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STABLE EQUIVALENCE BETWEEN UNIVERSAL COVERS OF TRIVIAL EXTENSION SELF-INJECTIVE ALGEBRAS

By

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Introduction.

Let A be an indecomposable basic artin algebra and T_A a basic tilting module with $B=\mathrm{End}(T_A)$. Let us denote by R and S the trivial extension self-injective algebras $A\ltimes DA$ and $B\ltimes DB$, respectively. In the papers [24] and [22], H. Tachikawa and the author have proved that there is a stably equivalent functor $\mathcal{S}: \underline{\mathrm{mod}}\text{-}R\to\underline{\mathrm{mod}}\text{-}S$ and the restriction of $\mathcal{S}$ to the tilting torsion class $\mathcal{T}=\{X\in\mathrm{mod}\text{-}A\,|\,\mathrm{Ext}_A^1(T, X)=0\}$ coincides with that of the tilting functor $\mathrm{Hom}_A(T, ?)$.

D. Hughes and J. Waschbüsch [18] introduced the following doubly infinite matrix algebra:

$$\hat{A}=\begin{pmatrix} \ddots & \ddots & & & 0 \\ & A_{n-1} & M_{n-1} & & \\ & & A_n & M_n & \\ & & & A_{n+1} & M_{n+1} \\ & & & & \ddots \quad \ddots \\ 0 & & & & \ddots \end{pmatrix}$$

in which matrices are assumed to have only finitely many entries different from zero, $A_n=A$ and $M_n=DA$ for all integers n, all the remaining entries are zero, and multiplication is induced from the canonical maps $A\otimes_A DA\xrightarrow{\sim}DA$, $DA\otimes_A A\xrightarrow{\sim}DA$ and zero maps $DA\otimes_A DA\to 0$.

The identity maps $A_n\to A_{n+1}$, $M_n\to M_{n+1}$ induce an algebra isomorphism ν_A of A. The orbit space $\hat{A}/\nu_A$ is easily seen to be the trivial extension algebra R. Similarly, we can consider the orbit space $A/(\nu_A)^n$ as a self-injective algebra and it is denoted by R_n for each $n=1, 2, \cdots, \infty$. Notice that $R_1=R$ and $R_\infty=\hat{A}$.

The aim of this article is to prove the existence of a stably equivalent functor $\mathcal{S}_n: \underline{\mathrm{mod}}\text{-}R_n\to\underline{\mathrm{mod}}\text{-}S_n$ for each n. Here S_n is an orbit space $B/(\nu_B)^n$. The desired functor $\mathcal{S}_n$ will be defined by slightly modifying the definition of the functor $\mathcal{S}=\mathcal{S}_1$ in [24] and [22].

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In order to relate the categories mod-$\hat{A}$ and mod-R, Hughes-Waschbüsch used the exact functor Φ: mod-$\hat{A}\to$mod-R which preserves the indecomposability and the composition length of a module and also almost split sequences and irreducible maps. Similarly to the functor Φ, we can define the functors Φ_n: mod-$\hat{A}\to$mod-R_n and $\Phi_{m,n}$: mod-$R_{m\cdot n}\to$mod-R_n. We shall show that the functors $\mathcal{S}_1=\mathcal{S}$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\cdots$, $\mathcal{S}_\infty$ make the following diagrams commutative:

$$\begin{tikzcd}
\text{mod-}R_\infty \arrow[dr, "\Phi_{m\cdot n}"] \arrow[drr, "\Phi_n"] \arrow[ddd, "\mathcal{S}_\infty"'] & & \\
& \text{mod-}R_{m\cdot n} \arrow[r, "\Phi_{m,n}"'] \arrow[d, "\mathcal{S}_{m\cdot n}"] & \text{mod-}R_n \arrow[d, "\mathcal{S}_n"] \\
& \text{mod-}S_{m\cdot n} \arrow[r, "\Phi_{m,n}"] & \text{mod-}S_n \\
\text{mod-}S_\infty \arrow[ur, "\Phi_{m\cdot n}"] \arrow[urr, "\Phi_n"'] & &
\end{tikzcd}$$

It should be noted that the functor Φ is not dense in general, though in the case where R is representation-finite or A is hereditary $\Phi=\Phi_1$ is dense and $\mathcal{S}=\mathcal{S}_1$ is induced from $\mathcal{S}_\infty$.

Recently, D. Happel [15] has proved that $\underline{\mathrm{mod}}$-$\hat{A}$ and $\underline{\mathrm{mod}}$-$\hat{B}$ are equivalent if gl. dim. $A<\infty$. But, since Φ is not dense in general even if gl. dim. $A<\infty$, our results does not follow from his one. At the end of this paper such an example will be given.

Throughout this paper, we fix a commutative artin ring K and all algebras are assumed to be artin K-algebras except R_∞ and S_∞, and modules are finitely generated over K and morphisms operate on the opposite side of the scalars. The ordinary duality functor is always denoted by D.

1. Preliminaries

In this section, we shall recall some of basic results on tilting theory and trivial extension algebras for the later use.

Let T_A be a tilting module in the sense of Happel-Ringel [16]. Put $B=\mathrm{End}(T_A)$, then ${}_BT$ is again a tilting module with $\mathrm{End}({}_BT)=A$. Let us put $\mathcal{T}=\{X\in\text{mod-}A\,|\,\mathrm{Ext}_A^1(T, X)=0\}$, $\mathcal{F}=\{X\in\text{mod-}A\,|\,\mathrm{Hom}_A(T, X)=0\}$, $\mathcal{X}=\{Y\in\text{mod-}B\,|\,Y\otimes_BT=0\}$ and $\mathcal{Y}=\{Y\in\text{mod-}B\,|\,\mathrm{Tor}_1^B(Y, T)=0\}$. Further, let $F=\mathrm{Hom}_A(T, ?)$, $F'=\mathrm{Ext}_A^1(T, ?)$ (resp. $G=(?\otimes_BT)$, $G'=\mathrm{Tor}_1^B(?, T)$) be functors from mod-$A$ (resp. mod-B) to mod-B (resp. mod-A). Then there are short exact sequences of functors

$$0 \longrightarrow GF \xrightarrow{\varepsilon} 1_{\text{mod-}A} \longrightarrow G'F' \longrightarrow 0,$$

$$0 \longrightarrow F'G' \longrightarrow 1_{\text{mod-}B} \xrightarrow{\eta} FG \longrightarrow 0,$$

where ε and η denote the counit and the unit of the adjunction (F, G), respectively. Hence the restrictions of the functors F and G (resp. F' and G') give a category equivalence $\mathcal{T} \cong \mathcal{Y}$ (resp. $\mathcal{F} \cong \mathcal{X}$).

We call a short exact sequence $0 \to X_A \to V_A \to L_A \to 0$ a torsion resolution of X_A if $V \in \mathcal{T}$ and $L \in \operatorname{add}(T_A)$. There is the minimal torsion resolution $0 \to X \xrightarrow{\alpha_X} V(X) \xrightarrow{\beta_X} T(X) \to 0$ for any A-module X and every torsion resolution of X is of the form

$$0 \longrightarrow X \xrightarrow{\begin{pmatrix}\alpha_X \\ 0\end{pmatrix}} V(X) \oplus T_0 \xrightarrow{\begin{pmatrix}\beta_X & 0 \\ 0 & 1_{T_0}\end{pmatrix}} T(X) \oplus T_0 \longrightarrow 0.$$

Similarly, a short exact sequence $0 \to W_B \to U_B \to Y_B \to 0$ is said to be a torsion-free resolution of Y_B if $U \in \mathcal{Y}$ and $W \in \operatorname{add}(DT_B)$. It is easy to see that the sequence $0 \to W_B \to U_B \to Y_B \to 0$ in the category mod-B is a torsion-free resolution iff the corresponding sequence $0 \to {}_BDY \to {}_BDU \to {}_BDW \to 0$ in the category B-mod is a torsion resolution. Therefore, there is the minimal torsion-free resolution $0 \to W(Y) \xrightarrow{\delta_Y} U(Y) \xrightarrow{\gamma_Y} Y \to 0$ and every torsion-free resolution is of the form

$$0 \longrightarrow W(Y) \oplus W_0 \xrightarrow{\begin{pmatrix}\delta_Y & 0 \\ 0 & 1_{W_0}\end{pmatrix}} U(Y) \oplus W_0 \xrightarrow{(\gamma_Y,\ 0)} Y \longrightarrow 0.$$

