TSUKUBA J. MATH.
Vol. 11 No. 2 (1987). 227—239

A CONSISTENCY PROOF OF A SYSTEM INCLUDING FEFERMAN'S ID_ξ BY TAKEUTI'S REDUCTION METHOD

By

Toshiyasu ARAI

This paper is a sequel to our [1] and [2].

Let $\prec$ be a p.r. (primitive recursive) well-ordering on a p.r. subset of the set of natural numbers $\mathbf{N}$, with the least element 0 and the largest element ξ which is used to denote the order type of the initial segment of $\prec$ determined by ξ. Let $\lambda x.x\oplus 1$ and $\lambda x.x\ominus 1$ be p.r. successor and predecessor functions with respect to $\prec$, respectively. Strictly speaking, we should suppose that some fixed p.r. definitions (indices) of $\prec$, $\lambda x.x\oplus 1$ and $\lambda x.x\ominus 1$ are given instead of their graphs. And we will assume that formulae which express the above facts except the well-orderedness of $<$ by using p.r. definitions of $\prec$, $\lambda x.x\oplus 1$ and $\lambda x.x\ominus 1$, are all derivable in a weak fragment of arithmetic, say, primitive recursive arithmetic. A complete list of formulae which should be derivable for our purpose can be found in [1, p. 20].

For such an ordering $\prec$, we define a first order theory AI_ξ^-. The lahgugage of the theory AI_ξ^- is described as follows. Let X be a unary predicate variable and Y a binary one. For each arithmetical formula $\mathfrak{B}(X, Y, a, b)$ having no free variables except X, Y, a and b, we introduce a binary predicate constant $Q^{\mathfrak{B}}$ whose intended meaning is the disjoint union of the family $\{Q_\zeta^{\mathfrak{B}}\}_{\zeta\prec\xi}$, where $Q_\zeta^{\mathfrak{B}}$ $(\zeta\prec\xi)$ are subsets of $\mathbf{N}$ defined by the following transfinite recursion on the ordinals (natural numbers) $\zeta\prec\xi$:

$$n\in_\zeta^{\mathfrak{B}} \text{ iff } \mathfrak{B}(\mathcal{X}, Q_{\prec\zeta}^{\mathfrak{B}}, \zeta, n) \text{ holds for every subset } \mathcal{X} \text{ of } \mathbf{N},$$

where $Q_{\prec\zeta}^{\mathfrak{B}}$ is the disjoint union of the family $\{Q_\nu^{\mathfrak{B}}\}_{\nu\prec\zeta}$.
Then the theory AI_ξ^- is obtained from the Peano Arithmetic PA in this language by adding an axiom scheme ($Q^{\mathfrak{B}}$-initial sequent in 1. 41. 21, below) and an inference rule ($Q^{\mathfrak{B}}$: right in 1. 41. 22, below) corresponding to the above mentioned meaning of $Q^{\mathfrak{B}}$

As is expected, Feferman's theory ID_ξ for the ξ-times iterated inductive definitions is interpretable in our AI_ξ^-. This is shown in 1.

In 2, we give a consistency proof of AI_ξ^- by the accessibility of the system of ordinal diagrams $O(\xi+1, 1)$ with respect to $<_o$. This is done by Takeuti's

Received October 14, 1986.

reduction method.

On the other hand, we showed in [1] that the transfinite induction up to each ordinal diagram from $O(\xi+1, 1)$ with respect to $<_o$, is derivable in an intuitionistic accessible-part theory $\mathrm{ID}_\xi^i(\mathfrak{A})$. Hence we have that the system of ordinal diagrams $O(\xi+1, 1)$ with respect to $<_o$ gives proof theoretic ordinal of Feferman's theory ID_ξ.

Applications such as provable well-orderings, reflection principles and conservation results, and generalization to the autonomous closure Aut(ID) will be reported elsewhere.

We will give an outline of proof, because it can be obtained by minor modifications of Takeuti's original proof.

The author wishes to express his heart-felt thanks to Prof. N. Motohashi for his invaluable advices and comments.

1. Preliminary

Firstly we specify the language LPA of the Peano Arithmetic PA.

DEFINITION 1.1. The language LPA consists of the following symbols:

1.11. Function constants: 0 (zero), ′ (successor), and the function constant f_e for each index e of each p.r. function.

1.12. Predicate constant: = (equality)

1.13. Variables:

Free number variables: a, b, $\cdots$.

Bound number variables: x, y, $\cdots$.

1.14. Logical symbols: $\neg$, $\wedge$, $\vee$, $\supset$, $\forall$ and $\exists$.

1.15. Auxiliary symbols: (,), ,(comma) and $\rightarrow$.

Terms, formulea and sequents in LPA are defined as usual [*cf.* PT, pp. 6–9].

DEFINITION 1.2. Let L be a first order language obtained from LPA by adding some predicate constants and variables. Then PA(L) denotes the formal system defined as follows:

1.21. Initial sequents of PA(L)·

1.21.1. *Logical initial sequent*:

$$D \rightarrow D$$

where D is an arbitrary formula of L.

