $\mathfrak{A}$ tells us that

$$\begin{aligned}-\frac{1}{2\pi i}\int_\Gamma \mathfrak{A}\lambda^m e^{z\lambda}(\mathfrak{R}-\lambda)^{-1}d\lambda &= \mathfrak{A}\Big(-\frac{1}{2\pi i}\int_\Gamma \lambda^m e^{z\lambda}(\mathfrak{A}-\lambda)^{-1}d\lambda\Big)\\ &= \mathfrak{A}\frac{d^m}{dz^m}(e^{z\mathfrak{A}})\\ &= \mathfrak{A}^{m+1}e^{z\mathfrak{A}}.\end{aligned}$$

Note that

$$\int_\Gamma \lambda^m e^{z\lambda}d\lambda=0 \qquad \text{for} \quad m\geqq 1.$$

Hence it follows that

$$\frac{d^{m+1}}{dz^{m+1}}(e^{z\mathfrak{A}})=\mathfrak{A}^{m+1}e^{z\mathfrak{A}}, \qquad z\in\Delta(\varepsilon).$$

The statements (3.7) and (3.8) imply that

$$\|\mathfrak{A}^m e^{z\mathfrak{A}}\|\leqq M_m|z|^{\theta-1-m}, \qquad z\in\Delta(\varepsilon), \qquad m\geqq 1$$

with a constant $M_m>0$ depending on m and ω.

The proof of Theorem 3.1 is complete.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by following the proof of Theorem B of Muramatu [1]. If there exists a solution u of the problem (*) for $u_0\in \mathcal{D}((-\mathfrak{A})^\alpha)$ with $1-\theta<\alpha<\theta$, we can uniquely write the solution in the following form:

$$u(t)=e^{t\mathfrak{A}}u_0+\int_0^t e^{(t-s)\mathfrak{A}}f(s)ds, \qquad 0\leqq t\leqq T.$$

First we verify that u satisfies the condition (1.1). Theorem 1.3 tells us that

$$e^{t\mathfrak{A}}u_0\in C([0,T];X)\cap C^1((0,T];X).$$

So, it suffices to show that

$$F(\cdot)=\int_0^\cdot e^{(\cdot-s)\mathfrak{A}}f(s)ds\in C([0,T];X)\cap C^1((0,T];X).$$

Since it is clear that $f\in B^{1-\theta}_{\infty,1}(I;X)$ implies $F\in C([0,T];X)$, we have only to verify that $F\in C^1((0,T];X)$. By Corollary 2.1, we have

$$B^1_{\infty,1}((\varepsilon,T);X)\subset C^1([\varepsilon,T];X) \qquad \text{for any} \quad 0<\varepsilon<T.$$

Therefore, if $F\in B^{1}_{\infty,1}((\varepsilon, T); X)$ for any $0<\varepsilon<T$, it follows that $F\in C^{1}((0, T]; X)$.

Let I_ε be the open interval (ε, T). In the following we simply write $\int_R$ as $\int$. In order to verify that $F\in B^{1}_{\infty,1}(I_\varepsilon; X)$, we apply Theorem 2.1 with $I=I_\varepsilon$ and $m=4$. That is, we show that the function F satisfies the following conditions for $0<c\leqq 1$:

$$\int\phi\Big(\cdot, \frac{\cdot-s}{c}\Big)F(s)ds\in L^{\infty}(I_\varepsilon; X) \quad \text{for } \phi\in\mathcal{K}_0(I_\varepsilon), \tag{4.1}$$

$$\tau^{-1}\int\frac{1}{\tau}\phi\Big(\cdot, \frac{\cdot-s}{\tau}\Big)F(s)ds\in L^{1}_{*}((0, c); L^{\infty}(I_\varepsilon; X)) \tag{4.2}$$

for $\phi\in\mathcal{K}_4(I_\varepsilon)\cap\mathcal{K}_4(I)$ (cf. Lemma 2.1 and Remark 2.1(A)).

