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COMPLETE 2-TRANSNORMAL HYPERSURFACES IN A KAEHLER MANIFOLD OF NEGATIVE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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§ 1. Introduction

The idea of constant width has been developed in a somewhat different spirit, as a topic in differential geometry, and the concept of "transnormality" has been introduced as the generalized one of constant width in a Riemannian manifold.

Let M be a connected complete hypersurface of a connected complete Riemannian manifold $\bar{M}$. For each $x\in M$, there exists, up to parametrization, a unique geodesic τ_x of $\bar{M}$ which intersects M orthogonally at x. M is called *a transnormal hypersurface* of $\bar{M}$ if, for each pair $x, y\in M$, the relation $y\in\tau_x$ implies that $\tau_x=\tau_y$. For a transnormal hypersurface M, we define an equivalence relation $\sim$ on M as follows; for $x, y\in M$, $x\sim y$ means that $y\in\tau_x$. Then we can consider the quotient space $\hat{M}=M/\sim$ with the quotient topology with respect to this relation. We call M an *r-transnormal* hypersurface if the natural projection of M onto $\hat{M}$ is an r-fold covering map.

Topological structures of transnormal submanifolds are full of interest and have been investigated from various angles (for example, see [3]). On the other hand, differential geometric structures of 2-transnormal hypersurfaces in a space form have been given in [2] and [4].

Recentry, the author has studied in [5] differential geometric structures of compact 2-transnormal hypersurfaces in a complex space form. The purpose of this paper is to generalize the result in [5] to the case where 2-transnormal hypersurfaces are complete. Namely we shall prove that 2-transnormal hypersurfaces in a Kaehler manifold of negative constant holomorphic sectional curvature are tubes over some submanifolds or geodesic hyperspheres if any principal curvature is constant.

§ 2. Preliminalies

First we shall review the definition of the function L_p on M for some point

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$p \in M$, which plays an impotant part to investigate the properties of transnormal submanifolds.

If M is an r-transnormal hypersurface and if there exists a point $p \in M$ satisfying the condition $C(p) \cap M = \Phi$, then the differential function L_p on M is defined by

$$L_p(x) = d_{\bar{M}}(p, x)^2 \quad \text{for any } x \in M,$$

where $C(p)$ is the cut locus of p in $\bar{M}$ and $d_{\bar{M}}$ denotes the distance function in $\bar{M}$. It is well known that any transnormal hypersurface has no intersection with its focal set. Therefore, the function L_p is the Morse function.

Next we describe relevant concept and formulas used for the proof of Main Theorem.

From now on, let $\bar{M}$ be a simply connected complete Kaehler manifold of negative constant holomorphic sectional curvature k (for convenience, we will assume $k=-4$), $\dim_C \bar{M} = m$ and M be a connected complete 2-transnormal real hypersurface in $\bar{M}$. Note that the cut locus $C(p)$ of any point $p \in M$ is empty because of the negativity of the holomorphic sectional curvature of $\bar{M}$. Then, for any point $p \in M$, the Morse function L_p can be defined.

Since M is 2-transnormal, for any point $x \in M$, there exists the unique point $\tilde{x} \in M$ such that $\tilde{x} \sim x$ and $\tilde{x} \neq x$. It is known that $\tilde{x}$ is a critical point of L_x, which is called *an antipodal point* of x, and we call $d_{\bar{M}}(x, \tilde{x})$ the width of M as a subset of $\bar{M}$, which is constant on M.

Let $\gamma(x, \tilde{x})$ be the minimizing normal geodesic segment from x to the antipodal point $\tilde{x}$ of x. We denote by $N(x)$ the initial vector $\gamma'(0)$ of $\gamma(x, \tilde{x})$ and $E(x) = JN(x)$, where J is the complex structure of $\bar{M}$. We call $N(x)$ *an inward unit normal vector* at x and $E(x)$ *an almost contact structure vector* at x.