1.21.2 *Equality axiom*:

$$s=t,\ F(s)\to F(t)$$

where s and t are arbitrary terms and F is an arbitrary formula of L.

1.21.3. *Mathematical initial sequents*:

$$\to t=t\,;\ 0'=0\to$$

and defining equations for p.r. functions.
For example, if a p.r. function f is defined from p.r. functions g and h by equations,

$$\begin{cases} f(a_0,\ \cdots,\ a_{n-1},\ 0)=g(a_0,\ \cdots,\ a_{n-1}), \\ f(a_0,\ \cdots,\ a_{n-1},\ b+1)=h(a_0,\ \cdots,\ a_{n-1},\ b,\ f(a_0,\ \ldots,\ a_{n-1},\ b)), \end{cases}$$

and $\bar{f}$, $\bar{g}$ and $\bar{h}$ are function constants corresponding to the definitions of f, g and h, respectively, then

$$\to \bar{f}(t_0,\ \cdots,\ t_{n-1},\ 0)=\bar{g}(t_0,\ \cdots,\ t_{n-1})$$

and

$$\to \bar{f}(t_0,\ \cdots,\ t_{n-1},\ s')=\bar{h}(t_0,\ \cdots,\ t_{n-1},\ s,\ \bar{f}(t_0,\ \cdots,\ t_{n-1},\ s)$$

are mathematical initial sequents for all terms $t_0,\ \cdots,\ t_{n-1}$ and s.

1.21.4. *Induction axiom*:

$$F(0),\ \forall x(F(x)\supset F(x'))\to F(t)$$

where t is an arbitrary term and F is an arbitrary formula of L.

1.22. The inference rules of PA(L) are those of Gentzen's LK in [PT, DEFINITION 2.1].

DEFINITION 1.3. Let LPA+ $\{X,\ Y,\ c_0,\ c_1\}$ be the language obtained from LPA by adding a unary predicate variable X, a binary predicate variable Y and two new individual constants c_0 and c_1.

1.31. A formula $\mathfrak{B}(X,\ Y,\ c_0,\ c_1)$ of LPA+ $\{X,\ Y,\ c_0,\ c_1\}$, where X, Y, c_0 and c_1 are fully indicated in $\mathfrak{B}(X,\ Y,\ c_0,\ c_1)$ (*cf.* [PT, DEFINITION 1.6]), is said to be an *arithmetical form* if it has no free number variables.

1.32. An arithmetical form $\mathfrak{A}(X,\ Y,\ c_0,\ c_1)$ is said to be a *positive operator form* if every occurrence of X in it is positive.

Let $\prec$ be a p.r. well-ordering with the largest element ξ as in the introduction. For such an ordering $\prec$ we define a formal system $\mathrm{AI}_\xi^-(\mathscr{B})$, where $\mathscr{B}$ is a set of arithmetical forms. Also, for readers' convenience, we repeat the definition of the formal system ID_ξ.

DEFINITION 1.4.

1.41. *Definition of* $\mathrm{AI}_\xi^-(\mathcal{B})$.

1.41.1. The language $\mathrm{LAI}_{\xi(\mathcal{B})}^-$ is obtained from LPA by adding a binary predibate constant $Q^{\mathfrak{B}}$ for each arithmetiɔal form $\mathfrak{B}$ in $\mathcal{B}$ and a countable list of unary predicate variables X_0, X_1, $\cdots$. For brevity we write

$$\mathrm{LAI}_{\xi(\mathcal{B})}^- : = \mathrm{LPA} + \{Q^{\mathfrak{B}} : \mathfrak{B} \in \mathcal{B}\} + \{X_i : i < \omega\}.$$

In the following, we use the letter X to denote one of the variables X_0, X_1, $\cdots$.

1.41.2. The system $\mathrm{AI}_\xi^-(\mathcal{B})$ is obtained from $\mathrm{PA}(\mathrm{LAI}_{\xi(\mathcal{B})}^-)$ by adding the following extra initial sequents called $Q^{\mathfrak{B}}$-initial sequent and the following new rule of inference called $Q^{\mathfrak{B}}$: right for each $\mathfrak{B}$ in $\mathcal{B}$.

1.41.21. $Q^{\mathfrak{B}}$*-initial sequent*:

$$t \prec \xi,\ Q^{\mathfrak{B}} ts \rightarrow \mathfrak{B}(V,\ Q^{\mathfrak{B}}_{\prec t},\ t,\ s)$$

where t and s are arbitrary terms, V is an arbitrary unary abstract of $\mathrm{LAI}_{\xi(\mathcal{B})}^-$, $t \prec \xi$ denotes the formula $f_e(t,\ \xi) = 0$ for a characteristic function f_e of $\prec$, ξ in the formula $f_e(t,\ \xi) = 0$ denotes the numeral corresponding to the number ξ and $Q^{\mathfrak{B}}_{\prec}t$ is the binary abstract defined by

$$Q^{\mathfrak{B}}_{\prec t} : = \{x,\ y\}(x \prec t \wedge Q^{\mathfrak{B}} xy).$$

Here $Q^{\mathfrak{B}} ts$ is called the principal formula of this sequent.