Any module X_R over the trivial extension self-injective algebra $R = A \ltimes DA$ is defined by giving its underlying A-module X_A and the A-morphism $\phi: X \otimes_A DA \to X$ such that $\phi \cdot (\phi \otimes DA) = 0$ and any R-morphism $f: X_R = (X_A, \phi) \to X'_A = (X'_A, \phi')$ can be considered as an A-morphism $f: X_A \to X'_A$ making the following diagram commutative:

$$\begin{array}{ccc} X \otimes_A DA & \xrightarrow{\ \phi\ } & X \\ {\scriptstyle f \otimes DA}\downarrow & & \downarrow{\scriptstyle f} \\ X' \otimes_A DA & \xrightarrow[\ \phi'\ ]{} & X'. \end{array}$$

See [12] for details. If the underlying A-module X_A is decomposed as a direct sum $X_0 \oplus X_1$ and the morphism ϕ is of the form $\begin{pmatrix}0 & 0 \\ \phi_0 & 0\end{pmatrix}: (X_0 \oplus X_1) \otimes DA \to (X_0 \oplus X_1)$, we shall denote $X_R = (X_A, \phi)$ by $\dfrac{X_0}{X_1}$. It should be noted that any indecomposable projective (= injective) R-module has to be of the form $fR = \dfrac{fA}{fDA}$ with a primitive idempotent $f \in A \subset R$.

Similarly, by the definition, any object X in the category mod-$R_\infty (R_\infty = \hat{A})$ is is defined by giving a family of A-modules $\{X_i\}_{i \in Z}$ ($X_i \neq 0$ for only finite number of integers $i \in \boldsymbol{Z}$) and a family of A-morphisms $\{\phi_i : X_i \otimes DA \to X_{i+1}\}_{i \in Z}$ satisfying $\phi_{i+1} \cdot (\phi_i \otimes DA) = 0$ for all $i \in \boldsymbol{Z}$. Any morphism in the category mod-R_∞ from $X = \{X_i, \phi_i\}$ to $X' = \{X'_i, \phi'_i\}$ is a family of A-morphisms $\{f_i : X_i \to X'_i\}$ such that the following diagrams are commutative for all $i \in \boldsymbol{Z}$:

$$\begin{tikzcd} X_i \otimes_A DA \arrow[r, "\phi_i"] \arrow[d, "f_i \otimes DA"'] & X_{i+1} \arrow[d, "f_{i+1}"] \\ X'_i \otimes_A DA \arrow[r, "\phi'_i"'] & X'_{i+1}. \end{tikzcd}$$

Similarly to the above also, for any positive integer n, an R_n-module X is defined by giving a family of A-modules $X_0, X_1 \cdots, X_{n-1}$ and a family of A-morphisms $\phi_0 : X_0 \otimes_A DA \to X_1, \cdots, \phi_{n-2} : X_{n-2} \otimes_A DA \to X_{n-1}$ and $\phi_{n-1} : X_{n-1} \otimes_A DA \to X_0$ satisfying $\phi_{i+1} \cdot (\phi_i \otimes DA) = 0$ for each $i = 0, 1, \cdots, n-1$. An R_n-morphism from $X = \{X_i, \phi_i\}$ to $X' = \{X'_i, \phi'_i\}$ is a family of A-morphisms $f = \{f_i : X_i \to X'_i\}$ such that $\phi'_i \cdot f_i \otimes DA = f_{i+1} \cdot \phi_i$ for each i. Where we put $X_{i+s \cdot n} = X_i$ and $\phi_{i+s \cdot n} = \phi_i$ ($1 \leqq i \leqq n$, $s \in \boldsymbol{N}$), for convenience.

Then the functors $\Phi_n : \text{mod-}R_\infty \to \text{mod-}R_n$ and $\Phi_{m,n} : \text{mod-}R_{m \cdot n} \to \text{mod-}R_n$ are defined as follows:

$$\Phi_n(\{X_j, \phi_j\}) = \{Y_i, \psi_i\}_{i=1}^n, \qquad Y_i = \bigoplus_{j \equiv i (\text{mod } n)} X_j$$

and

$$\psi_i | X_j \otimes DA = \phi_j \qquad \text{for all} \quad j \equiv i (\text{mod } n).$$

$$\Phi_n(\{f_j : X_j \to X'_j\}) = \{g_i : \bigoplus_{j \equiv i} X_j \to \bigoplus X'_j\}, \qquad g_i = \bigoplus_{j \equiv i} f_j.$$

It is easy to verify that the functors Φ_n, $\Phi_{m,n}$ are exact and preserve the projectivity (= injectivity), indecomposability and composition length of a module and almost split sequences and irreducible maps. Further they make the following commutative diagrams:

$$\begin{tikzcd} & \text{mod-}R_\infty \arrow[dl, "\Phi_{m \cdot n}"'] \arrow[dr, "\Phi_n"] & \\ \text{mod-}R_{m \cdot n} \arrow[rr, "\Phi_{m,n}"'] & & \text{mod-}R_n \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \underline{\text{mod}}\text{-}R_\infty \arrow[dl, "\Phi_{m \cdot n}"'] \arrow[dr, "\Phi_n"] & \\ \underline{\text{mod}}\text{-}R_{m \cdot n} \arrow[rr, "\Phi_{m,n}"'] & & \underline{\text{mod}}\text{-}R_n . \end{tikzcd}$$

Here $\underline{\text{mod}}$-* denotes the projectively (= injectively) stable category of mod-* in the sense of M. Auslander, for each self-injective algebra *.

2. The functors $\mathcal{S}_n : \underline{\mathrm{mod}}\text{-}R_n \to \underline{\mathrm{mod}}\text{-}S_n$ and $\mathcal{Q}_n : \underline{\mathrm{mod}}\text{-}S_n \to \underline{\mathrm{mod}}\text{-}R_n$

In this section, we shall define the functor $\mathcal{S}_n : \underline{\mathrm{mod}}\text{-}R_n \to \underline{\mathrm{mod}}\text{-}S_n$ first and then, by making use of this functor $\mathcal{S}_n$, the functor $\mathcal{Q}_n$ will be defined as the composite $\underline{\mathrm{mod}}\text{-}S_n \xrightarrow{D} S_n\text{-}\underline{\mathrm{mod}} \xrightarrow{\mathcal{S}_n} R_n\underline{\mathrm{mod}} \xrightarrow{D} \underline{\mathrm{mod}}\text{-}R_n$. Notice that, since R_n and S_n are self-injective, the duality functor $D : \mathrm{mod}\text{-}R_n \rightleftarrows R_n\text{-mod}$, $\mathrm{mod}\text{-}S_n \rightleftarrows S_n\text{-mod}$ induces the duality functor $\underline{\mathrm{mod}}\text{-}R_n \rightleftarrows R_n\text{-}\underline{\mathrm{mod}}$, $\underline{\mathrm{mod}}\text{-}S_n \rightleftarrows S_n\text{-}\underline{\mathrm{mod}}$ (we denote this functor also by D). The functor $\mathcal{S}_n : S_n\text{-}\underline{\mathrm{mod}} \to R_n\text{-}\underline{\mathrm{mod}}$ can be defined similarly to $\mathcal{S}_n : \underline{\mathrm{mod}}\text{-}R_n \to \underline{\mathrm{mod}}\text{-}S_n$.