Then, the Hessian H of $L_{\tilde{x}}$ at critical point x is given by

$$\begin{aligned} H(x, y) = {} & 2d\langle\{\coth(d)\cdot I - S_{N(x)}\}X, Y\rangle \\ & + 2d\cdot\tanh(d)\langle E(x), X\rangle\langle E(x), Y\rangle \\ & \qquad\qquad \text{for } X, Y \in M_x, \end{aligned}$$

where $d = d_{\bar{M}}(x, \tilde{x})$ and I denotes the identity transformation and S is the second fundamental tensor. See [5] for details.

In the sequel we assume that the almost contact structure vector $E(x)$ is a principal vector with the principal curvature $\lambda(x)$ at each point $x \in M$. Furthermore, we denote by $\nu(x, X)$ the principal curvature of M at x associated with the principal vector X orthogonal to $E(x)$. Then we have the following proposition.

PROPOSITION 2.1 (Lemma 4.3 of [5]) *At the antipodal point $\tilde{x}$ of x,*

$$(1) \qquad \lambda(\tilde{x}) = \frac{-2\sinh(2d) + \lambda(x)\cosh(2d)}{(\lambda(x)/2)\sinh(2d) - \cosh(2d)}$$

$$(2) \qquad \nu(\tilde{x}, \tilde{X}) = \frac{-\sinh(d) + \nu(x, X)\cosh(d)}{\nu(x, X)\sinh(d) - \cosh(d)}$$

where $\tilde{X}$ is the tangent vector of M at $\tilde{x}$ given by the parallel translation of X along $\gamma(x, \tilde{x})$ and $d = d_{\bar{M}}(x, \tilde{x})$.

Finally we shall consider some properties of a focal point of M. For each $p \in M$, let γ_p be the normal geodesic starting from p perpendicularly to M such that $\gamma'(0) = N(p)$.

PROPOSITION 2.2 *A point $x \in \bar{M}$ is a focal point of M along geodesic γ_p if and only if $x = \gamma_p(r)$ where $2\coth(2r) = \lambda(p)$ or $\coth(r) = \nu(p, X)$ for some non-zero principal curvature of M at p.*

PROOF. $\gamma_p(r)$ is a focal point of M along γ_p if and only if there exists a non-trivial (M, p)-Jacobi field along γ_p which vanishes at $\gamma_p(r)$. For a non-zero principal curvature of M at p, we can consider the (M, p)-Jacobi field

$$Y(t) = (\cosh(2t) - (\lambda(p)/2)\sinh(2t))J\gamma'(t) \qquad \text{or}$$
$$Z(t) = (\cosh(t) - \nu(p, X)\sinh(t))X(t),$$

where $X(t)$ is the parallel vector field along γ_p with $X(0) = X$ which is principal vector orthogonal to $E(p)$. Then we obtain the assertion. q.e.d.

REMARK 2.1 Since any transnormal hypersurface has no intersection with its focal set, for any point $x \in M$ the folloings are true;

$$2\cosh(2d) - \lambda(x)\sinh(2d) \neq 0$$
$$\cosh(d) - \nu(x, X)\sinh(d) \neq 0,$$

where d is a width of M as a subset of $\bar{M}$.

REMARK 2.2 From the form of Hessian of $L_{\tilde{x}}$ at critical point x, the index of $L_{\tilde{x}}$ at x is equal to the number of principal curvatures λ and ν of M at x with respect to $N(x)$ such that $\lambda > 2\coth(2d)$ or $\nu > \coth(d)$.

In the sequel, we label the principal curvatures ν from 1 to $2m-2$ as followings; $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_{2m-2}$.