1.41.22. $Q^{\mathfrak{B}}$: *right.*

$$\frac{\Gamma \rightarrow \Delta,\ \mathfrak{B}(X,\ Q^{\mathfrak{B}}_{\prec t},\ t,\ s)}{\Gamma \rightarrow \Delta,\ Q^{\mathfrak{B}} ts}$$

where t and s are arbitrary terms, Γ, Δ are arbitrary finite sequences of formulae and the predicate variable X dose not occur in the lower sequent. X is called the eigenvariable of this inference and $Q^{\mathfrak{B}} ts$ is called the principal formula of this inference.

1.42. *Definition of* ID_ξ.

1.42.1. The language is defined by
$\mathrm{LID}_\xi : = \mathrm{LPA} + \{P^{\mathfrak{A}} : \mathfrak{A}$ is a positive operator form$\}$.

Here $P^{\mathfrak{A}}$ is a binary predicate constant.

1.42.2. The system ID_ξ is obtained from $\mathrm{PA}(\mathrm{LID}_\xi)$ by adding the following initial sequents $(P^{\mathfrak{A}}.\ 1)_\xi$, $(P^{\mathfrak{A}}.\ 2)_\xi$ and $(\mathrm{TI})_\xi$ for each positive operator form $\mathfrak{A}$:

$$(P^{\mathfrak{A}}.\ 1)_\xi \quad \longrightarrow \quad \forall x \prec \xi (\mathfrak{A}(P^{\mathfrak{A}}_x,\ P^{\mathfrak{A}}_{\prec x},\ x) \subseteq P^{\mathfrak{A}}_x)$$

where $\forall x \prec \xi (\mathfrak{A}(P^{\mathfrak{A}}_x,\ P^{\mathfrak{A}}_{\prec x},\ x) \subseteq P^{\mathfrak{A}}_x)$ is an abbreviation for the formula $\forall x \prec \xi \forall y \mathfrak{A}(P^{\mathfrak{A}}_x,\ P^{\mathfrak{A}}_{\prec x},\ x,\ y) \supset P^{\mathfrak{A}}_x y)$ and $P^{\mathfrak{A}}_a$, $P^{\mathfrak{A}}_{\prec a}$ are abstracts defined by

$$P^{\mathfrak{A}}_a : = \{x\}(P^{\mathfrak{A}} ax)$$

$P^{\mathfrak{A}}_{\prec a} := \{x, y\}(x \prec a \wedge P^{\mathfrak{A}}xy)$.

$(P^{\mathfrak{A}}.\ 2)_\xi \qquad \rightarrow \forall x \prec \xi(\mathfrak{A}(V, P^{\mathfrak{A}}_{\prec x}, x) \subseteq V.\ \supset.\ P^{\mathfrak{A}}_x \subseteq V)$

for each unary abstract V of LID_ξ.

$(\mathrm{TI})_\xi \qquad \rightarrow \forall x \prec \xi(\forall y \prec x F(y) \supset F(x)) \supset \forall x \prec \xi F(z)$

for each formula F of LID_ξ.

If $\mathscr{B}$ is the set of all arithmetical forms, then we write AI_ξ^- for $\mathrm{AI}_\xi^-(\mathscr{B})$. If $\mathscr{B}$ is a singleton $\{\mathfrak{B}\}$, then we write $\mathrm{AI}_\xi^-(\mathfrak{B})$ for $\mathrm{AI}_\xi^-(\mathscr{B})$

Let $\mathfrak{B}_0$ be the arithmetical form defined by

$\mathfrak{B}_0(X, Y, c_0, c_1) := \forall x \prec c_0 Yxc_1 \supset T(X, c_0)$

where $T(X, c_0)$ denotes the formula $\forall x \prec \xi(\forall y \prec x Xy \supset Xx) \supset Xc_0$ (*cf.* [3, p. 334]).

Then the transfinite induction up to ξ is derivable in $\mathrm{AI}_\xi^-(\mathscr{B})$ for every $\mathscr{B}$ containing $\mathfrak{B}_0$.

PROPOSITION 1.5 (Takeuti) *The formula* $\forall x \prec \xi T(\{y\}(F(y)), x)$ *is derivable in* $AI_\xi^-(\mathscr{B})$ *for every formula* F *of* $L_{AI_\xi^-(\mathscr{B})}$ *if* $\mathscr{B}$ *contains* $\mathfrak{B}_0$.

PROOF. Put

$$C(a) := Q^{\mathfrak{B}_0}a_0.$$

From the $Q^{\mathfrak{B}_0}$-initial sequent we have

$$a \prec \xi,\ C(a) \rightarrow \forall x \prec a C(x) \supset T(\{y\}(F(y)), a)$$

for every formula F of $\mathrm{LAI}_{\xi(\mathscr{B})}^-$.