First, we show that F satisfies the condition (4.1). Since ϕ satisfies the condition (2.3), we have

$$\int\phi\Big(t, \frac{t-s}{c}\Big)F(s)ds=\int_0^T\phi\Big(t, \frac{t-s}{c}\Big)\Big(\int_0^s e^{(s-r)\mathfrak{A}}f(r)dr\Big)ds. \tag{4.3}$$

By interchanging the integral order of s and r and by integration by substitution with $s-r=s'$, the right hand of (4.3) becomes

$$\begin{aligned}&\int_0^T\phi\Big(t, \frac{t-s}{c}\Big)\Big(\int_0^s e^{(s-r)\mathfrak{A}}f(r)dr\Big)ds\\ &=\int_0^T\Big(\int_r^T\phi\Big(t, \frac{t-s}{c}\Big)e^{(s-r)\mathfrak{A}}ds\Big)f(r)dr\\ &=\int_0^T\Big(\int_0^{T-r}\phi\Big(t, \frac{t-s'-r}{c}\Big)e^{s'\mathfrak{A}}ds'\Big)f(r)dr.\end{aligned}$$

Again, by interchanging the integral order of s and r, it follows that

$$\begin{aligned}&\int_0^T\Big(\int_0^{T-r}\phi\Big(t, \frac{t-s'-r}{c}\Big)e^{s'\mathfrak{A}}ds'\Big)f(r)dr\\ &=\int_0^T e^{s'\mathfrak{A}}ds'\int_0^{T-s'}\phi\Big(t, \frac{t-s'-r}{c}\Big)f(r)dr.\end{aligned}$$

Hence we have

$$\int\phi\Big(t, \frac{t-s}{c}\Big)F(s)ds=\int_0^T e^{s\mathfrak{A}}ds\int_0^{T-s}\phi\Big(t, \frac{t-s-r}{c}\Big)f(r)dr.$$

Now we cite a lemma which we use in order to estimate the right term (cf. Muramatu [1], Lemma 3).

LEMMA 4.1. *Suppose that* $1\leqq p\leqq\infty$, $0<\tau\leqq 1$, $f\in L^{1}(I; X)$ *and* $\phi\in\mathcal{K}_0(I_\varepsilon)$.

Then there exists a constant $M_1>0$ *such that*

$$\left\|\int_0^{T-s}\frac{1}{\tau}\phi\left(\cdot,\ \frac{\cdot-s-r}{\tau}\right)f(r)dr\right\|_{L^p(I;\,X)}\leqq M_1\tau^{-1+1/p}\|f\|_{L^1(I;\,X)}$$

for $0\leqq s\leqq T$.

By making use of Lemma 4.1 and the estimate:

$$\|e^{s\mathfrak{A}}\|\leqq Ms^{\theta-1},\qquad s>0,$$

it follows that

(4.4)
$$\left\|\int\phi\left(\cdot,\ \frac{\cdot-s}{c}\right)F(s)ds\right\|_{L^\infty(I_\varepsilon;\,X)}$$
$$\leqq\int_0^T\|e^{s\mathfrak{A}}\|ds\left\|\int_0^{T-s}\phi\left(\cdot,\ \frac{\cdot-s-r}{c}\right)f(r)dr\right\|_{L^\infty(I;\,X)}$$
$$\leqq C\|f\|_{L^1(I;\,X)}.$$

Here and in the following the letter C is a general constant independent of f.

Next we show that F satisfies the condition (4.2). Let $0<\tau\leqq c$, $\phi\in\mathcal{K}_4(I_\varepsilon)\cap\mathcal{K}_4(I)$ and

$$U(\tau,\ t)=\int\frac{1}{\tau}\phi\left(t,\ \frac{t-s}{\tau}\right)F(s)ds.$$

We divide the integral with respect to s into two parts as follows:

$$U(\tau,\ t)=\int\frac{1}{\tau}\phi\left(t,\ \frac{t-s}{\tau}\right)F(s)ds$$
$$=\int_0^T e^{s\mathfrak{A}}ds\int_0^{T-s}\frac{1}{\tau}\phi\left(t,\ \frac{t-s-r}{\tau}\right)f(r)dr$$
$$=\left(\int_0^\tau+\int_\tau^T\right)e^{s\mathfrak{A}}ds\int_0^{T-s}\frac{1}{\tau}\phi\left(t,\ \frac{t-s-r}{\tau}\right)f(r)dr$$
$$\equiv U_1(\tau,\ t)+U_2(\tau,\ t).$$

We cite a lemma which is used in order to estimate U_1 and U_2 (cf. Muramatu [1], Lemma 4).