PROPOSITION 2.3 *If for some point $x \in M$, the index of L_x at antipodal point $\tilde{x}$ is n, then, for any point $y \in M$, the index of L_y at $\tilde{y}$ is also n.*

PROOF. We assume that $\lambda(\tilde{x}) > 2\coth(2d)$. Then $\nu_i(\tilde{x}) > \coth(d)$ and $\nu_j(\tilde{x}) < \coth(d)$ $(1 \leqq i \leqq n-1,\ n \leqq j \leqq 2m-2)$ from Remark 2.2. In the sequel, adopt that

$1 \leqq a \leqq 2m-2$, $1 \leqq i \leqq n-1$ and $n \leqq j \leqq 2m-2$. Now we shall consider the set D of M such that

$$D=\{y \in M;\ \lambda(y)>2\coth(2d),\ \nu_i(y)>\coth(d) \text{ and } \nu_j(y)<\coth(d)\}.$$

Then D is open and closed. In fact, each λ and ν_a being continuous on M, for any point $x \in D$, there exists an open neighborhood of x in M contained in D. Thus D is open. Next, for $x \in \bar{D}$ (closure of D), let $\{x_m\}$ be a sequence in D such that $\lim_{m \to \infty} x_m = x$. Then, by the continuity of λ and ν_a, we have $\lim_{m \to \infty} \lambda(x_m)=\lambda(x) \geqq 2\coth(2d)$, $\lim_{m \to \infty} \nu_i(x_m)=\nu_i(x) \geqq \coth(d)$ and $\lim_{m \to \infty} \nu_j(x_m)=\nu_j(x) \leqq \coth(d)$. By Remark 2.1, we obtain that $\lambda(x)>2\coth(2d)$, $\nu_i(x)>\coth(d)$ and $\nu_j(x)<\coth(d)$. Thus D is closed. Hence $D=M$.

If $\lambda(x)<2\coth(2d)$, then it holds that $\nu_i(x)>\coth(d)$ and $\nu_j(x)<\coth(d)$ for $1 \leqq i \leqq n$ and $n+1 \leqq j \leqq 2m-2$. By the same way as above,

$$D=\{y \in M;\ \lambda(y)<2\coth(2d),\ \nu_i(y)>\coth(d) \text{ and } \nu_j(y)<\coth(d) \text{ for } 1 \leqq i \leqq n,\ n+1 \leqq j \leqq 2m-2\}$$

is open and closed. Hence $D=M$. q.e.d.

§ 3. Main Theorem

Now, we shall prove the following theorem using the results prepared.

THEOREM *Let $\bar{M}$ be a simply connected complete Kaehler manifold of negative constant holomorphic sectional curvature -4 and $\dim_C \bar{M}=m$. Let M be a connected complete 2-transnormal hypersurface of $\bar{M}$ and d be the width of M as a subset of $\bar{M}$. Suppose that, for a point $x \in M$, the index of L_x at the antipodal point $\tilde{x}$ is $n(\geqq 1)$. For each point $x \in M$, the almost contact structure vector $E(x)$ is assumed to be a principal vector with principal curvature $\lambda(x)$. Let $\nu_1(x) \geqq \nu_2(x) \geqq \cdots \geqq \nu_{2m-2}(x)$ be other principal curvatures at $x \in M$. Then we have followings.*

(1) *For each point of M, if $\lambda(>2\coth(2d))$, ν_i(for $1 \leqq i \leqq n-1$) and ν_j(for $n \leqq j \leqq 2m-2$) are bounded from either above or below by* $2\coth(d)$, $\coth(d/2)$ *and* $\tanh(d/2)$ *respectively, then M is a tube of radius $d/2$ over $(2m-n-1)/2$-dimensional complex totally geodesic submanifold.*

(2) *For each point of M, if $\lambda(<2\coth(2d))$, ν_i(for $1 \leqq i \leqq n$) and ν_j(for $n+1 \leqq j \leqq 2m-2$) are bounded from either above or below by* $2\tanh(d)$, $\coth(d/2)$ *and* $\tanh(d/2)$ *respectively, then M is a tube of radius $d/2$ over $(2m-n-1)$-dimensional anti-holomorphic totally geodesic submanifold. In particular, if $n=2m-1$ then this implies that M is a geodesic hypersphere with radius $d/2$.*

PROOF. First we consider only the following case of (1);

$$\lambda \geqq 2\coth(d),\ \nu_i \geqq \coth(d/2)\ \ (1 \leqq i \leqq n-1) \text{ and } \nu_j \geqq \tanh(d/2)\ \ (n \leqq j \leqq 2m-2).$$