If follows that

$$a \prec \xi,\ \forall x \prec a C(x) \rightarrow T(\{y\}(F(y)), a)$$

and

$$a \prec \xi,\ \forall x \prec a C(x) \rightarrow \forall x \prec a T(\{y\}(F(y)), x).$$

From the definition of the formula T, we have

$$a \prec \xi,\ \forall x \prec a T(\{y\}(F(y)), x) \rightarrow T(\{y\}(F(y)), a).$$

Hence

$$a \prec \xi,\ \forall x \prec a C(x) \rightarrow T(\{y\}(F(y)), a). \qquad (1)$$

Since F is arbitrary, we could take X instead of F,

$$a \prec \xi,\ \forall x \prec a C(x) \rightarrow T(X, a),$$

nomely

$$a \prec \xi \rightarrow \mathfrak{B}_0(X, Q^{\mathfrak{B}_0}_{\prec a}, a, 0)$$

By $Q^{\mathfrak{B}_0}$: right

$$a \prec \xi \rightarrow Q^{\mathfrak{B}_0}a_0.$$

Hence we have

$$\rightarrow \forall x \prec \xi C(x) \qquad (2)$$

From (1) and (2) we conclude that

$$\to \forall x \prec^{\xi} T(\{y\}(F(y)),\ x).$$

q.e.d.

Let $\{\mathfrak{A}_1, \mathfrak{A}_2, \cdots\}$ be an enumeaation of all positive operator forms. Now we define the arithmetical form $\mathfrak{B}_n$ for each $n \geq 1$ by

$$\begin{aligned}\mathfrak{B}_n(X,\ Y,\ c_0,\ c_1) &: = \mathfrak{A}_n(X,\ Y,\ c_0) \subseteq X . \supset X c_1 \\ &: = \forall y(\mathfrak{A}_n(X,\ Y,\ c_0,\ y) \supset Xy) \supset X c_1.\end{aligned}$$

Let $\mathcal{B}_0$ be $\{\mathfrak{B}_i : i < \omega\}$, where $\mathfrak{B}_0$ is the arithmetical form defined above. Then we have:

COROLLARY 1.6. *For each sequent* $\Gamma(\cdots, P^{\mathfrak{A}_n}, \cdots) \to \Delta(\cdots, P^{\mathfrak{A}_n}, \cdots)$ *of* L_{ID_ξ}, *Let* $\Gamma(\cdots, Q^{\mathfrak{B}_n}, \cdots) \to \Delta(\cdots, Q^{\mathfrak{B}_n}, \cdots)$ *denote the sequent of* $L_{AI_\xi^-}(\mathcal{B}_0/\{\mathfrak{B}_0\})$ *obtained from* $\Gamma(\cdots, P^{\mathfrak{A}_n}, \cdots) \to \Delta(\cdots, P^{\mathfrak{A}_n}, \cdots)$ *by replacing every* $P^{\mathfrak{A}_n}$ *by* $Q^{\mathfrak{B}_n}$. *Then*

$\Gamma(\cdots, P^{\mathfrak{A}_n}, \cdots) \to \Delta(\cdots, P^{\mathfrak{A}_n}, \cdots)$ *is derivable in* ID_ξ

iff

$\Gamma(\cdots, Q^{\mathfrak{B}_n}, \cdots) \to \Delta(\cdots, Q^{\mathfrak{B}_n}, \cdots)$ *is derivable in* $AI_\xi^-(\mathcal{B}_0)$.

We omit a proof of this corollary because its only-if part readily follows from Proposition 1.5 and a usual argument, and the other half will not be used in the following.

Note that we can easily see ID_ξ is interpretable in $\mathrm{AI}_\xi^-(\mathcal{B}_0)$ by this Corollary.

2. A consistency proof of AI_ξ^-

Let $\mathfrak{B}$ be an arbitrary but fixed arithmetical form. In this section we will give a consistency proof of $\mathrm{AI}_\xi^-(\mathfrak{B})$ by Takeuti's reduction method. Here note the following well-known propositiom.

PROPOSITION 2.1. *Let F be a formula of* L_{PA}. *If F is derivable in* AI_ξ^-, *then there exists an arithmetical farm* $\mathfrak{B}$ *such that F is derivable in* $AI_\xi^-(\mathfrak{B})$.

For simplicity we will write Q for $Q^{\mathfrak{B}}$.

We add the following sequents to the mathematical initial sequents of $\mathrm{AI}_\xi^-(\mathfrak{B})$.

$$s = t \to$$

where s and t are closed terms and under the standard interpretation $s \neq t$ holds. Since $s \prec t$ denotes the formula $f_e(s,\ t) = 0$, if $s \not\prec t$ holds, then $s \prec t \to$ one of the mathematical initial sequents.

