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THE GENERALIZATIONS OF FIRST COUNTABLE SPACES

By

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Abstract. In this paper we consider some generalizations of first countable spaces, called w_κ-spaces. When $\kappa=1$, ω_1, ∞, the spaces are respectively Fréchet spaces, w-spaces in the sense of G. Gruenhage [5] and first countable spaces. We show that the w_κ-spaces are the images of metric spaces under certain kind of continuous maps, called w_κ-maps. For any cardinals $\kappa_1<\kappa_2$, we construct by forcing a model in which there is a countable space with character ω_1 which is a w_{κ_1}-space but not w_{κ_2}-space.

1. Introduction

Generalizations of first countable spaces have been one of the traditional topics in general topology. G. Gruenhage [5] defined the class of w-spaces by topological games. P.L. Sharma [9] gave out a very useful characterization of w-spaces. In this paper we introduce w_κ-spaces which establish an interesting relationship among Fréchet spaces, w-spaces and first countable spaces.

It is well-known that Fréchet spaces and first countable spaces are respectively the images of metric spaces under pseudo-open and almost open maps (see [7]). The author [10] proved that w-spaces are the images of metric spaces under w-maps. Theorem 3.2 in this paper unifies all of these results.

Assuming $\boldsymbol{MA}$, F. Galvin [4] constructed a w-space which is not a c^*-space, i.e. a space X with countable tightness and every countable subspace of X is first countable. In this paper we show that for any cardinals $\kappa_1<\kappa_2$, it is consistent that there is a countable space with character ω_1 which is a w_{κ_1}-space but not w_{κ_2}-space.

2. Notations, Definitions and Basic Properties

All spaces considered are assumed to be Hausdorff and maps continuous onto. The notation $\{A_\alpha : \alpha<\kappa\}$ is not necessarily faithful. For the terminology

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and basic facts about forcing see [6], for the weak versions of Martin's axiom see [3]. We use p to denote the least cardinality of a centered family $\mathcal{F}$ of subsets of ω such that there is no $A\in[\omega]^{\omega}$ such that $A\subset^* B$ for any $B\in\mathcal{F}$. $\alpha, \beta, \cdots$ denote ordinals and $\kappa, \lambda, \cdots$ cardinals.

DEFINITION 2.1. We call a map $f: X\to Y$ a w_κ-map, if for any $y\in Y$ and any open cover $\{U_\alpha: \alpha<\kappa\}$ of $f^{-1}(y)$, there exists an α such that $y\in \mathrm{int}(f(U_\alpha))$. If a map is a w_κ-map for any κ, we call it a w_∞-map. From now on, κ is a nonzero cardinal or ∞.

It is obvious that the class of w_1-maps equals to the class of pseudo-open maps. We can easily construct for any $\kappa_1<\kappa_2$ a map which is a w_{κ_1}-map but not w_{κ_2}-map.

LEMMA 2.1. *Let $f: X\to Y$, then the following are equivalent:*

(1) *f is a w_κ-map;*

(2) *If $\{A_\alpha: \alpha<\kappa\}$ is a family of subsets of Y, $y\in\cap\{cl(A_\alpha): \alpha<\kappa\}$, then there exists an $x\in f^{-1}(y)$, $x\in\cap\{cl(f^{-1}(A_\alpha)): \alpha<\kappa\}$.*

PROOF. (1)→(2) Suppose that there exists a family $\{A_\alpha: \alpha<\kappa\}$ of subsets of Y and $y\in\cap\{cl(A_\alpha): \alpha<\kappa\}$ such that for any $x\in f^{-1}(y)$, $x\notin\cap\{cl(f^{-1}(A_\alpha)): \alpha<\kappa\}$. Then if $x\in f^{-1}(y)$, there are an open neighbourhood U_x of x and an $\alpha_x<\kappa$ such that $U_x\cap f^{-1}(A_{\alpha_x})=0$. Let $U_\alpha=\cup\{U_x: x\in f^{-1}(y)\ \&\ \alpha_x=\alpha\}$ for any $\alpha<\kappa$ $\{U_\alpha: \alpha<\kappa\}$ is clearly an open cover of $f^{-1}(y)$. Since f is a w_κ-map, there is a U_α such that $y\in\mathrm{int}(f(U_\alpha))$. However, $U_\alpha\cap f^{-1}(A_\alpha)=\cup\{U_x\cap f^{-1}(A_\alpha): x\in f^{-1}(y)\ \&\ \alpha_x=\alpha\}=0$. So $f(U_\alpha)\cap A_\alpha=0$, but $y\in cl(A_\alpha)$. This is a contradiction.

