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FIBER SHAPE THEORY

By

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Abstract. In this paper we develop a fiber shape theory for maps between metric spaces. Our approach is based on the Mardešić-Segal method and, instead of ANR's, their fiber preserving analogues are used. A fiber preserving version of Chapman's complement theorem is proved.

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§0. Introduction.

The purpose of this paper is to develop a fiber shape theory for maps between metric spaces. There are several approaches to the fiber shape theory for maps between compact metric spaces ([CM], [$\mathrm{Ka}_{1,3}$]), which correspond to those to the shape theory ([B_0], [Ch_1], [MS]). The description of our fiber shape category is based on the general construction of shape categories in [MS].

In shape theory ([DS], [MS]), the shape of a space is represented by an ANR-system associated with the space. In our setting, the same role will be played by a *fiber preserving* version of ANR's (cf. [CM]). §1 contains the definition and some examples of such fibered ANR's.

In §2 we will give the description of our fiber shape category. It is proved that our approach is particularly useful to treat *proper* maps and many results in [$\mathrm{Ka}_{1,2,3}$] have natural generalizations in our setting. For example, among proper maps, hereditary shape equivalences, shape fibrations or the notion of movability introduced in [Y_2] are shown to be fiber shape invariant.

In §3, we will prove a fiber preserving version of Chapman's complement theorem, which gives the fiber shape classification of proper maps over a separable metric base space. The same statement is also found in [CM], where the base space is restricted to ENR's.

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Finally, we will list some notations and conventions used throughout this paper. All spaces are metric spaces and in § 3 they are assumed to be separable. ANR's are ones for the metric spaces ([Hu]). id_X denotes the identity map on the space X and $\pi_B: B\times M\to B$, $\pi_M: B\times M\to M$ always denote the projections onto appropriate factors. Given maps $p: X\to B$ and $q: Y\to B$, a map $f: X\to Y$ is said to be *fiber preserving* (f. p.) if $qf=p$. Similarly a homotopy $f_t: X\to Y$ $(0\leqq t\leqq 1)$ is f. p. if $qf_t=p$ $(0\leqq t\leqq 1)$. In particular, a map $f: X\to B\times M$ is f. p. if $\pi_B f=p$. A map $p: X\to B$ is *proper* if $p^{-1}(K)$ is compact for each compact $K\subset B$. For a subset $C\subset B$, p_C denotes the restriction $p|_{p^{-1}(C)}: p^{-1}(C)\to C$. Let $\mathcal{V}$ be an open cover of a space Y. We say that the maps $f, g: X\to Y$ are $\mathcal{V}$-near, written $(f, g)\leqq\mathcal{V}$, if each $x\in X$ admits a $V\in\mathcal{V}$ with $f(x), g(x)\in V$. A homotopy $F: X\times[0, 1]\to Y$ is a $\mathcal{V}$-homotopy if for each $x\in X$ there exists a $V\in\mathcal{V}$ with $F(\{x\}\times[0, 1])\subset V$.

We refer to [DS] and [MS] for shape theory and related topics, and to [CM] and [$\mathrm{Ka}_{1,3}$] for fiber shape theory.

§ 1. Absolute neighborhood fiber retracts.

In this section we will define an f. p. version of ANR's and prove their elementary properties, which will be used in the next section to define a fiber shape category.

Let B be a fixed space. A map $p: E\to B$ is said to be an *absolute neighborhood fiber retract* (ANFR) over B provided for any map $q: X\to B$ and any f. p. closed embedding $i: E\to X$, there exist a neighborhood U of $i(E)$ in X and a map $r: U\to E$ such that $ri=id_E$ and $pr=q|_U$. In addition, if we can always take $U=X$, then we say p is an *absolute fiber retract* (AFR) over B.