And we add the inference rule, called *term-replacement*:

$$\frac{\Gamma(s)\to\Delta(s)}{\Gamma(t)\to\Delta(t)}$$

where s and t are closed terms whose values under the standard interpretation coincide, and $\Gamma(t)\to\Delta(t)$ denotes the sequent obtained from $\Gamma(s)\to\Delta(s)$ by replacing some occurrences of s by t.

Furthermore we add the inference rule, called *substitution*:

$$\frac{\Gamma \quad \to \Delta}{\Gamma\binom{X}{V}\to \Delta\binom{X}{V}}$$

where $\Gamma\binom{X}{V}\to \Delta\binom{X}{V}$ denotes the sequent obtained from $\Gamma\to\Delta$ by substituting a unary abstract V for X in $\Gamma\to\Delta$. V may be an arbitrary abstract of $\mathrm{LAI}_{\xi}^{-}(\mathfrak{B})$. Here X is called the eigenvariable of this substitution.

If any confusion dose not likely to occur, the system modified in this way is also denoted by $\mathrm{AI}_{\xi}^{-}(\mathfrak{B})$.

DEFINITION 2.2.

2.21. The *grade* of a formula F, denoted by $g(F)$, is the number of occurrences of logical symbols in it.

2.22. Let P be a proof in $\mathrm{AI}_{\xi}^{-}(\mathfrak{B})$ and S a sequent in P. The *hsight of S in P*, denoted by $h(S;P)$ or simply $h(S)$, is defined inductively 'from below to above', as follows:

2.22.1. $h(S)=0$

if S is the end-sequent of P, or S is the upper sequent of a substitution.

2.22.2. $h(S)=h(S')$

if S is an upper sequent of an inference except substitution and cut, where S' is the lower sequent of the inference.

2.22.3. $h(S)=max\{h(S'), g(D)\}$

if S is an upper sequent of a cut, where S' is the lower sequent of the cut and D is the cut formula of the cut.

DEFINITION 2.3. A semi-term t_1 is said to be *numequivalent* to a semi-term t_2 if there exist a semi-term $t(x_0, \cdots, x_{n-1})$ and closed terms $s_0, r_0, \cdots, s_{n-1}, r_{n-1}$ such that for every $m<n$ the value of s_m is equal to that of r_m, t_1 is $t(s_0, \cdots, s_{n-1})$ and t_2 is $t(r_0, \cdots, r_{n-1})$.

DEFINITION 2.4. The *degree* of a semi-formula F, denoted by $d(F)$, is defined inductively as follows:

2.41. $d(t=s):=d(Xt):=0$

for all semi-terms t, s and variable X.

2.42. $d(Qts):=\begin{cases} i\oplus 1 & \text{if } t \text{ is a closed term whose value is } i \text{ and } i\prec\xi \text{ holds,}\\ \xi & \text{otherwise.}\end{cases}$

2.43. $d(t_1\prec s\wedge Qt_2r):=\begin{cases} i & \text{if } s \text{ is a closed term whose value is } i\prec\xi \text{ and } t_1 \text{ is numequevalent to } t_2,\\ \xi & \text{otherwise.}\end{cases}$

2.44. $d(B\wedge C):=max_{\prec}\{d(B),\ d(C)\}$

if $B\wedge C$ is not of the form in 2.43, where $max_{\prec}$ denotes the maximum with respect to $\prec$.

2.45. $d(\neg B):=d(B)$; $d(B\vee C)$; $=d(B\supset C):max_{\prec}\{d(B),\ d(C)\}$

2.46. $d(\forall xB):=d(\exists xB):=d(B)$.

The degree of a semi-formula is an ordinal$\preceq\xi$ (in fact it is a natural number). Note that the following holds for every semi-formula $B(x)$:

$$d(B(x))\prec\xi\Rightarrow d(B(t))=d(B(x)) \text{ for every semi-term } t.$$

In what follows, we assume that P is a proof of the empty sequent $\rightarrow$ and d is a mapping from the set of substitutions in P to the set of ordinals$\prec\xi$ (natural numbers).

DEFINITION 2.5. We call the pair $\langle P, d\rangle$ *a proof with degree* if the following condition is satisfied:

For every substitution J in P and every formula B in the upper sequent of J,

$$d(B)\preceq d(J)$$

holds.

Here note that $d(J)$ is the value of the mapping d at J.

Let $O(\xi+1,\ 1)$ be the system of o.d.'s (ordinal diagrams) based on I and 1, where I is the field of $\prec$, i.e., $I=\{n\in\mathbf{N}: n\preceq\xi\}$. Then we assign an o.d. from $O(\xi+1,\ 1)$ to a proof with degree. For simplicity we write $(i,\ \mu)$ for each non-zero connected o.d. $(i,\ 0,\ \mu)$.

DEFINITION 2.6.