For an R_n-module $X = \{X_i, \phi_i\}$, we shall define S_n-modules $\mathcal{A}(X)$ and $\mathcal{B}(X)$ and S_n-monomorphism $u(X) : \mathcal{A}(X) \to \mathcal{B}(X)$ and the module $\mathcal{S}_n(X)$ is defined as its cokernel Cok $u(X)$. In order to define those S_n-modules and S_n-morphism, the following lemma is necessary

LEMMA 2.1. ${}_A DA_A \cong {}_A DT \otimes_B T_A$ and ${}_B DB_B \cong {}_B T \otimes_A DT_B$.

PROOF. Since ${}_B T_A$ is a balanced bimodule, we have ${}_A DA_A \cong {}_A D\,\mathrm{Hom}({}_B T, {}_B T)_A \cong {}_A DT \otimes_B T_A$ and ${}_B DB_B \cong {}_B D\,\mathrm{Hom}(T_A, T_A)_B \cong {}_B T \otimes_A DT_B$.

In the following, we can identify DA (resp. DB) with $DT \otimes T$ (resp. $T \otimes DT$). Further, from the lemma, it follows that ${}_A DA \otimes_A DT_B \cong {}_A DT \otimes_B T_A$ and ${}_B T \otimes_A DA_A \cong {}_B DB \otimes_B T_A$ and we shall identify these bimodules respectively.

Now let us put $\mathcal{A}(X) = \{X_i \otimes DT, -\phi_i \otimes DT : X_i \otimes DT \otimes DB = X_i \otimes DA \otimes DT \to X_{i+1} \otimes DT\}$ and $\mathcal{B}(X) = \left\{F(V(X_{i+1})) \oplus F(V(X_i)) \otimes DB, \begin{pmatrix} 0 & 0 \\ 1_{F(V(X_{i+1}))\otimes DB} & 0 \end{pmatrix} : F(V(X_{i+1})) \otimes DB \oplus F(V(X_i)) \otimes DB \otimes DB \to F(V(X_{i+2})) \oplus F(V(X_{i+1})) \otimes DB\right\}$. Then it is not hard to see that $\mathcal{A}(X)$ and $\mathcal{B}(X)$ become S_n-modules. We shall define the map $u(X)$ by the following:

$$u(X)_i = \begin{pmatrix} F(\alpha_{X_{i+1}} \cdot -\phi_i) \cdot \eta_{X_i \otimes DT} \\ (\varepsilon_{V(X)_i} \otimes DT)^{-1} \cdot \alpha_{X_i} \otimes DT \end{pmatrix} :$$

$$\mathcal{A}(X)_i = X_i \otimes DT \longrightarrow F(V(X_{i+1})) \oplus F(V(X_i)) \otimes DB = \mathcal{B}(X)_i .$$

To see that the above map $u(X)$ is an S_n-morphism, it is enough to show the commutativity of the following diagram:

$$
\begin{tikzcd}[column sep=small]
 & F(X_i\otimes DA) \arrow[r, "F(-\phi_i)"] & F(X_{i+1}) \arrow[dr, hook, "F(\alpha_{X_{i+1}})"] & \\
X_i\otimes DT \arrow[ur, "\eta_{X_i}\otimes DT"] \arrow[drr, hook, "\alpha_{X_i}\otimes DT"'] \arrow[ddd, "-\phi_i\otimes DT"'] & & & F(V(X_{i+1})) \oplus F(V(X_i))\otimes DB \quad \otimes DB \arrow[dl, "\varepsilon_{V(X_i)}\otimes DT", "\sim"'] \arrow[ddd, "\mathrm{id.}"] \\
 & & V(X_i)\otimes DT & \\
 & F(X_{i+1}\otimes DA) \arrow[r, "F(-\phi_{i+1})"] & F(X_{i+2}) \arrow[dr, hook, "F(\alpha_{X_{i+2}})"] & \\
X_{i+1}\otimes DT \arrow[ur, "\eta_{X_{i+1}}\otimes DT"] \arrow[drr, hook, "\alpha_{X_{i+1}}\otimes DT"'] & & & F(V(X_{i+2})) \oplus F(V(X_{i+1}))\otimes DB \arrow[dl, "\varepsilon_{V(X_{i+1})}\otimes DT", "\sim"'] \\
 & & V(X_{i+1})\otimes DT &
\end{tikzcd}
$$

LEMMA 2.2. The above diagram is commutative.

PROOF. From the naturality of the ε and η, we have the following equalities:

$$
\begin{aligned}
&\varepsilon_{V(X_{i+1})}\otimes DT\cdot F(\alpha_{X_{i+1}}\cdot -\phi_i)\otimes DB\cdot \eta_{X_i\otimes DT}\otimes DB \\
&\quad =(\alpha_{X_{i+1}}\cdot -\phi_i)\otimes DT\cdot \varepsilon_{X_i\otimes DA}\otimes DT\cdot \eta_{X_i\otimes DT}\otimes DB \\
&\quad =(\alpha_{X_{i+1}}\cdot -\phi_i)\otimes DT\cdot (\varepsilon_{X_i\otimes DT\otimes T}\cdot \eta_{X_i\otimes DT}\otimes T)\otimes DT \\
&\quad =(\alpha_{X_{i+1}}\cdot -\phi_i)\otimes DT\cdot 1_{X_i\otimes DT\otimes DB} \\
&\quad =\alpha_{X_{i+1}}\otimes DT\cdot(-\phi_i\otimes DT), \\
&F(\alpha_{X_{i+2}}\cdot -\phi_{i+1})\cdot \eta_{X_{i+1}\otimes DT}\cdot -\phi_i\otimes DT \\
&\quad =F(\alpha_{X_{i+2}}\cdot -\phi_{i+1})\cdot F(-\phi_i\otimes DA)\cdot \eta_{X_i\otimes DA\otimes DT} \\
&\quad =F(\alpha_{X_{i+2}})\cdot F(\phi_{i+1}\cdot \phi_i\otimes DA)\cdot \eta_{X_i\otimes DA\otimes DT}=0.
\end{aligned}
$$

The desired commutativity follows from the above equalities.

Since $\alpha_{X_i}\otimes DT$ is an injection and $\varepsilon_{V(X_i)}\otimes DT$ is a bijection, $u(X)_i$ is also an injection for each i. Therefore, $u(X)$ is an S_n-monomorphism. Thus we can define the S_n-module $\mathcal{S}_n(X)$ as the cokernel Cok $u(X)$ of this S_n-monomorphism $u(X)$.

From the definition of $\mathcal{S}_n$, the following lemma is easily checked.

LEMMA 2.3. *For any projective R_n-module P, the S-module $\mathcal{S}_n(P)$ is also* *projective.*

The remaining part of this section is devoted to the proof of the following proposition.

PROPOSITION 2.4. *The correspondence $\mathcal{S}_n$ can be seen as a stable functor from* $\underline{\text{mod}}$-R_n *to* $\underline{\text{mod}}$-S_n.

It is necessary to define the S_n-morphism $\mathcal{S}_n(f)$ for any R_n-morphism $f=\{f_i\} : X=\{X_i, \phi_i\}\to\{X'_i, \phi'_i\}=X'$, at first. In order to define such a morphism, it is sufficient to define S_n-morphisms $\mathcal{A}(f) : \mathcal{A}(X)\to\mathcal{A}(X')$ and $\mathcal{B}(f) : \mathcal{B}(X)\to\mathcal{B}(X')$ such that $u(X')\cdot\mathcal{A}(f)=\mathcal{B}(f)\cdot u(X)$.

Let us put $\mathcal{A}(f)$ and $\mathcal{B}(f)$ as follows:

$$\mathcal{A}(f)_i=f_i\otimes DT : \mathcal{A}(X)_i=X_i\otimes DT \longrightarrow X'_i\otimes DT=\mathcal{A}(X')_i,$$

$$\mathcal{B}(f)_i=\begin{pmatrix} F(f^*_{i+1}) & \\ 0 & F(f^*_i)\otimes DB \end{pmatrix}:$$

$$\mathcal{B}(X)_i=F(V(X_{i+1}))\oplus F(V(X_i))\otimes DB \longrightarrow F(V(X'_{i+1}))\oplus F(V(X'_i))\otimes DB=\mathcal{B}(X')_i,$$

where f^*_i is defined by the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_i & \xrightarrow{\alpha_{X_i}} & V(X_i) & \xrightarrow{\beta_{X_i}} & T(X_i) & \longrightarrow & 0 \\
 & & {\scriptstyle f_i}\downarrow & & {\scriptstyle f^*_i}\downarrow & & {\scriptstyle f^{**}_i}\downarrow & & \\
0 & \longrightarrow & X'_i & \xrightarrow[\alpha_{X'_i}]{} & V(X'_i) & \xrightarrow[\beta_{X'_i}]{} & T(X'_i) & \longrightarrow & 0.
\end{array}$$

The fact that $\mathcal{A}(f)$ and $\mathcal{B}(f)$ are S_n-morphisms is clear.