LEMMA 4.2. *Assume that* $1\leqq p\leqq\infty$, $0<\tau\leqq c$, $f\in L^1(I;\ X)$ *and* $\phi\in\mathcal{K}_0(I_\varepsilon)$. *Then there exists a constant* $M_2>0$ *such that*

$$\left\|\int_0^{T-s}\frac{1}{\tau}\phi\left(\cdot,\ \frac{\cdot-s-r}{\tau}\right)f(r)dr\right\|_{L^p(I_\varepsilon;\,X)}$$
$$\leqq\sum_{j=0}^{2}\frac{s^j}{j!}\|u_j(\tau,\ \cdot)\|_{L^p(I;\,X)}+M_2s^3\tau^{-1+1/p}\|f\|_{L^1(I;\,X)}$$

for $0\leqq s\leqq\varepsilon$. *Here*

$$u_j(\tau, t)=\int_0^T \frac{1}{\tau}\phi_{j,0}\Big(t, \frac{t-r}{\tau}\Big)f(r)dr,$$

$$\phi_{j,k}(t, s)=\partial_t^j\partial_s^k\phi(t, s).$$

Now we may assume that $0<c\leqq\varepsilon$. Lemma 4.2 gives that

$$\begin{aligned}
(4.5)\qquad &\|U_1(\tau, \cdot)\|_{L^\infty(I_\varepsilon;\, X)} \\
&\leqq\int_0^\tau\|e^{s\mathfrak{A}}\|ds\Big\|\int_0^{T-s}\frac{1}{\tau}\phi\Big(\cdot, \frac{\cdot-s-r}{\tau}\Big)f(r)dr\Big\|_{L^\infty(I_\varepsilon;\, X)} \\
&\leqq\int_0^\tau Ms^{\theta-1}\Big(\sum_{j=0}^{2}\frac{s^j}{j\,!}\|u_j(\tau, \cdot)\|_{L^\infty(I;\, X)}+M_2s^3\tau^{-1}\|f\|_{L^1(I;\, X)}\Big)ds \\
&\leqq C\tau^\theta\Big(\sum_{j=0}^{2}\tau^j\|u_j(\tau, \cdot)\|_{L^\infty(I;\, X)}+\tau^2\|f\|_{L^1(I;\, X)}\Big).
\end{aligned}$$

Since $\phi\in\mathcal{K}_4(I_\varepsilon)\cap\mathcal{K}_4(I)$, we can represent ϕ as $\phi(t, s)=\partial_s^4\psi(t, s)$ where $\psi\in\mathcal{K}_0(I_\varepsilon)\cap\mathcal{K}_0(I)$. By interchanging the integral order, we have

$$\begin{aligned}
U_2(\tau, t)&=\int_\tau^T e^{s\mathfrak{A}}ds\int_0^{T-s}\frac{1}{\tau}\phi\Big(t, \frac{t-s-r}{\tau}\Big)f(r)dr \\
&=\int_0^{T-\tau}\Big(\int_\tau^{T-r}\frac{1}{\tau}\phi\Big(t, \frac{t-s-r}{\tau}\Big)e^{s\mathfrak{A}}ds\Big)f(r)dr \\
&=\int_0^{T-\tau}\Big(\int_\tau^{T-r}\frac{1}{\tau}\psi_{0,4}\Big(t, \frac{t-s-r}{\tau}\Big)e^{s\mathfrak{A}}ds\Big)f(r)dr
\end{aligned}$$

where $\psi_{i,j}(t, s)=\partial_t^i\partial_s^j\psi(t, s)$. By integration by parts, it follows that