From Proposition 2.1 and the above assumption,

$$\begin{aligned}\lambda(\tilde{x}) &= \frac{-2\sinh(2d)+\lambda(x)\cosh(2d)}{(\lambda(x)/2)\sinh(2d)-\cosh(2d)}.\\ &\geqq 2\coth(d)\\ &= 2(1+\cosh(2d))/\sinh(2d).\end{aligned}$$

Note here that $\lambda > 2\coth(2d)$, i.e. $(\lambda/2)\sinh(2d)-\cosh(2d)>0$. Then this inequality implies

$$\lambda(x) \leqq 2(1+\cosh(2d))/\sinh(2d) = 2\coth(d).$$

Thus we obtain $\lambda \equiv 2\coth(d)$.

Next we shall discuss ν_a. To begin with, we should note that $\nu(x, X) > \coth(d)$ implies $\nu(\tilde{x}, \tilde{X}) > \coth(d)$. In fact, we have the following inequality;

$$\nu(\tilde{x}, \tilde{X})\sinh(d) - \cosh(d) = 1/\{(\nu(x, X) - \coth(d))\sinh(d)\}.$$

Furthermore note that $\nu_i > \coth(d)$ and $\nu_j < \coth(d)$. Then, by the same way as above together with Proposition 2.1, we get $\nu_i \equiv \coth(d/2)$ and $\nu_j \equiv \tanh(d/2)$.

In seven other cases of (1) and all cases of (2), we can prove similarly that λ and $\nu_a (1 \leqq a \leqq 2m-2)$ are all constant.

Now, for $r \in \boldsymbol{R}$, we consider a map $F_r : M \longrightarrow \bar{M}$ by

$$F_r(x) = \exp(rN(x)) \qquad x \in M,$$

where $N(x)$ denotes the inward unit normal vector at x and exp is the exponential map on the normal bundle of M. By the way, if $\lambda = 2\coth(d)$ or $\nu = \coth(d/2)$, then (M, x)-Jacobi fields $Y(t)$ and $Z(t)$ along γ_x in the proof of Proposition 2.2 vanish in $t = d/2$. Hence the exponential map on the normal bundle of M is degenerate at $(d/2)N(x)$ for any point $x \in M$ in above situation, whose nullity is n. Therefore $F_{d/2}$ has constant rank $2m-n-1$. By the inverse function theorem, for $x_0 \in M$, there exists an open neighborhood W of x_0 such that $F_{d/2}(W) = V$ is a $(2m-n-1)$-dimensional real submanifold embedded in $\bar{M}$. Now, from Theorem 4.2 in [1] we can get the following fact; if $\lambda = 2\coth(d)$, then $JT_p^{\perp}V \subset T_p^{\perp}V$, that is, V is complex, or if $\lambda \neq 2\coth(d)$, then $JT_p^{\perp}V \subset T_pV$, that is, V is anti-holomorphic, where $T_p^{\perp}V$ is the complement of the tangent space T_pV of V at $p \in V$. From the completeness of M a global version can be obtained. Namely, in the case of (1) (resp. (2)) M is a tube of radius $d/2$ over $(2m-n-1)/2$-dimensional complex submanifold (resp. over $(2m-n-1)$-dimensional anti-holomorphic submanifold). Furthermore also we have the following facts in general. (See section 5 in [1]) ; principal curvatures of $F_r(M)$ are $2(\lambda\coth(2r)-2)/(2\coth(2r)-\lambda)$ and $(\nu_a\coth(r)-1)/(\coth(r)-\nu_a)$ for $\lambda\neq 2\coth(2r)$ and $\nu_a\neq\coth(r)$. Hence, substituting $r=d/2$, $\lambda=2\tanh(d)$ and $\nu_a=\tanh(d/2)$, we have that $(2m-n-1)$-principal curvatures of $F_{d/2}(M)$ are all zero in any cases. So $F_{d/2}(M)$ is totally geodesic and we can get the theorem. q.e.d.

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