Similarly we may say a map $p: E\to B$ is an *absolute neighborhood fiber extensor* (ANFE) over B provided for any map $q: X\to B$ and any map $f: A\to E$ from a closed subset A of X with $pf=q|_A$, there exists an extension $\tilde{f}: U\to E$ of f to a neighborhood U of A in X with $p\tilde{f}=q|_U$.

We will list some elementary properties of ANFR's, which are f. p. analogues of ones of ANR's ([Hu]).

1.1. PROPOSITION. (i) ([CM]) *Let M be an* ANR *and U be an open set in $B\times M$. Then the projection $\pi_B|_U: U\to B$ is an* ANFR. *A map $p: E\to B$ is an* ANFR *iff p is an f. p. retract of such a projection $\pi_B|_U$.*

(ii) *A map $p: E\to B$ is an* ANFR *iff p is an* ANFE.

(iii) *Every f. p. neighborhood retract of an* ANFR *is also an* ANFR.

(iv) (*The f. p. homotopy extension property*) *Suppose $q: E\to B$ is an* ANFR,

$p: X \to B$ is a map and A is a closed subset of X. Then for any f.p. map $\phi: X \to E$ and any f.p. homotopy $\psi_t: A \to E$ such that $\psi_0 = \phi|_A$ and $q\psi_t = p|_A$ $(0 \leqq t \leqq 1)$, there exists an f.p. homotopy $\phi_t: X \to E$ such that $\phi_0 = \phi$ and $\phi_t|_A = \psi_t$ $(0 \leqq t \leqq 1)$. Furthermore, if ψ_t is a $\mathcal{V}$-homotopy for an open cover $\mathcal{V}$ of E, then we can take ϕ_t a $\mathcal{V}$-homotopy.

PROOF. (i)-(iii) follow from the following observations:

(a) $\pi_B: B \times M \to B$ is an ANFE. (If M is an AR, then π_B is an AFR.)

(b) Every f.p. neighborhood retract of an ANFE is also an ANFE.

(c) Every map $p: E \to B$ admits an f.p. closed embedding $i: E \to B \times M$ for some ANR M.

(iv) follows from the same argument as in [Hu, p. 116].

1.2. COMMENTS AND EXAMPLES. (i) *Every fiber of an* ANFR (AFR) *is an* ANR (AR).

(ii) *If an onto map $p: E \to B$ is an* ANFR, *then p admits local sections (i.e., for each $b \in B$ and each $e \in p^{-1}(b)$, there exists a map $s: V \to E$ from a neighborhood V of b such that $ps = id_V$ and $s(b) = e$). In particular, if E is an* ANR *then so is B.*

(iii) *If $p: E \to B$ is a proper* ANFR, *then p is a Hurewicz fibration. Conversely if $p: E \to B$ is a Hurewicz fibration between* ANR's *then p is an* ANFR.

(iv) ([Fe$_1$, Y$_2$]) *If $p: E \to B$ is a proper strongly regular map with* ANR *fibers and* $\dim B < \infty$, *then p is an* ANFR.

(v) *Every bundle map with* ANR *fibers is an* ANFR.

PROOF. (i) This follows from 1.1 (i). If p is an AFR, then p is an f.p. retract of a projection $\pi_B: B \times M \to B$, with M an AR. Therefore each fiber $p^{-1}(b)$ is a retract of the AR M.

(ii) By 1.1. (i), p is an f.p. retract of some $\pi_B|_U$ as in 1.1. (i). Since $\pi_B|_U$ admits local sections, so does p. The second assertion follows from [Hu, p. 98, Theorem 8.1].

(iii) Suppose p is an ANFR. Embed E into an ANR M as a closed subset and consider the f.p. embedding $i: E \to B \times M$, $i(e) = (p(e), e)$ $(e \in E)$. ($i(E)$ is the graph of p.) By the definition there exists an f.p. retraction $r: U \to E$ from some open neighborhood U of $i(E)$. Since p is proper, each $b_0 \in B$ admits neighborhoods W of b_0 in B and V of $p^{-1}(b_0)$ in M such that $W \times V \subset U$ and $p^{-1}(W) \subset V$. Since $r|_{W \times V}$ is an f.p. retraction from the projection $\pi|_W: W \times V \to W$ to p_W, p_W is a fibration. By [Du, p. 403], p is a fibration.