$$\begin{aligned}
\int_\tau^{T-r}\frac{1}{\tau}\psi_{0,4}\Big(t, \frac{t-s-r}{\tau}\Big)e^{s\mathfrak{A}}ds=&\sum_{k=0}^{3}\psi_{0,k}\Big(t, \frac{t-\tau-r}{\tau}\Big)(\tau\mathfrak{A})^{3-k}e^{\tau\mathfrak{A}} \\
&+\int_\tau^{T-r}\tau^3\psi\Big(t, \frac{t-s-r}{\tau}\Big)\mathfrak{A}^4e^{s\mathfrak{A}}ds.
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
(4.6)\qquad U_2(\tau, t)=\int_0^{T-\tau}\Big(&\sum_{k=0}^{3}\psi_{0,k}\Big(t, \frac{t-\tau-r}{\tau}\Big)(\tau\mathfrak{A})^{3-k}e^{\tau\mathfrak{A}} \\
&+\int_\tau^{T-r}\tau^3\psi\Big(t, \frac{t-s-r}{\tau}\Big)\mathfrak{A}^4e^{s\mathfrak{A}}ds\Big)f(r)dr.
\end{aligned}$$

We write the first and second terms of (4.6) as

$$V_k(\tau, t)=\tau(\tau\mathfrak{A})^{3-k}e^{\tau\mathfrak{A}}\int_0^{T-\tau}\frac{1}{\tau}\psi_{0,k}\Big(t, \frac{t-\tau-r}{\tau}\Big)f(r)dr,\quad k=0, 1, 2, 3,$$

$$V_4(\tau, t)=\int_0^{T-\tau}\Big(\int_0^{T-r}\tau^3\psi\Big(t, \frac{t-s-r}{\tau}\Big)\mathfrak{A}^4e^{s\mathfrak{A}}ds\Big)f(r)dr,$$

respectively. That is, $U_2(\tau, t)$ is written as

$$U_2(\tau, t)=\sum_{k=0}^{3} V_k(\tau, t)+V_4(\tau, t).$$

By noting that

$$\|\mathfrak{A}^m e^{t\mathfrak{A}}\| \leqq M_m t^{\theta-1-m}, \qquad t>0$$

with a constant $M_m>0$ for $m=0, 1, 2, \cdots$, Lemma 4.2 gives that

$$(4.7) \qquad \|V_k(\tau, \cdot)\|_{L^\infty(I_\varepsilon; X)} \leqq C\tau^\theta\Big(\sum_{j=0}^{2}\|\tau^j v_{jk}(\tau, \cdot)\|_{L^\infty(I; X)}+\tau^2\|f\|_{L^1(I; X)}\Big) \quad \text{for} \quad k=0, 1, 2, 3.$$

Here

$$v_{jk}(\tau, t)=\int_0^T \frac{1}{\tau}\phi_{j,k}\Big(t, \frac{t-r}{\tau}\Big)f(r)dr, \qquad j=0, 1, 2,\ k=0, 1, 2, 3.$$

$V_4(\tau, t)$ is, by interchanging the integral order of s and r, written by the following form:

$$V_4(\tau, t)=\tau^4\int_\tau^T \mathfrak{A}^4 e^{s\mathfrak{A}}ds\int_0^{T-s}\frac{1}{\tau}\phi\Big(t, \frac{t-s-r}{\tau}\Big)f(r)dr.$$