LEMMA 2.5. *The above morphisms $\mathcal{A}(f)$ and $\mathcal{B}(f)$ satisfy $u(X')\cdot\mathcal{A}(f)=\mathcal{B}(f)\cdot u(X)$.*

PROOF. We have to verify the following two equalities:

(a) $\varepsilon_{V(X'_i)}\otimes DT\cdot F(f^*_i)\otimes DB\cdot(\varepsilon_{V(X_i)}\otimes DT)^{-1}\cdot\alpha_{X_i}\otimes DT=\alpha_{X'_i}\otimes DT\cdot f_i\otimes DT$

and

(b) $F(f^*_{i+1}\cdot\alpha_{X_{i+1}}\cdot -\phi_i)\cdot\eta_{X_i\otimes DT}=F(\alpha_{X'_{i+1}}\cdot -\phi'_i)\cdot\eta_{X'_i\otimes DT}\cdot f_i\otimes DT$.

The above two equalities (a) and (b) follow from the naturality of ε and η and the following three equalities: $f_{i+1}\cdot\phi_i=\phi'_i\cdot f_i\otimes DA$, $f^*_i\cdot\alpha_{X_i}=\alpha_{X'_i}\cdot f_i$ and $f^*_{i+1}\cdot\alpha_{X_{i+1}}=\alpha_{X'_i}\cdot f_{i+1}$

Therefore we have defined the S_n-morphism $\mathcal{S}(f)$ by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(X) \arrow[r, "u(X)"] \arrow[d, "\mathcal{A}(f)"'] & \mathcal{B}(X) \arrow[r] \arrow[d, "\mathcal{B}(f)"'] & \mathcal{S}_n(X) \arrow[r] \arrow[d, "\mathcal{S}_n(f)"] & 0 \\
0 \arrow[r] & \mathcal{A}(X') \arrow[r, "u(X')"'] & \mathcal{B}(X') \arrow[r] & \mathcal{S}_n(X') \arrow[r] & 0.
\end{tikzcd}$$

By the definition, $\mathcal{A}(f)$ is uniquely determined by f but $\mathcal{B}(f)$ is not and so $\mathcal{S}_n(f)$ is not uniquely determined by f. However, in the stable category $\underline{\text{mod}}$-S_n, we can prove the singleness of the morphism $\mathcal{S}_n(f)$. To show this fact, we shall prove that $\mathcal{S}_n(f)$ factors through projective S_n-modules if $f=0$.

Since $f_i=0$, there is a morphism $\delta_i: T(X_i)\rightarrow V(X_i')$ and $f_i^*=\delta_i\cdot\beta_{X_i}$. Let $\mathcal{P}(X)$ be a projective S_n-module defined by

$$\mathcal{P}(X)_i=F(T(X_{i+1}))\oplus F(T(X_i))\otimes DB$$

and

$$(\mathcal{P}(X)_i\otimes DB\rightarrow\mathcal{P}(X)_{i+1})=\begin{pmatrix}0 & 0\\ 1_{F(T(X_{i+1}))\otimes DB} & 0\end{pmatrix}:$$

$$F(T(X_{i+1}))\otimes DB\oplus F(T(X_i))\otimes DB\otimes DB\longrightarrow F(T(X_{i+2}))\oplus F(T(X_{i+1}))\otimes DB.$$

It is possible to define S_n-morphisms $\beta(X)$ from $\mathcal{B}(X)$ to $\mathcal{P}(X)$ and Δ from $\mathcal{P}(X)$ to $\mathcal{B}(X')$ so that $\mathcal{B}(f)=\Delta\cdot\beta(X)$ by putting:

$$\beta(X)_i=F(\beta_{X_{i+1}})\oplus F(\beta_{X_i})\otimes DB:$$

$$F(V(X_{i+1}))\oplus F(V(X_i))\otimes DB\longrightarrow F(T(X_{i+1}))\oplus F(T(X_i))\otimes DB$$

and

$$\Delta_i=F(\delta_{i+1})\oplus F(\delta_i)\otimes DB:$$

$$F(T(X_{i+1}))\oplus F(T(X_i))\otimes DB\longrightarrow F(V(X'_{i+1}))\oplus F(V(X_i'))\otimes DB.$$

It is easy to see that $\beta(X)\cdot u(X)=0$ and $\mathcal{S}_n(f)$ factors through projective S_n-module $\mathcal{P}(X)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(X) \arrow[r, "u(X)"] & \mathcal{B}(X) \arrow[r] \arrow[d, "\beta(X)"'] & \mathcal{S}_n(X) \arrow[r] \arrow[dl, dashed] & 0 \\
 & & \mathcal{P}(X) \arrow[d, "\Delta"'] & & \\
0 \arrow[r] & \mathcal{A}(X') \arrow[r, "u(X')"'] & \mathcal{B}(X') \arrow[r] & \mathcal{S}_n(X') \arrow[r] & 0.
\end{tikzcd}$$

Therefore, we have defined the functor nod-$R_n\rightarrow\underline{\text{mod}}$-$S_n$ and this functor induces the desired stable functor $\bar{\mathcal{S}}_n$: $\underline{\text{mod}}$-$R_n\rightarrow\underline{\text{mod}}$-$S_n$ by Lemma 2.3. This completes the proof of Proposition 2.4.

From the definition of the functors Φ_n, $\Phi_{m,n}$ and $\mathcal{S}_n$, the commutativity of the diagram in Introduction is now obvious.

3. The functor $Q_n : \underline{\text{mod}}\text{-}S_n \to \underline{\text{mod}}\text{-}R_n$

The functor Q_n has defined as the composite $\underline{\text{mod}}\text{-}S_n \xrightarrow{D} S_n\text{-}\underline{\text{mod}} \xrightarrow{\mathcal{S}_n} R_n\text{-}\underline{\text{mod}} \xrightarrow{D} \underline{\text{mod}}\text{-}R_n$. In this section, we shall show the construction of this functor in an explicit way, for the later use.

In the definition of the functor $\mathcal{S}_n$, we expressed R_n- and S_n-modules as the tensor forms: $\{X_i, \phi_i : X_i \otimes DA \to X_{i+1}\}$ and $\{Y_i, \psi_i : Y_i \otimes DB \to Y_{i+1}\}$. But for the definition of the functor Q_n, it is convenient to express the modules as the hom-forms: $\{X_i, \bar{\phi}_i : X_i \to \mathrm{Hom}_A(DA, X_{i+1})\}$ and $\{Y_i, \bar{\psi}_i : Y_i \to \mathrm{Hom}_B(DB, Y_{i+1})\}$, where $\bar{\phi}_i$ (resp. $\bar{\psi}_i$) is the adjoint of ϕ_i (resp. ψ_i) which corresponds to ϕ_i (rep. ψ_i) by the canonical isomorphism $\mathrm{Hom}_A(X_i \otimes_A DA, X_{i+1}) \cong \mathrm{Hom}_A(X_i, \mathrm{Hom}_A(DA, X_{i+1}))$ (resp. $\mathrm{Hom}_B(Y_i \otimes_B DB, Y_{i+1}) \cong \mathrm{Hom}_B(Y_i, \mathrm{Hom}_B(DB, Y_{i+1}))$).

In the following we shall sometimes abbroviate Hom(?, ?) by [?, ?].