Conversely suppose $p: E \to B$ is a fibration between ANR's. The ANR B admits a *local equiconnecting function* $\lambda: V \times [0, 1] \to B$ ([Fo]), that is,

(a) V is an open neighborhood of the diagonal $\Delta(B)=\{(b, b): b\in B\}$ in $B\times B$.

(b) $\lambda(b, b', 0)=b'$, $\lambda(b, b', 1)=b$ $((b, b')\in V)$ and $\lambda(b, b, t)=b$ $(b\in B, 0\leqq t\leqq 1)$.

Let $U=(id_B\times p)^{-1}(V)$. Since p is a *regular* fibration ([Du, p. 397]), there exists a homotopy $H: U\times[0, 1]\to E$ such that $pH(b, e, t)=\lambda(b, p(e), t)$, $H(b, e, 0)=e$ $((b, e)\in U)$ and $H(p(e), e, t)=e$ $(e\in E, 0\leqq t\leqq 1)$. Then $H_1: U\to E$ is an f.p. retraction and by 1.1. (i) p is an ANFR.

(iv) See [Y_2, Theorem 1.4] and also [Fe_1, Theorem 1].

(v) This follows from the next proposition.

1.3. PROPOSITION. *Let $p: E\to B$ be an onto map.*

(i) *If $p: E\to B$ is an* ANFR *and $C\subset B$ is a subset, then p_C is an* ANFR *over C.*

(ii) *If $B=B_1\cup B_2$, $B_i\subset B$ closed and p_{B_i} is an* ANFR *over B_i $(i=1, 2)$, then p is an* ANFR.

(iii) *If each $b\in B$ admits a neighborhood U for which p_U is an* ANFR *over U then p is an* ANFR.

PROOF. (i) If p is an f.p. retract of the projection $\pi_B|_U$ as in 1.1. (i), then p_C is an f.p. retract of $\pi_C|_{U\cap C\times M}$. Therefore (i) follows from 1.1. (i).

(ii) We may assume that E is a closed subset in $B\times M$, M is a ANR, and that $p=\pi_B|_E$. Since p_{B_1} is an ANFR, there exist an open neighborhood U_1 of $E|_{B_1}=E\cap B_1\times M$ in $B_1\times M$ and an f.p. retraction $s: U_1\to E|_{B_1}$. Similarly $E|_{B_2}$ is an f.p. retract of an open neighborhood U_2 in $B_2\times M$. Since M is an ANR, replacing U_2 by a smaller one, we have an f.p. deformation retraction

$$\phi: U_2\times[0, 1]\to U_1|_{B_1\cap B_2}\cup(B_2-B_1)\times M$$

such that $\phi_0=id$, $\phi_1(U_2)\subset E$ and $\phi_t|_{E|_{B_2}}=id$ $(0\leqq t\leqq 1)$. Since $p|_{U_1}$ is an ANFR, by 1.1 (iv) we can extend ϕ_1 to an f.p. map $\tilde{\phi}_1: U=U_1|_{B-B_2}\cup U_2\to U_1|_{B-B_2}\cup E$ such that $\tilde{\phi}_1|_E=id$. Define an f.p. retraction $r: U\to E$ by

$$r(b, m)=\begin{cases} s\tilde{\phi}_1(b, m) & (b\in B_1) \\ \phi_1(b, m) & (b\in B_2). \end{cases}$$

By 1.1. (i), p is an ANFR.

(iii) This follows from (i), (ii) and [Mi, Theorem 5.5].