Lemma 4.1 and Lemma 4.2 give that

$$\begin{aligned}
(4.8) \qquad & \|V_4(\tau, \cdot)\|_{L^\infty(I_\varepsilon; X)} \\
&\leqq \tau^4\Big(\int_\tau^\varepsilon+\int_\varepsilon^T\Big)\|\mathfrak{A}^4 e^{s\mathfrak{A}}\|ds\Big\|\int_0^{T-s}\frac{1}{\tau}\phi\Big(\cdot, \frac{\cdot-s-r}{\tau}\Big)f(r)dr\Big\|_{L^\infty(I_\varepsilon; X)} \\
&\leqq \tau^4\int_\tau^\varepsilon M_4 s^{\theta-5}\Big(\sum_{j=0}^{2}\frac{s^j}{j!}\|v_{j0}(\tau, \cdot)\|_{L^\infty(I; X)}+M_2 s^3\tau^{-1}\|f\|_{L^1(I; X)}\Big)ds \\
&\quad +\tau^4\int_\varepsilon^T M_4 s^{\theta-5}\tau^{-1}\|f\|_{L^1(I; X)}ds \\
&\leqq C\tau^4\int_\tau^\infty s^{\theta-5}\Big(\sum_{j=0}^{2}s^j\|v_{j0}(\tau, \cdot)\|_{L^\infty(I; X)}+s^3\tau^{-1}\|f\|_{L^1(I; X)}\Big)ds \\
&\quad C\tau^4\int_\varepsilon^T s^{\theta-5}\tau^{-1}\|f\|_{L^1(I; X)}ds \\
&\leqq C\tau^\theta\Big(\sum_{j=1}^{2}\tau^j\|v_{j0}(\tau, \cdot)\|_{L^\infty(I; X)}+(\tau^2+\tau^{3-\theta})\|f\|_{L^1(I; X)}\Big).
\end{aligned}$$

Hence we have

$$\begin{aligned}
(4.9) \quad \|U_2(\tau, \cdot)\|_{L^\infty(I_\varepsilon; X)} &\leqq \sum_{k=0}^{4}\|V_k(\tau, \cdot)\|_{L^\infty(I_\varepsilon; X)} \\
&\leqq C\sum_{k=0}^{3}\tau^\theta\Big(\sum_{j=0}^{2}\tau^j\|v_{jk}(\tau, \cdot)\|_{L^\infty(I; X)}+(\tau^2+\tau^{3-\theta})\|f\|_{L^1(I; X)}\Big).
\end{aligned}$$

By the estimates (4.5) and (4.9), we have

$$\begin{aligned}(4.10)\qquad &\|\tau^{-1}U(\tau, t)\|_{L^1_*((0,c);\, L^\infty(I_\varepsilon;\, X))} \\ &=\int_0^c \tau^{-1}\|U(\tau, \cdot)\|_{L^\infty(I_\varepsilon; X)}\frac{d\tau}{\tau} \\ &\leqq C\Big(\sum_{j=0}^{2}\|\tau^{-(1-\theta)}u_j(\tau, t)\|_{L^1_*((0,c);\, L^\infty(I;\, X))} \\ &\quad+\sum_{j=0}^{2}\sum_{k=0}^{3}\|\tau^{-(1-\theta)}v_{jk}(\tau, t)\|_{L^1_*((0,c);\, L^\infty(I;\, X))}+\|f\|_{L^1(I;\, X)}\Big).\end{aligned}$$

By Remark 2.1 (B), it follows that

$$(4.11)\qquad \|\tau^{-1}U(\tau, t)\|_{L^1_*((0,c);\, L^\infty(I_\varepsilon;\, X))}\leqq C(\|f\|_{B^{1-\theta}_{\infty,1}(I;\, X)}+\|f\|_{L^1(I;\, X)}).$$

It has been proved that F satisfies the condition (4.2).

Now, by making use of Remark 2.1 (B), the estimates (4.6) and (4.11) imply that

$$(4.12)\qquad \|F\|_{B^1_{\infty,1}(I_\varepsilon;\, X)}\leqq C(\|f\|_{B^{1-\theta}_{\infty,1}(I;\, X)}+\|f\|_{L^1(I;\, X)}).$$

Now we verify that u, given by the formula

$$u(t)=e^{t\mathfrak{A}}u_0+\int_0^t e^{(t-s)\mathfrak{A}}f(s)ds,$$

satisfies the conditions (1.2) and (1.3). Theorem 1.3 tells us that $e^{t\mathfrak{A}}u_0$ satisfies the conditions (1.2) and (1.3). By virtue of Theorem 2.2, there exists a sequence $\{f_n\}_{n=1}^\infty$ such that

$$(4.13)\qquad f_n\in B^{1-\theta}_{\infty,1}(I;\, X)\cap C^1([0, T];\, X),$$

$$(4.14)\qquad f_n\longrightarrow f \quad\text{in}\quad B^{1-\theta}_{\infty,1}(I;\, X)\cap L^1(I;\, X).$$