For an S_n-module $Y = \{Y_i, \bar{\psi}_i\}$, let us put

$$\mathcal{C}(Y) = \{[DT, Y_i], [DT, -\bar{\psi}_i] : [DT, Y_i] \longrightarrow [DT, [DB, Y_{i+1}]] = [DA, [DT, Y_{i+1}]]\}$$

and

$$\mathcal{D}(Y) = \left\{ [DA, G(U(Y_i))] \oplus G(U(Y_{i-1})), \begin{pmatrix} 0 & 0 \\ [DA, G(U(Y_i))] & 0 \end{pmatrix} : [DA, G(U(Y_i))] \oplus G(U(Y_{i-1})) \longrightarrow [DA, [DA, G(U(Y_{i+1}))]] \oplus [DA, G(U(Y_i))] \right\}$$

and define the map $p(Y) : \mathcal{D}(Y) \to \mathcal{C}(Y)$ as follows:

$$p(Y)_i = ([DT, \gamma_{Y_i} \cdot (\eta_{U(Y_i)})^{-1}, \varepsilon_{[DT, Y_i]} \cdot G(-\bar{\psi}_i \cdot \gamma_{Y_{i-1}}) : \mathcal{D}(Y)_i = [DA, G(U(Y_i))] \oplus G(U(Y_{i-1})) \longrightarrow [DT, Y_i] = \mathcal{C}(Y)_i .$$

Then $\mathcal{C}(Y)$ and $\mathcal{D}(Y)$ become R_n-modules and $p(Y)$ is an R_n-morphism. The mhdule $Q_n(Y)$ coincides with the kernel Ker $p(Y)$ of the above morphism $p(Y)$.

For an S_n-morphism $g = \{g_i : Y_i \to Y'_i\} : Y = \{Y_i, \bar{\psi}_i\} \to Y' = \{Y'_i, \bar{\psi}'_i\}$, we put $\mathcal{C}(g) : \mathcal{C}(Y) \to \mathcal{C}(Y')$ and $\mathcal{D}(g) : \mathcal{D}(Y) \to \mathcal{D}(Y')$ as follows:

$$\mathcal{C}(g)_i = [DT, g_i] : [DT, Y_i] \longrightarrow [DT, Y'_i],$$

$$\mathcal{D}(g)_i = \begin{pmatrix} [DA, G(g_i^*)] & 0 \\ 0 & G(g_{i-1}^*) \end{pmatrix} : [DA, G(U(Y_i))] \oplus G(U(Y_{i-1})) \longrightarrow [DA, G(U(Y'_i))] \oplus G(U(Y'_{i-1})),$$

where g_i^* is defined by the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(Y_i) & \xrightarrow{\delta_{Y_i}} & U(Y_i) & \xrightarrow{\gamma_{Y_i}} & Y_i & \longrightarrow & 0 \\ & & \downarrow g_i^{**} & & \downarrow g_i^* & & \downarrow g_i & & \\ 0 & \longrightarrow & W(Y_i') & \xrightarrow[\delta_{Y_i'}]{} & U(Y_i') & \xrightarrow[\gamma_{Y_i'}]{} & Y_i' & \longrightarrow & 0. \end{array}$$

Then $\mathcal{C}(g)$ and $\mathcal{D}(g)$ become R_n-morphisms and satisfy $\mathcal{C}(g)\cdot p(Y)=p(Y')\cdot\mathcal{D}(g)$. The R_n-morphism $Q_n(g): Q_n(Y)\to Q_n(Y')$ is defined by the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_n(Y) & \longrightarrow & \mathcal{D}(Y) & \xrightarrow{p(Y)} & \mathcal{C}(Y) & \longrightarrow & 0 \\ & & \downarrow Q_n(g) & & \downarrow D(g) & & \downarrow \mathcal{C}(g) & & \\ 0 & \longrightarrow & Q_n(Y') & \longrightarrow & \mathcal{D}(Y') & \xrightarrow[p(Y')]{} & \mathcal{C}(Y') & \longrightarrow & 0. \end{array}$$

Similarly to the functor $\mathcal{S}_n$, Q_n can be considered as a functor mod-$S_n\to$ $\underline{\text{mod}}$-R_n and it induces $\underline{\text{mod}}$-$S_n\to\underline{\text{mod}}$-$R_n$.

4. The proof of the isomorphism $Q_n\cdot\mathcal{S}_n\cong 1_{\underline{\text{mod}}\text{-}R_n}$

We begin with the survey of the torsion-free resolution of the component of $\mathcal{S}_n(X)$, in order to investigate the module $Q_n\mathcal{S}_n(X)$.

Let us denote the morphism cok $u(X)$ by $\theta(X)$:

$$\theta(X)_i=(x_i, y_i): F(V(X_{i+1}))\oplus F(V(X_i))\otimes DB \longrightarrow \mathcal{S}_n(X)_i.$$

Let $P_0^i \xrightarrow{p_0^i} V(X_i)\to 0$ be the projective cover, then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^i & \xrightarrow{\alpha^i} & P_0^i & \xrightarrow{\beta^i=\beta_{X_i}\cdot p_0^i} & T(X_i) & \longrightarrow & 0 \\ & & \downarrow p_1^i & & \downarrow p_0^i & & \| & & \\ 0 & \longrightarrow & X_i & \xrightarrow[\alpha_{X_i}]{} & V(X_i) & \xrightarrow[\beta_{X_i}]{} & T(X_i) & \longrightarrow & 0 \end{array}$$

Since proj. dim. $T_A\leqq 1$, P_1^i has to be projective. Applying $(?\otimes_A DT)$ to the above diagram, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^i\otimes DT & \xrightarrow{\alpha^i\otimes DT} & P_0^i\otimes DT & \xrightarrow{\beta^i\otimes DT} & T(X_i)\otimes DT & \longrightarrow & 0 \\ & & \downarrow p_1^i\otimes DT & & \downarrow p_0^i\otimes DT & & \| & & \\ 0 & \longrightarrow & X_i\otimes DT & \xrightarrow[\alpha_{X_i}\otimes DT]{} & V(X_i)\otimes DT & \xrightarrow[\beta_{X_i}\otimes DT]{} & T(X_i)\otimes DT & \longrightarrow & 0. \end{array}$$

Here we used the fact that $\mathrm{Tor}_1^A(T(X_i), DT) \cong D\,\mathrm{Ext}_A^1(T(X_i), T) = 0$. Hence we know $\mathrm{Ker}(p_1^i \otimes DT)_B \cong \mathrm{Ker}(p_0^i \otimes DT)_B$ by the Snake Lemma.

Consider the following diagram of B-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^i \otimes DT & \xrightarrow{\zeta} & \begin{matrix} F(V(X_{i+1})) \\ \oplus \\ F(P_0^i \otimes DA) \end{matrix} & & & & \\ & & \Big\downarrow {\scriptstyle p_1^i \otimes DT} & & \Big\downarrow {\scriptstyle \chi} & & & & \\ 0 & \longrightarrow & \mathcal{A}(X)_i & \xrightarrow[u(X)_i]{} & \mathcal{B}(X)_i & \xrightarrow[\theta(X)_i]{} & \mathcal{S}_n(X)_i & \longrightarrow & 0, \end{array}$$

where ζ and χ are defined as follows:

$$\zeta = \begin{pmatrix} F(\alpha_{X_i} \cdot -\phi_i) \cdot \eta_{X_i \otimes DT} \cdot p_1^i \otimes DT \\ (\eta_{P_0^i \otimes DT})^{-1} \cdot \alpha^i \otimes DT \end{pmatrix},$$

$$\chi = \begin{pmatrix} 1_{F(V(X_{i+1}))} & 0 \\ 0 & (\varepsilon_{V(X_i)} \otimes DT)^{-1} \cdot p_0^i \otimes DT \cdot (\eta_{P_0^i \otimes DT})^{-1} \end{pmatrix}.$$

From the fact that $\mathrm{Ker}(p_0^i \otimes DT)_B \cong \mathrm{Ker}(p_1^i \otimes DT)_B$, it follows that $\mathrm{Ker}\,\chi \cong \mathrm{Ker}((\varepsilon_{V(X_i)} \otimes DT)^{-1} \cdot p_0^i \otimes DT \cdot (\eta_{P_0^i \otimes DT})^{-1}) \cong \mathrm{Ker}(p_0^i \otimes DT) \cong \mathrm{Ker}(p_1^i \otimes DT)$. Therefore we have $\mathrm{Cok}\,\zeta \cong \mathcal{S}_n(X)_i$ and we have a torsion-free resolution of $\mathcal{S}_n(X)_i$.