2.61. For each $i\prec\xi$, we define a binary relation $\ll_i$ on the set of o.d.'s by

$$\mu\ll_i\nu \text{ iff } \mu<_j\nu \text{ holds for every } j \text{ with } i\preceq j\preceq\xi.$$

2.62. Let μ be an o.d. and n a natural number. Then an o.d. $\xi(n,\ \mu)$ is defined inductively, as follows:

$$\xi(0,\ \mu):=\mu\quad \xi(n+1,\ \mu):=(\xi,\ \xi(n,\ \mu)).$$

The following proposition is easily verified.

PROPOSITION 2.7. *For all o.d.'s* μ, ν, θ *and every* $i<\xi$,

2.71. *if* $\mu\leqslant_i\nu$, *then* $\xi(n, \mu)\leqslant_i\xi(n, \nu)$ *and* $\xi(n, \mu\#\theta)\leqslant_i\xi(n, \nu\#\theta)$.

2.72. *if* $\mu\leqslant_i\nu$ *and* $i\leq j<\xi$, *then* $(j, \mu)\leqslant_i(j, \nu)$.

2.73. *if* $\mu\#0\leqslant_0\nu$, *then* $\xi(n, \mu\#\theta)\#0\leqslant_0\xi(n, \nu\#\theta)$ *and* $(i, \mu)\#0\leqslant_0(i\ \nu)$.

DEFINITION 2.8. Let $\langle P, d\rangle$ be a proof with degree. To each sequent S and each line of an inferench J in P, we will assign o.d.'s in $O(\xi+1, 1)$, denoted by $O(S; P, d)$ and $O(J; P, d)$, or simply $O(S)$ and $O(J)$, inductively 'from aboved to below', as follows:

2.81. Let S be an initial sequent in P.

2.81.1. $O(S)=(0, 0)$
if S is an induction axiom.

2.81.2. $O(S)=(\xi, 0)$
if S is a Q-initial sequent.

2.81.3. $O(S)=0$ otherwise.

2.82. Suppose that the o.d.'s of the upper sequents of an inference J has been assigned, and let J be of the form

$$\frac{S'\ \ (S'')}{S}\ J.$$

The o.d.'s $O(J)$ and $O(S)$ are then determined, as follows:

2.82.1. If J is a weak structural inference or term-replacement, then $O(J)$ is $O(S')$

2.82.2. If J is a logical inference with one upper sequent or Q : right, then $O(J)$ is $O(S')\#0$.

2.82.3. If J is a cut, $\vee$: left or $\supset$: left, then $O(J)$ is $O(S')\#O(S'')$.

2.82.4. If J is a $\wedge$: right, then $O(J)$ is $O(S')\#O(S'')\#0$.

2.82.5. If J is a substitution, then $O(J)$ is $(\xi, O(S'))$.

2.82.6. If J is not a substitution, then $O(S)$ is $\xi(h(S')-h(S), O(J))$.

2.82.7. If J is a substitution, then $O(S)$ is $(d(J), O(J))$.

And the o.d. $O(P, d)$ of $\langle P, d\rangle$ is defined to be $(\xi, O(S; P, d))$ where S is the end-sequent of P.

MAIN LEMMA. *If* $\langle P, d\rangle$ *is a proof with degree, then we can construct another proof with degree* $\langle P', d'\rangle$ *such that*

$$O(P', d')\leqslant_0 O(P, d)$$

hence a fortiori

$$O(P', d')<_0 O(P, d).$$

From this lemma the consistency of $\mathrm{AI}_{\xi}^{-}(\mathfrak{B})$ follows, since the system $O(\xi+1, 1)$ with respect to $<_0$ is accessible. (*cf.* Corollary 1.6 and Proposition 2.1.)

PROOF of the Main Lemma.

M1. We substitute 0 for every free number variable in P except if it is used as an eigenvariable. Then the resulting figure under the 'same' degree-assignment, is also a proof with degree and the o.d. does not change. Here note the remark after Definition 2.4.

M2. Remember the definition of the end-piece of a proof P ending with $\rightarrow$. The end-piece of P consists of the following trunk of proof tree:

i) the end-sequent of P belongs to the end-piece of P;

ii) if the lower sequent of a structural inference, term-replacement or substitution belongs to the end-piece of P, so do its upper sequents.

Suppose P contains an induction axiom in its end-piece. Then P' is defined by an obvious way. The o.d. decreases.

M3. Suppose P contains an equality axiom, logical initial sequent or weakening in its end-piece. Then the reduction steps are defined as usual (*cf.* [2, p. 26]).

M4. Suppose the end-piece of P contains neither weakening nor initial sequent other than mathematical or Q-initial one. Then P differs from its end-piece and contains a suitable cut J. Here a suitable cut is a cut in the end-piece of P satisfying:

both of its cut formulae have ancestors which are principal formulae of

i) boundary logical inferences,

or

ii) a boundary Q: right and Q-initial sequent.

Remember that a boundary inference in P is an inference whose lower sequent belongs to the end-piece of P but not its upper sequents.

Let D be the cut formula of a suitable cut J.

M41. D is of the form Qts.