(2)→(1) Suppose that f is not a w_α-map, i.e. we have a $y_0\in Y$ and a cover $\{U_\alpha: \alpha<\kappa\}$ of $f^{-1}(y_0)$ such that for any α, U_α is open and $y_0\notin\mathrm{int}(f(U_\alpha))$. Therefore, we have $y_0\in\cap\{cl(Y-f(U_\alpha)): \alpha<\kappa\}$. By (2), there exists an $x\in f^{-1}(y_0)$ such that $x\in\cap\{cl(f^{-1}(Y-f(U_\alpha))): \alpha<\kappa\}$. However, since $\{U_\alpha: \alpha<\kappa\}$ is a cover of $f^{-1}(y_0)$, there is a U_α, $x\in U_\alpha$. Since $U_\alpha\cap f^{-1}(Y-f(U_\alpha))=0$, $x\notin cl(f^{-1}((Y-f(U_\alpha)))$. This contradiction completes the proof. □

DEFINITION [7] 2.2. $f: X\to Y$ is called almost open, if for any $y\in Y$, there is an $x\in f^{-1}(y)$ such that for any neighbourhood U of x, $f(U)$ is a neighbourhood of y.

THEOREM 2.1. *Let $f: X\to Y$. The following are equivalent:*

(1) *f is an almost open map;*

(2) *f is a w_{μ_1}-map, where* $\mu_1=\sup\{L(f^{-1}(y)): y\in Y\}$, *$L$ denotes the Lindelöf degree;*

(3) *f is a w_{μ_1}-map, where* $\mu_2=2^{|Y|}$.

The proof is routine by the definitions and Lemma 2.1.

3. Theorems on w_κ-spaces

DEFINITION 3.1. *A* space Y is called a w_κ-space, if for any family $\{A_\alpha : \alpha<\kappa\}$. of subsets of Y and $y\in\cap\{cl(A_\alpha): \alpha<\kappa\}$, there exists a decreasing sequence $\{F_n : n\in\omega\}$ of subsets of Y satisfying that $F_n\cap A_\alpha\neq 0$ for any n and α and for any open neighbourhood U of y there is an n such that $F_m\subset U$ for any $m>n$, i.e., $\{F_n : n\in\omega\}$ converges to y. What a w_∞-space means is obvious.

We can see easily from the definition that when κ is finite, w_κ-spaces are exactly Fréchet spaces. By the trick of repeatedly enumerating, if necessary, we can see from [9] that w_ω-spaces are exactly the w-spaces in the sense of G. Gruenhage [5].

THEOREM 3.1. *Let Y be a space. The following are equivalent:*

(1) *Y is a first countable space;*

(2) *Y is a w_∞-space;*

(3) *Y is a $w_{2^{|Y|}}$-space.*

PROOF. We need only to proof (3)→(1). Take $y\in Y$. We enumerate $\{A : y\in cl(A)\ \&\ A\subset Y\}$ as $\{A_\alpha : \alpha<2^{|Y|}\}$. Since Y is a $w_{2^{|Y|}}$-space, there must be a decreasing sequence $\{F_n : n\in\omega\}$ converging to y such that $F_n\cap A_\alpha\neq 0$ for any n and α. Let $U_n=\mathrm{int}(F_n)$. Then $\{U_n : n\in\omega\}$ is a neighbourhood base at y. □

We generalize A.V. Arhangelskii's sheaf (see [8]) to any cardinals. We need it in the proof of Theorem 3.2.

DEFINITION 3.2. If $\{r_\alpha : \alpha<\lambda\}$ is a family of convergent sequences with a common limit point y, we call it κ-sheaf with the vertex y. Let $r_\alpha=\{y_{\alpha n} : n\in\omega\}$. If for any neighbourhood U of y, there is an n_0 such that $y_{\alpha n}\in U$ for any $n>n_0$ and α, we call it a uniform κ-sheaf. If for any κ-sheaf $\{r_\alpha : \alpha<\kappa\}$ in Y there is a uniform κ-sheaf $\{r'_\alpha : \alpha<\kappa\}$ such that r'_α is a subsequence of r_α, we call Y a κ-sheafed space.

PROPOSITION 3.1. *A space Y is a w_κ-space if and only if Y is a Fréchet κ-sheafed space. Consequently, w_κ-spaces are almost countably productive for any $\kappa \geqq \omega$.*

The last part of Proposition 3.1 follows from the fact that w-spaces are almost countably productive [8].

THEOREM 3.2. *A space Y is a w_κ-space if and only if Y is an image of a metric space under a w_κ-map.*

PROOF. On the part of "only if" needs to be proven here, since w_κ-spaces are preserved by w_κ-maps by Lemma 2.1.