LEMMA 4.1. *The following exact sequence is a torsion-free resolution of* $\mathcal{S}_n(X)_i$:

$$0 \longrightarrow P_1^i \otimes DT \xrightarrow{\zeta} F(V(X_{i+1})) \oplus F(P_0^i \otimes DA) \longrightarrow \mathcal{S}_n(X)_i \longrightarrow 0.$$

It is clear that $P_1^i \otimes DT \in \mathrm{add}(DT_B)$ and $F(F(X_{i+1})) \oplus F(P_0^i \otimes DA) \in \mathcal{Y}$. We shall denote $\mathrm{coker}\,\zeta$ by $\hat{\theta}_i = (x_i, \hat{y}_i)$.

To define the modules $\mathcal{S}_n(X)$ and $Q_n(Y)$, we have used the minimal torsion and torsion-free resolutions. But by the remark on torsion and torsion-free resolutions in section one, we may use any such resolutions since we consider modules in the stable categories.

Now, using the torsion-free resolutions given by Lemma 4.1, let us calculate the module $Q_n \mathcal{S}_n(X)$.

The routine verification shows the following lemma.

LEMMA 4.2. *The map $p(\mathcal{S}_n(X))$ is expressed as follows:*

(a) $\mathcal{C}\mathcal{S}_n(X)_i = [DT, \mathcal{S}_n(X)_i]$,

(b) $\mathcal{D}\mathcal{S}_n(X)_i = [DA, GF(V(X_{i+1}))] \oplus [DA, GF(P_0^i \otimes DA)]$
$\oplus GF(V(X_i)) \oplus GF(P_0^{i-1} \otimes DA)$

(c) $p(\mathcal{S}_n(X))_i=([DT,\ x_i\cdot(\eta_{F(V(X_{i+1}))})^{-1}],\ [DT,\ \hat{y}_i\cdot(\eta_{F(P_0^i\otimes DA)})^{-1}],$

$$\varepsilon_{[DT,\mathcal{S}_n(X)_i]}\cdot F(-\bar{\phi}_{i-1}\cdot x_{i-1}),\ \varepsilon_{[DT,\mathcal{S}_n(X)_i]}\cdot F(-\bar{\phi}_{i-1}\cdot\hat{y}_{i-1})),$$

where we identify $[DA, ?]$ (*resp.* $[DB, ?]$) *with* $[DT, [T, ?]]$ (*resp.* $[T, [DT, ?]]$) *and* $\bar{\phi}_i: \mathcal{S}_n(X)_i \to [DB, \mathcal{S}_n(X)_{i+1}]$ *denotes the i-th structure map of the* S_n*-module* $\mathcal{S}_n(X)$ *in the hom-form.*

The remaining part of this section is devoted to prove that $\operatorname{Ker} p(\mathcal{S}_n(X))$ is isomorphic to X as an object in the stable category $\underline{\mathrm{mod}}$-R_n. In fact, we shall show $\operatorname{Ker} p(\mathcal{S}_n(X)) \cong X\oplus P$ for the projective (= injective) R_n-module P defined as follows:

$$P=\left\{P_0^i\oplus P_0^{i-1}\otimes DA,\ \begin{pmatrix} 0 & 0 \\ 1_{P_0^i}\otimes_{DA} & 0\end{pmatrix}\right\}.$$

LEMMA 4.3. $|\operatorname{Ker} p(\mathcal{S}_n(X))|=|X|+|P|$, *where* $|*|$ *denotes the K-composition length of a module* *.

PROOF. By Lemma 4.1, we have

$$\begin{aligned}|\operatorname{Ker} p(\mathcal{S}_n(X))_i| &= |[DA,\ V(X_{i+1})]|+|P_0^i|+|V(X_i)| \\ &\quad +|P_0^{i-1}\otimes DA|-|[DT,\ \mathcal{S}_n(X)_i]|,\end{aligned}$$

since $\varepsilon_V: FG(V)\cong V$ for a torsion A-module V and $\eta_U: U\cong GF(U)$ for a torsion-free B-module U by Brenner-Butler's theorem. On the other hand, from the exact sequence

$$0 \longrightarrow P_1^i\otimes DT \longrightarrow F(V(X_{i+1}))\oplus F(P_0^i\otimes DA) \longrightarrow \mathcal{S}_n(X)_i \longrightarrow 0$$

we have the exact sequence

$$\begin{aligned}0 \longrightarrow [DT,\ P_1^i\otimes DT] &\longrightarrow [DT,\ F(V(X_{i+1}))]\oplus[DT,\ F(P_0^i\otimes DA)] \\ &\longrightarrow [DT,\ \mathcal{S}_n(X)_i] \longrightarrow 0\end{aligned}$$

and $[DT,\ F(P_0^i\otimes DA)]\cong[DA,\ P_0^i\otimes DA]\cong P_0^i$, $[DT,\ P_1^i\otimes DT]\cong[DT,\ D[P_1^i,\ T]]\cong [[P_1^i{}^{\prime}T], T]\cong P_1^i$, as well. Therefore, it follows $|[DT, \mathcal{S}_n(X)_i]|=|[DA, V(X_{i+1})]| +|P_0^i|-|P_1^i|$. Hence we have $|\operatorname{Ker} p(\mathcal{S}_n(X))_i|=|V(X_i)|+|P_0^{i-1}\otimes DA|+|P_1^i|$. Further, from the exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^i & \longrightarrow & P_0^i & \longrightarrow & T(X_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_i & \longrightarrow & V(X_i) & \longrightarrow & T(X_i) & \longrightarrow & 0,\end{array}$$

we know $|P_0^i|-|P_1^i|=|T(X_i)|=|V(X_i)|-|X_i|$, i.e., $|V(X_i)|+|P_1^i|=|P_0^i|+|X_i|$.

Finally, we have $|\mathrm{Ker}\, p(\mathcal{S}_n(X))_i| = |P_0^{i-1}\otimes DA| + |P_0^i| + |X_i| = |X_i| + |P_i|$ and this means that $|\mathrm{Ker}\, p(\mathcal{S}_n(X))| = |X| + |P|$ as desired.

By the above lemma, in order to prove the isomorphism $\mathrm{Ker}\, p(\mathcal{S}_n(X)) \cong X\oplus P$, it suffices to show the existence of an R_n-monomorphism $(e(X), f(X)) : X\oplus P \to \mathcal{D}\mathcal{S}_n(X)$ such that the composition $p(\mathcal{S}_n(X))\cdot(e(X), f(X))$ is a zero map. To define such morphisms $e(X)$ and $f(X)$, it is necessary to introduce a notation: For a bimodule ${}_{E_1}M_{E_2}$ over algebras E_1 and E_2, we can always consider the adjoint pair of functors $\mathrm{Hom}_{E_2}(M, ?)$: mod-$E_2\to$mod-E_1 and $(?\otimes_{E_1}M)$: mod-$E_1\to$ mod-E_2. We shall denote the unit and counit of this adjunction by $\eta^M : 1_{\text{mod-}E_1} \to \mathrm{Hom}_{E_2}(M, ?\otimes_{E_1}M)$ and $\varepsilon^M : \mathrm{Hom}_{E_2}(M, ?)\otimes_{E_1}M \to 1_{\text{mod-}E_2}$, respectively. Then it is noted that $\eta=\eta^T$ and $\varepsilon=\varepsilon^T$.