Then t is a closed term by M1. Let j be the value of t. Let P be the following form:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \xrightarrow{\lambda} \Delta_1, \mathfrak{B}(X, Q_{<t_1}, t_1, s_1)\end{array}}{\begin{array}{c}\Gamma_1 \xrightarrow{\lambda \# 0} \Delta_1, Qt_1s_1 \\ \vdots \\ \Gamma_2 \xrightarrow{\tau} \Delta_2, Qts\end{array}} \qquad \begin{array}{c} t_2 < \xi, Qt_2s_2 \xrightarrow{(\xi, 0)} \mathfrak{B}(V, Q_{<t_2}, t_2, s_2) \\ \vdots \\ Qts, \Pi \xrightarrow{\rho} \Lambda \end{array}}{\begin{array}{c}\Gamma_2, \Pi \to \Delta_2, \Lambda \\ \vdots \\ \Gamma_3 \underset{(\xi, \nu)}{\xrightarrow{\nu}} \Delta_3 \\ \vdots \\ \underset{(\xi, \sigma)}{\xrightarrow{\sigma}} \end{array}}
$$

where $\Gamma_3 \to \Delta_3$ is the i-resolvent of $\Gamma_2, \Pi \to \Delta_2, \Lambda$, i being $d(\mathfrak{B}(X, Q_{<t}, t, s))$.

Remember the definition of the i-resolvent of a sequent S in a proof with degree $<P, d>$. The i-resolvent of S is the upper sequent of the uppermost substitution J under S whose degree $d(J)$ is not greater than i, i.e., $d(J) \leq i$, if such exists; otherwise, the i-resolvent of S is the end-sequent of P.

M41.1 $j < \xi$.

Let P' be the following:

$$
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots \\ \Gamma_1 \xrightarrow{\lambda} \Delta_1, \mathfrak{B}(X, Q_{<t_1}, t_1, s_1)\end{array}}{\begin{array}{c}\Gamma_1 \to \mathfrak{B}(X, Q_{<t_1}, t_1, s_1), \Delta_1, Qt_1s_1 \\ \vdots \\ \Gamma_2 \xrightarrow{\tau'} \mathfrak{B}(X, Q_{<t}, t, s), \Delta_2\, Qts\end{array}} \qquad \begin{array}{c}\vdots \\ Qts, \Pi \xrightarrow{\rho} \Lambda\end{array}\end{array}}{\begin{array}{c}\Gamma_2, \Pi \to \mathfrak{B}(X, Q_{<t}, t, s), \Delta_2\, \Lambda \\ \vdots \\ \dfrac{\Gamma_3 \underset{(\xi, \theta)}{\xrightarrow{\theta}} \Delta_3, \mathfrak{Q}(X, Q_{<t}, t, s)}{\dfrac{\Gamma_3 \xrightarrow{(i, (\xi, \theta))} \Delta_3\, \mathfrak{B}(V, Q_{<t}, t, s)}{t_2 < \xi, Qt_2s_2, \Gamma_3 \to \Delta_3\, \mathfrak{B}(V, Q_{<t_2}, t_2, s_2)}}\end{array}}
$$

$$
\dfrac{\begin{array}{c}\vdots \\ \Gamma_2 \xrightarrow{\tau} \Delta_2, Qts\end{array} \qquad \begin{array}{c}\vdots \\ Qts, \Pi, \Gamma_3 \xrightarrow{\rho'} \Delta_3, \Lambda\end{array}}{\dfrac{\Gamma_2, \Pi, \Gamma_3 \to \Delta_2, \Delta_3, \Lambda}{\begin{array}{c}\Gamma_2, \Pi, \Gamma_3 \to \Delta_3, \Delta_2, \Lambda \\ \vdots \\ \dfrac{\Gamma_3, \Gamma_3 \to \Delta_3, \Delta_3}{\begin{array}{c}\Gamma_3 \underset{(\xi, \nu)}{\xrightarrow{\nu'}} \Delta_3 \\ \vdots \\ \underset{(\xi, \sigma')}{\xrightarrow{\sigma'}}\end{array}}\end{array}}}
$$

From $j \underline{\succ} i$ we see that $(\xi, \sigma') \leqslant_0 (\xi, \sigma)$ as usual (*cf.* [2, p. 28]).

M41.2 $j \not\prec \xi$.

Replace the Q-initial sequent by

$$\frac{t_2 \prec \xi \qquad \overset{0}{\rightarrow}}{t_2 \prec \xi,\ Q t_2 s_2 \rightarrow \mathfrak{B}(V, Q_{<t_2}, t_2, s_2)}$$

M42. D is of the form $t_1 \prec s \wedge Q t_2 r$ and t_1 is numequivalent to t_2.