We let

$$F_n(t)=\int_0^t e^{(t-s)\mathfrak{A}}f_n(s)ds.$$

Then we have by Theorem 1.3

$$F_n\in C^1((0, T];\, X),$$

$$F_n(t)\in\mathcal{D}(\mathfrak{A}),\qquad 0<t\leqq T,$$

$$\frac{dF_n}{dt}(t)=\mathfrak{A}F_n(t)+f_n(t),\qquad 0<t\leqq T.$$

By applying the inequality (4.12) to $f-f_n$ and $F-F_n$, we have

$$\|F-F_n\|_{B^1_{\infty,1}(I_\varepsilon;\, X)}\leqq C(\|f-f_n\|_{B^{1-\theta}_{\infty,1}(I;\, X)}+\|f-f_n\|_{L^1(I;\, X)}).$$

Using the statement (4.14), we obtain that

$$\|F-F_n\|_{B^1_{\infty,1}(I_\varepsilon;\,X)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{4.15}$$

On the other hand, we have

$$\begin{aligned}
&\left\|\mathfrak{A}F_n-\frac{dF}{dt}+f\right\|_{B^0_{\infty,1}(I_\varepsilon;\,X)} \\
&=\left\|-f_n+\frac{dF_n}{dt}-\frac{dF}{dt}+f\right\|_{B^0_{\infty,1}(I_\varepsilon;\,X)} \\
&\leqq \|f_n-f\|_{B^0_{\infty,1}(I_\varepsilon;\,X)}+\left\|\frac{dF_n}{dt}-\frac{dF}{dt}\right\|_{B^0_{\infty,1}(I_\varepsilon;\,X)}.
\end{aligned}$$

We estimate the two terms of the right. The inclusion (2.6) and the statement (4.14) tell us that

$$\begin{aligned}
\|f_n-f\|_{B^0_{\infty,1}(I_\varepsilon;\,X)} &\leqq C\|f_n-f\|_{B^{1-\theta}_{\infty,1}(I_\varepsilon;\,X)} \\
&\leqq C\|f_n-f\|_{B^{1-\theta}_{\infty,1}(I;\,X)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned} \tag{4.16}$$

The definition of Besov spaces and (4.15) give that

$$\left\|\frac{dF_n}{dt}-\frac{dF}{dt}\right\|_{B^0_{\infty,1}(I_\varepsilon;\,X)} \leqq \|F_n-F\|_{B^1_{\infty,1}(I_\varepsilon;\,X)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{4.17}$$

From (4.16) and (4.17), it follows that

$$\left\|\mathfrak{A}F_n-\frac{dF}{dt}-f\right\|_{B^0_{\infty,1}(I_\varepsilon;\,X)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{4.18}$$

By using the inclusion (2.8), if $t \in I_\varepsilon$, the statements (4.15) and (4.18) imply that as $n \to \infty$

$$\begin{aligned}
&F_n(t) \longrightarrow F(t) \quad \text{in} \quad X, \\
&\mathfrak{A}F_n(t) \longrightarrow \frac{dF}{dt}(t)-f(t) \quad \text{in} \quad X.
\end{aligned}$$

By virtue of the closedness of $\mathfrak{A}$, it follows that

$$\begin{aligned}
&F(t) \in \mathcal{D}(\mathfrak{A}), \qquad 0<t\leqq T, \\
&\mathfrak{A}F(t)=\frac{dF}{dt}(t)-f(t), \qquad 0<t\leqq T.
\end{aligned}$$

The proof of Theorem 1.4 is now complete.

REMARK 4.1. The proof of Theorem 1.4 tells us that for any $\varepsilon>0$

$$f \in B^{1-\theta}_{\infty,1}((0,\,T);\,X) \Longrightarrow F \in B^1_{\infty,1}((\varepsilon,\,T);\,X).$$

This implies that the regularity of F is as maximal as possible. In other words, if $\sigma>1$ and $1\leqq q\leqq\infty$, it does not necessarily hold that $F\in B^\sigma_{\infty,q}((\varepsilon,\,T);\,X)$ if

$f\in B_{\infty,1}^{1-\theta}((0, T); X)$.

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