Now let us put $e(X) : X\to\mathcal{D}\mathcal{S}_n(X)$ and $f(X) : P\to\mathcal{D}\mathcal{S}_n(X)$ as follows:

$$e(X)_i = \begin{pmatrix} [DA, (\varepsilon_{V(X_{i+1})})^{-1}\cdot\alpha_{X_{i+1}}\cdot\phi_i]\cdot\eta_{X_i}^{DA} \\ 0 \\ (\varepsilon_{V(X_i)})^{-1}\cdot\alpha_{X_i} \\ 0 \end{pmatrix}$$

and

$$f(X)_i = \begin{pmatrix} 0 & 0 \\ [DT, \eta_{[T, P_0^i\otimes DA]}] & 0 \\ (\varepsilon_{V(X_i)})^{-1}\cdot p_0^i & 0 \\ 0 & (\varepsilon_{P_0^{i-1}\otimes DA})^{-1} \end{pmatrix}.$$

In the following, we shall show that $e(X)$ and $f(X)$ are R_n-homomorphisms and $p(\mathcal{S}_n(X))\cdot e(X) = 0 = p(\mathcal{S}_n(X))\cdot f(X)$. To do so, it is necessary to provide the following lemma.

LEMMA 4.4. *The following diagrams are commutative for any A-module X and B-morphism $g : Z\otimes DB\to Y$.*

(a)
$$\begin{array}{ccc} [DT, F(X)] & \xrightarrow{[DT,\ \eta_{F(X)}]} & [DT, FGF(X)] \\ \| & & \| \\ [DA, X] & \xrightarrow[{[DA,\ \varepsilon_X]}]{} & [DA, GF(X)] \end{array}$$

(b)
$$\begin{array}{ccc} X & \xrightarrow{\eta_X^{DA}} & [DA, X\otimes DA] \\ {\scriptstyle \eta_X^{DT}}\downarrow & & \| \\ [DT, X\otimes DT] & \xrightarrow[{[DT,\ \eta_{X\otimes DT}]}]{} & [DT, F(X\otimes DA)] \end{array}$$

(c)
$$\begin{array}{ccc}
[DT, Z\otimes DB] & \xrightarrow{[DT,\, g]} & [DT, Y] \\
\uparrow{\scriptstyle \eta^{DT}_{G(Z)}} & & \uparrow{\scriptstyle \varepsilon_{[DT,Y]}} \\
G(Z) & & GF([DT, Y]) \\
\downarrow{\scriptstyle G(\eta^{DB}_Z)} & & \| \\
G([DB, Z\otimes DB]) & \xrightarrow[G([DB,\, g])]{} & G([DB, Y])
\end{array}$$

PROOF. Rutine verification.

LEMMA 4.5. *The map $e(X)$ is an R_n-morphism.*

PROOF. At first, we have to verify the equality

$$\varepsilon^{DA}_{GF(V(X_{i+1}))}\cdot([DA, \varepsilon_{V(X_{i+1})}]\otimes DA)^{-1}\cdot[DA, \alpha_{X_{i+1}}\cdot\phi_i]\otimes DA\cdot\eta^{DA}_{X_i}\otimes DA$$
$$=(\varepsilon_{V(X_{i+1})})^{-1}\cdot\alpha_{X_{i+1}}\cdot\phi_i .$$

By the naturality of ε^{DA} and the relation $\varepsilon^{DA}_{X_i\otimes DA}\cdot\eta^{DA}_{X_i}\otimes DA=1_{X_i}$, we have $\alpha_{X_{i+1}}\cdot\phi_i=\varepsilon^{DA}_{V(X_{i+1})}\cdot[DA, \alpha_{X_{i+1}}\cdot\phi_i]\otimes DA\cdot\eta^{DA}_{X_i}\otimes DA$. Hence it is sufficient to show $(\varepsilon_{V(X_{i+1})})^{-1}\cdot\varepsilon^{DA}_{V(X_{i+1})}=\varepsilon^{DA}_{GF(V(X_{i+1}))}\cdot([DA, \varepsilon_{V(X_{i+1})}]\otimes DA)^{-1}$, i.e., $\varepsilon_{V(X_{i+1})}\cdot\varepsilon^{DA}_{GF(V(X_{i+1}))}=\varepsilon^{DA}_{V(X_{i+1})}\cdot[DA, \varepsilon_{V(X_{i+1})}]\otimes DA$. But this follows again from the naturality of ε^{DA}.

The another necessary condition $([DA, \varepsilon_{V(X_{i+2})}])^{-1}\cdot[DA, \alpha_{X_{i+2}}\cdot\phi_{i+1}]\cdot\eta^{DA}_{X_{i+1}}\cdot\phi_i=0$ is obvious.

LEMMA 4.6. *The map $f(X)$ is an R_n-morphism, as well.*

PROOF. We have to verify the equality

$$(\varepsilon_{P^i_0\otimes DA})^{-1}=\eta^{DA}_{GF(P^i_0\otimes DA)}\cdot[DT, \eta_{F(P^i_0\otimes DA)}]\otimes DA\cdot\eta^{DA}_{P^i}.$$

Since $\varepsilon^{DA}_{P^i_0\otimes DA}\cdot\eta^{DA}_{P^i_0}\otimes DA=1_{P^i_0\otimes DA}$ and ε^{DA} is a natural transformation, we have

$$(\varepsilon_{P^i_0\otimes DA})^{-1}=\varepsilon^{DA}_{GF(F^i_0\otimes DA)}\cdot([DA, \varepsilon_{P^i_0\otimes DA}]\otimes DA)^{-1}\cdot\eta^{DA}_{P^i_0}\otimes DA.$$

Hence we have the desired result since $([DA, \varepsilon_{P^i_0\otimes DA}])^{-1}=[DT, \eta_{F(P^i_0\otimes DA)}]\otimes DA$ by Lemma 4.4 (a).

For the proof of $p(X)\cdot(e(X), f(X))=0$, we note that the i-th structure map $\psi_i: \mathcal{S}_n(X)_i\otimes DB\to\mathcal{S}_n(X)_{i+1}$ satisfies $\psi_i\cdot y_i\otimes DB=0$ and $y_{i+1}=\psi_i\cdot x_i\otimes DB$ and its adjoint $\bar{\psi}_i$ is the same with the composition: $[DB, \psi_i]\cdot\eta^{DB}_{\mathcal{S}_n(X)_i}: \mathcal{S}_n(X)_i\to[DB, \mathcal{S}_n(X)_i\otimes DB]\to[DB, \mathcal{S}_n(X)_{i+1}]$. Then it is easy to prove the following lemma, by definition.

LEMMA 4.7. *The following hold.*

(a) $G(\bar{\phi}_{i-1}\cdot x_{i-1})=G([DB,\ y_i]\cdot\eta^{DA}_{F(V(X_i))})$

(b) $G(\bar{\phi}_{i-1}\cdot\hat{y}_{i-1})=0$

LEMMA 4.8. $p(X)\cdot e(X)=0$.

PROOF. By Lemma 4.7 (a), it is sufficient to prove the commutativity of the following diagram:

$$\begin{tikzcd}
X_i \arrow[r, "\eta^{DA}_{X_i}"] \arrow[d, hook, "\alpha_{X_i}"'] & {[DA,\ X_i\otimes DA]} \arrow[r, "{[DA,\ \phi_i]}"] & {[DA,\ X_{i+1}]} \arrow[r, "{[DA,\ \alpha_{X_{i+1}}]}"] & {[DT,\ F(V(X_{i+1}))]} \arrow[d, "{[DT,\ x_i]}"] \\
V(X_i) & & & {[DT,\ \mathcal{S}_n(X_i)]} \\
GF(V(X_i)) \arrow[u, "\varepsilon_{V(X_i)}", "\wr"'] \arrow[r, "G(\eta^{DB}_{F(V(X_i))})"'] & G([DB,\ F(V(X_i))\otimes DB]) \arrow[rr, "{G([DB,\ y_i])}"'] & & G([DB,\ \mathcal{S}_n(X)_i]) . \arrow[u, "\varepsilon_{[DT,\mathcal{S}_n(X)_i]}"']
\end{tikzcd}$$