Then t_1, t_2 and s are closed terms by M1, and the value of t_1 equals to that of t_2. Let i and j be the values of t_1, s, respectively. Let P be the following:

$$J\ \frac{\begin{array}{c}\vdots \\ \dfrac{\begin{array}{cc}\vdots & \vdots \\ \Gamma_1 \rightarrow \Delta_1, B_1 & \Gamma_1 \rightarrow \Delta_1, B_2\end{array}}{\Gamma_1 \rightarrow \Delta_1, B_1 \wedge B_2} \\ \vdots \\ \Gamma_2 \rightarrow \Delta_2, t_1 \prec s \wedge Q t_2 r\end{array} \qquad \begin{array}{c}\vdots \\ \dfrac{C_n, \Pi_1 \rightarrow \Lambda_1}{C_1 \wedge C_2, \Pi_1 \rightarrow \Lambda_1} \\ \vdots \\ t_1 \prec s \wedge Q t_2 r, \Pi_2 \rightarrow \Lambda_2\end{array}}{\begin{array}{c}\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2 \\ \vdots \\ \Phi \rightarrow \Psi \\ \vdots \\ \rightarrow\end{array}}$$

where $\Phi \rightarrow \Psi$ denotes the uppermost sequent below J whose height is less than that of the upper sequent of J $(n=1, 2)$.

M42.1 $i \not\prec j$.

Let P' be the following:

$$\frac{\dfrac{\dfrac{\begin{array}{c}\vdots \\ \dfrac{\begin{array}{c}\vdots \\ \Gamma_1 \rightarrow \Delta_1, B_1\end{array}}{\Gamma_1 \rightarrow B_1, \Delta_1, B_1 \wedge B_2} \\ \vdots \\ \Gamma_2 \rightarrow t_1 \prec s, \Delta_2, t_1 \prec s \wedge Q t_2 r\end{array} \qquad \begin{array}{c}\vdots \\ t_1 \prec s \wedge Q t_2 r, \Pi_2, \rightarrow \Lambda_2\end{array}}{\Gamma_2, \Pi_2 \rightarrow t_1 \prec s, \Delta_2, \Lambda_2}}{\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2, t_1 \prec s} \qquad t_1 \prec s \rightarrow}{\begin{array}{c}\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2 \\ \vdots \\ \rightarrow\end{array}}$$

By Proposition 2.3, the o.d. decreases.

M42.2 $i \prec j$.

M42.21 $n=2$.

Let P' be the following:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\dfrac{\Gamma_1 \to \Delta_1, B_2}{\Gamma_1 \to B_2, \Delta_1 \; B_1 \wedge B_2} & & \dfrac{C_2, \Pi_1 \to \Lambda_1}{C_1 \wedge C_2, \Pi_1, C_2 \to \Lambda_1}
\end{array}
$$

$$
\begin{array}{cc}
\vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\
\dfrac{\Gamma_2 \to Qt_2r, \Delta_2, t_1 < s \wedge Qt_2r \quad t_1 < s \wedge Qt_2r, \Pi_2 \to \Lambda_2}{\Gamma_2, \Pi_2 \to Qt_2r, \Delta_2, \Lambda_2} & \dfrac{\Gamma_2 \to \Delta_2, t_1 < s \wedge Qt_2r \quad t_1 < s \wedge Qt_2r, \Pi_2, Qt_2r \to \Lambda_2}{\Gamma_2 \; \Pi_2, Qt_2r \to \Delta_2, \Lambda_2} \\
\vdots & \vdots
\end{array}
$$

$$
\dfrac{\dfrac{\Phi \to Qt_2r, \Psi}{\Phi \to \Psi, Qt_2r} \qquad \dfrac{\Phi, Qt_2r \to \Psi}{Qt_2r, \Phi \to \Psi}}{\dfrac{\Phi, \Phi \to \Psi, \Psi}{\Phi \to \Psi}}
$$

$$
\begin{array}{c}
\vdots \\
\to
\end{array}
$$

We assign the same degree as the corresponding substitution in P to every substitution in P'. To see that $<P', d'>$ is a proof with degree, note that if $j < \xi$, then

$$d(Qt_2r) = i \oplus 1 < j = d(t_1 < s \wedge Qt_2r).$$

We see that the o.d. decreases by the usual calculation.

M42.22 $n=1$.

The case is treated in the same way as M42.21 but simpler.

M43. D is of the from $B \wedge C$ but not the case in M42.

M44. D is one of the forms $\neg B$, $B \vee C$ and $B \supset C$.

M45. D is one of the forms $\forall x B$ and $\exists x B$.

These cases M43–45 are treated as usual. In M45, note the remark after Definition 2.4.

This completes a proof of Main Lemma.

References

[PT] Takeuti, G., Proof Theory, North-Holland, Amsterdam. 1975.

[1] Arai, T., An accessibility proof of ordinal diagrams in intuitionistic theories for iterated inductive definitions, this Journal **8** (1984), 209-218.

[2] Arai, T., A subsystem of classical analysis proper to Takeuti's reduction method for Π_1^1-analysis, this Journal **9** (1985), 21-29.

[3] Takeuti, G., On the inductive definition with quantifiers of second order, J. Math. Soc. Japan **13** (1961), 333-341.

Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki,
305 Japan

Current Adress: Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya, 464 Japan