Let $\{R_\eta : \eta \in \Lambda\}$ be an enumeration of all uniform κ-sheaves in Y. For any $\eta \in \Lambda$ we construct a metric space X_η as follows: Take κ disjoint countable infinite sets $\{s_{\alpha\eta} : \alpha < \kappa\}$ and $x_\eta \notin \cup\{s_{\alpha\eta} : \alpha < \kappa\}$. Let $X_\eta = \cup\{s_{\alpha\eta} : \alpha < \kappa\} \cup \{x_\eta\}$ and $s_{\alpha\eta} = \{x_{\alpha n}^\eta : n \in \omega\}$. We define

$$d_\eta(x_{\alpha m}^\eta, x_{\alpha n}^\eta) = \begin{cases} 1/m + 1/n & \alpha \neq \beta \\ |1/m - 1/n| & \alpha = \beta \end{cases};$$

$$d_\eta(x_{\alpha m}^\eta, x_\eta) = 1/m\,.$$

Then (X_η, d_η) is a metric space. Let X be the topological sum of $\{X_\eta : \eta \in \Lambda\}$ and $f : X \to Y$ be the map which maps X_η onto $\cup R_\eta$ in a natural way. Now we want to show that f is a w_κ-map. Suppose $\{A_\alpha : \alpha < \kappa\}$ is a family of subsets of Y and $y \in \cap\{cl((A_\alpha) : \alpha < \kappa\}$. Since Y is Fréchet, there is an $\eta \in \Lambda$ such that $r_{\alpha\eta} \subset A_\alpha$, where $R_\eta = \{r_{\alpha\eta} : \alpha < \kappa\}$, and the vertex of R_η is y. It is easily seen from the definition of f that $s_{\alpha\eta} \subset f^{-1}(A_\alpha)$ and $x_\eta \in f^{-1}(y)$. Therefore, $x_\eta \in cl(f^{-1}(A_\alpha))$. By Lemma 2.1, f is a w_κ-map. This completes the proof. □

THEOREM 3.3. *Let Y be a space with countable tightness and character less than p. Then Y is a w-space. In particular, if Y is countable, Y is a w_κ-space for any $\kappa < p$.*

PROOF. Let $\mathcal{U}$ be a local base at $y \in Y$ with cardinality less than p. Suppose that $\{A_n : n \in \omega\}$ is a family of subsets of Y such that $y \in \bar{A}_n$. Since Y has countable tightness, we can assume that A_n is countable. Let $\cup\{A_n : n \in \omega\} = \{y_n : n \in \omega\}$. We define $\mathrm{P} = \{(I, S) : I \in [\omega]^{<\omega}$ & $S \in [\mathcal{U}]^{<\omega}\}$ and $(I', S') \leqq (I, S)$ iff $I' \supset I$, $S' \supset S$ and $I' \setminus I \subset \cap\{U : U \in S\}$. It is easily seen that $(\mathrm{P}, \leqq)$ is a σ-centered poset. The conclusion follows from the standard density arguments. □

REMARK. It follows from Example 4.2 that it is consistent that $\omega_1 < p < 2_\omega$ and there is a countable space with character ω_1 which is not a w_p-space.

4. Examples of countable w_κ-spaces with character ω_1

It follows from Theorem 3.1 that every countable w_{2^ω}-space is first countable. Therefore, we are only interested in the models of $2^\omega > \omega_1$ in this section. We will construct some models of set theory in which there exist our desired examples.

EXAMPLE 4.1. A countable space which is Fréchet but not a w-space.

Let X be the quotient space of countably many copies of $\{0, 1/2, 1/3, \cdots. 1/\omega, \cdots\}$ with all limits adhering together. We adjoin ω_1 dominating reals to any model of $2^\omega > \omega_1$. Then in this model. X is a desired one.

EXAMPLE 4.2. A countable space with character ω_1 which is a w_{κ_1}-space but not w_{κ_2}-space, where $\omega \leqq \kappa_1 < \kappa_2 \leqq 2^\omega$.

We can assume that κ_2 is regular. We start with a model V of $MA + 2^\omega \geqq \kappa_2$. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be a family of infinite subsets of ω and well-ordered by $\subset^*$. We define a finite supports iteration $\{(P_\eta, Q_\eta) : \eta < \kappa_2\}$ of ccc forcing in the following way:

In V^{P_η}, we first take a ccc poset Q'_η so that in $V^{P_\eta * Q'_\eta}$ we have $MA + 2^\omega > \kappa_2$. Now we work in $V^{P_\eta * Q'_\eta}$. We define a poset $Q''_\eta = \{(a, S) : a \in [\omega]^{<\omega}, S \subset [\omega]^\omega$ is finite and for any $\alpha < \omega_1$, $\cup S \subset^* A_\alpha\}$ where $(a', S') \leqq (a, S)$ iff $a' \supset a$, $S' \supset S$ and $(a' \setminus a) \cap B = 0$ for any $B \in S$. Let $D_{\alpha, n} = \{(a, S)$: there exists an $m > n$ such that $m \in a \cap A_\alpha\}$ for any α and n. It is easily seen that $D_{\alpha n}$ is dense in Q''_η. So if G''_η is a generic filter of Q''_η then $B_\eta = \cup\{a$: there is an S with $(a, S) \in G''_\eta\}$ satisfies that $B_\eta \cap A_\alpha$ is infinite for any $\alpha < \omega_1$. By a similar density argument, if $B \in [\omega]^\omega \cap V^{P_\eta * Q'_\eta}$ satisfies $B \subset^* A_\alpha$ for any $\alpha < \omega_i$, then $B \cap B_\eta$ is finite. Let $Q_{\eta+1} = Q'_\eta * Q''_\eta$.

Let G_{κ_2} be a generic filter of P_{κ_2} over V. From now on, we work in $V[G_{\kappa_2}]$.

For any $\mathcal{U} \subset [\omega]^\omega$ and $|\mathcal{U}| < \kappa_2$ there is an $\alpha < \kappa_2$ such that $\mathcal{U} \in V[G_\alpha]$. So if $\mathcal{U}$ has the strong finite intersection property, there is a $W \in [\omega]^\omega$ such that $W \subset^* U$ for any $U \in \mathcal{U}$. Therefore, we have $p \geqq \kappa_2$ in $V[G_{\kappa_2}]$.

On the other hand, since there is no $U \in [\omega]^\omega$ such that $U \subset^* A_\alpha$ and $U \cap B_\eta$ is infinite for any $\alpha < \omega_1$ and $\eta < \kappa_2$, we have $p \leqq \kappa_2$ by Theorem 3.8 [2].

Now we begin to construct the countable space X with character ω_1 which is a w_{κ_1}-space but not w_{κ_2}-space. Let $X=\omega$. We define the topology in the following way: If $x\neq 0$, x is isolated; The neighbourhood base at 0 is $\{(A_\alpha\setminus s)\cup\{0\}: \alpha<\omega_1$ and $s\in[\omega]^{<\omega}\}$. By Theorem 3.3, X is a w_{κ_1}-space since $p=\kappa_2$. However, we can take $\{B'_\eta: \eta<\kappa_2\}\subset[\omega]^\omega$ so that $B'_\eta\subset^* A_\alpha\cap B_\eta$ for any $\alpha<\omega_1$ and $\eta<\kappa_2$. It is obvious that B'_η is a convergent sequence. Suppose that X is a w_{κ_2}-space. Then there exist $\{F_n: n\in\omega\}$ such that:

(1) $F_n\in[\omega]^\omega$ and $F_{n+1}\subset F_n$ for any $n\in\omega$;

(2) For any $\alpha<\omega_1$ there is an n such that $F_n\subset A_\alpha$;

(3) $F_n\cap B'_\eta\setminus m\neq 0$ for any $n, m\in\omega$ and $\eta<\kappa_2$.

Therefore, there is an n such that $F_n\subset^* A_\alpha$ and $F_n\cap B'_\eta$ is infinite for any $\alpha<\omega_1$ and $\eta<\kappa_2$. This is impossible by our choice of $\{B_\eta: \eta<\kappa_2\}$. □

QUESTION 4.1. *Is it consistent that every countable w-space is first countable? Moreover, is it consistent with* $\neg$***CH*** *that every countable Fréchet space with character less that* 2^ω *is first countable?*

REMARK. A. Dow and J. Steprans [2] have constructed a model in which every countable Fréchet α_1-space is first countable.

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References

[1] E.K. Van Douwen, The integers and topology, Handbook of set theoretic topology, North-Holland, Amsterdam, 1984.

[2] A. Dow and J. Steprans, A model in which countable Fréchet α_1-apaces are first countable, Math. Proc. Camb. Phil. Soc. **105** (1988), 473-480.

[3] D.H. Fremlin, Consequences of Martin's axiom, Combridge Univ. Press, 1984.

[4] F. Gavin, On Gruenhage's generalization of first countable spaces, Notices Amer. Math. Soc. **24** (1977), A-22.

[5] G. Gruenhage, Infinite games and first countable spaces, Gen. Top. Appl. **6** (1976), 339-352.

[6] K. Kunen, Set theory, North-Holland, Amsterdam, 1980.

[7] E. Michael, A quintuple quotient quest, Gen. Top. Appl. **2** (1972), 91-138.

[8] T. Nogura, Frechetness of inverse limits and products, Gen. Top. Appl. **20** (1985), 59-66.

[9] P.L. Sharma, Some characterizations of W-spaces and w-spaces, Gen. Top. Appl. **9** (1978), 289-293.
[10] J.P. Zhu, w-spaces and w-maps, Science Bulletin **32** (1987).

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