We know $\eta^{DA}_{X_i}=[DT,\ \eta_{X_i\otimes DT}]\cdot\eta^{DT}_{X_i}$: $X_i\to[DT,\ X_i\otimes DT]\to[DT,\ F(X_i\otimes DA)]=[DA,\ X_i\otimes DA]$ by Lemma 4.4 (b) and $\varepsilon_{[DT,\mathcal{S}_n(X)_i]}\cdot G([DB,\ y_i]\cdot\eta^{DB}_{F(V(X_i))})=[DT,\ y_i]\cdot\eta^{DT}_{GF(V(X_i))}$ by Lemma 4.4 (c). Further, by the definition of the map θ, it holds that

$$y_i\cdot G(\varepsilon_{V(X_i)})^{-1}\cdot\alpha_{X_i}\otimes DT=x_i\cdot F(\alpha_{X_{i+1}}\cdot\phi_i)\cdot\eta_{X_i\otimes DT}\,.$$

Hence we have

$$\begin{aligned}
&[DT,\ x_i]\cdot[DA,\ \alpha_{X_{i+1}}\cdot\phi_i]\cdot\eta^{DA}_{X_i}\\
&\quad=[DT,\ x_i\cdot F(\alpha_{X_{i+1}}\cdot\phi_i)]\cdot\eta^{DA}_{X_i}\\
&\quad=[DT,\ x_i\cdot F(\alpha_{X_{i+1}}\cdot\phi_i)\cdot\eta_{X_i\otimes DT}]\cdot\eta^{DT}_{X_i}\\
&\quad=[DT,\ y_i\cdot(\varepsilon_{V(X_i)}\otimes DT)^{-1}\cdot\alpha_{X_i}\otimes DT]\cdot\eta^{DT}_{X_i}\\
&\quad=[DT,\ y_i\cdot(\varepsilon_{V(X_i)}\otimes DT)^{-1}]\cdot\eta^{DT}_{V(X_i)}\cdot\alpha_{X_i}\\
&\quad=[DT,\ y_i]\cdot\eta^{DT}_{GF(V(X_i))}\cdot(\varepsilon_{V(X_i)})^{-1}\cdot\alpha_{X_i}\\
&\quad=\varepsilon_{[DT,\mathcal{S}_n(X)_i]}\cdot G([DB,\ y_i]\cdot\eta^{DB}_{F(V(X_i))})\cdot(\varepsilon_{V(X_i)})^{-1}\cdot\alpha_{X_i}\,.
\end{aligned}$$

LEMMA 4.9. $p(X)\cdot f(X)=0$.

PROOF. By Lemma 4.6 (a), it is enough to prove the commutativity of the diagram:

$$\begin{tikzcd}
P_0^i \arrow[r, "\eta^{DA}_{P_0^i}"] \arrow[d, "p_0^i"'] & {[DA,\ P_0^i\otimes DA]=[DT,\ F(P_0^i\otimes DA)]} \arrow[r, "{[DT,\ \hat{y}_i]}"] & {[DT,\ \mathcal{S}_n(X)_i]} \\
V(X_i) & & \\
GF(V(X_i)) \arrow[u, "\varepsilon_{V(X_i)}", "\wr"'] \arrow[r, "G(\eta^{DB}_{F(V(X_i))})"'] & {G([DB,\ F(V(X_i))\otimes DB])} \arrow[r, "{G([DB,\ y_i])}"'] & {G([DB,\ \mathcal{S}_n(X)_i])} \arrow[uu, hook, "\varepsilon_{[DT,\mathcal{S}_n(X)_i]}"']
\end{tikzcd}$$

By Lemma 4.4 (b) and (c), we know $\eta^{DA}_{P_0^i}=[DT,\ \eta_{P_0^i\otimes DT}]\cdot\eta^{DT}_{P_0^i}$ and $\eta^{DT}_{GF(V(X_i))}=\varepsilon_{[DT,F(V(X_i))\otimes DB]}\cdot G(\eta^{DB}_{F(V(X_i))})$. Hence we have

$$\begin{aligned}
[DT,\ \hat{y}_i]\cdot\eta^{DA}_{P_0^i}&=[DT,\ \hat{y}_i\cdot\eta_{P_0^i\otimes DT}]\cdot\eta^{DT}_{P_0^i}\\
&=[DT,\ y_i\cdot(\varepsilon_{V(X_i)}\otimes DT)^{-1}\cdot p_0^i\otimes DT]\cdot\eta^{DT}_{P_0^i}\\
&=[DT,\ y_i\cdot(\varepsilon_{V(X_i)}\otimes DT)^{-1}]\cdot\eta^{DT}_{V(X_i)}\cdot p_0^i\\
&=[DT,\ y_i]\cdot\eta^{DT}_{GF(V(X_i))}\cdot(\varepsilon_{V(X_i)})^{-1}\cdot p_0^i\\
&=[DT,\ y_i]\cdot\varepsilon_{[DT,E(V(X_i))\otimes DB]}\cdot G(\eta^{DB}_{F(V(X_i))})\cdot(\varepsilon_{V(X_i)})^{-1}\cdot p_0^i\\
&=\varepsilon_{[DT,\mathcal{S}_n(X)_i]}\cdot GF([DT,\ y_i])\cdot G(\eta^{DB}_{F(V(X_i))})\cdot(\varepsilon_{V(X_i)})^{-1}\cdot p_0^i\\
&=\varepsilon_{[DT,\mathcal{S}_n(X)_i]}\cdot G([DB,\ y_i]\cdot\eta^{DB}_{F(V(X_i))})\cdot(\varepsilon_{V(X_i)})^{-1}\cdot p_0^i.
\end{aligned}$$

Since $(e(X),\ f(X))$ is obviously an R_n-monomorphism, we have proved $Q_n\mathcal{S}_n(X)\cong X\oplus P$ as R_n-modules. It is easy to prove that the monomorphism $e(X)\colon X\to\mathcal{D}\mathcal{S}_n(X)$ has naturality on X. By the duality, we can prove the similar result on $\mathcal{S}_nQ_n$.

Thus we have

THEOREM 4.10. *$Q_n\mathcal{S}_n\cong 1_{\underline{\mathrm{mod}}\text{-}R_n}$ and $\mathcal{S}_nQ_n\cong 1_{\underline{\mathrm{mod}}\text{-}S_n}$, i.e., the stable categories $\underline{\mathrm{mod}}$-R_n and $\underline{\mathrm{mod}}$-S_n are always equivalent.*

REMARK. D. Happel [15] has proved that $\underline{\mathrm{mod}}$-$\hat{A}$ and $\underline{\mathrm{mod}}$-$\hat{B}$ are equivalent if gl. dim. $A<\infty$. And, in the case where $\Phi=\Phi_1\colon \underline{\mathrm{mod}}\text{-}\hat{A}\to\underline{\mathrm{mod}}\text{-}R$ is dense, the stable functor $\mathcal{S}_n$ is induced from $\mathcal{S}_\infty$ for each $n\neq\infty$. But, in general, Φ is not dense even if gl. dim. $A<\infty$, can not be induced from $\mathcal{S}_\infty$.

EXAMPLE 4.11. Let A be the bound quiver algebra of

$$\begin{tikzcd}
 & 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] & & 3 \arrow[dl] \\
 & 4 \arrow[ul] &
\end{tikzcd}
\qquad \text{with}\ \ \beta\cdot\alpha=0.$$

Then $\mathrm{soc}(e_4A) \cong \mathrm{top}(e_4A \otimes DA)$ and there is a non-zero map f from $e_4A \otimes DA$ to e_4A such that $\mathrm{Im}(f) = \mathrm{soc}(e_4A)$ and $\mathrm{Ker}(f) = \mathrm{rad}(e_4A \otimes DA)$. Hence we can define an indecomposable R-module $X = (e_4A, f)$. As is easily seen, for any R-module X', X can not be isomorphic to $\Phi(X')$. Thus $\Phi: \underline{\mathrm{mod}}\text{-}\hat{A} \to \underline{\mathrm{mod}}\text{-}R$ is not dense, even though g.l dim. $A = 2 < \infty$.

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