A map $p: X\to B$ is said to be *movable* ([Y_2]) provided for some ANFR $q: E\to B$ and some f.p. closed embedding $i: X\to E$, the following holds:

For each neighborhood U of $i(X)$ in E there exists a neighborhood V of $i(X)$ in U such that for each neighborhood W of $i(X)$ in V there exists an

f. p. deformation $\phi_t: V \to U$ such that $\phi_0 = id$, $\phi_1(V) \subset W$ and $q\phi_t = q|_V$ for $0 \leqq t \leqq 1$.

In addition, if we can take ϕ_t so that $\phi_t|_Z = id_Z$ $(0 \leqq t \leqq 1)$ for some neighborhood Z of $i(X)$, we say the map p is *strongly* movable.

For the definition of shape fibrations, see [$MR_{1,2}$], [Ma] and also 2.6 (iii).

1.4. PROPOSITION. *Let $p: E \to B$ be an* ANFR. *Then*

(i) *p is strongly movable.*

(ii) *If p is proper and B is separable then p is a shape fibration.*

PROOF. (i) This is obvious from the definition.

(ii) This follows from (i) and [Y_2, Theorem 1.1].

1.5. PROPOSITION. (cf. [Hu, p. 43, Theorem 7.1]) *A proper onto map $p: E \to B$ is an* AFR *iff p is an* ANFR *and each fiber of p is contractible.*

PROOF. By 1.2 (i), every fiber of an AFR is contractible.

Conversely suppose p is an ANFR and $p^{-1}(b) \cong *$ for each $b \in B$. Embed E into an AR M as a closed subset and define an f. p. closed embedding $i: E \to B \times M$ by $i(e) = (p(e), e)$. Let $r: U \to E$ be an f. p. retraction from a neighborhood U of $i(E)$ in $B \times M$ given by the assumption.

First we will show that p is *shrinkable* ([Do]), that is, there exists a map $f: B \to E$ and an f. p. homotopy $\phi: E \times [0, 1] \to E$ such that $pf = id_B$, $\phi_0 = id_E$ and $\phi_1 = fp$. To see this, by [Do, 3.2] it suffices to show that each $b \in B$ admits a neighborhood V in B such that p_V is shrinkable over V. Let $b \in B$. Since p is proper and $p^{-1}(b) \cong *$ (hence cell-like), there exist neighborhoods V of b in B and $W_1 \subset W_0$ of $p^{-1}(b)$ in M such that $V \times W_0 \subset U$, $p^{-1}(V) \subset W_1$ and $W_1 \simeq *$ in W_0 by a contraction $\psi: W_1 \times [0, 1] \to W_0$. Let $\psi_1(W_1) = \{m_1\}$. Then the desired section $f^V: V \to p^{-1}(V)$ and the f. p. homotopy $\phi^V: p^{-1}(V) \times [0, 1] \to p^{-1}(V)$ are defined by $f^V(b) = r(b, m_1)$ and $\phi^V(e, t) = r(p(e), \psi(e, t))$. This completes the proof of the shrinkability of p.

Now let f and ϕ be as above. Since i^{-1} is f. p. homotopic to $\phi_1 i^{-1}: i(E) \to E$ and $\phi_1 i^{-1}$ admits an extension $\tilde{\phi}_1: B \times M \to E$ defined by $\tilde{\phi}_1(b, m) = f(b)$, by 1.1 (iv) we have an f. p. retraction $r: B \times M \to E$ (i. e., an extension of i^{-1}). Since π_B is an AFR, so is p.

§ 2. Fiber shape category.

The purpose of this section is to describe a fiber shape category. Our construction is based on [MS, Ch I, §§ 1, 2], to which we refer for definitions of basic terms (pro-category, expansion, etc.).

Fix a space B. $\mathcal{FH}_B$ will denote the usual fiber homotopy category over B, whose objects are maps from metric spaces to B. By $\mathcal{FA}_B$ we denote the full subcategory of $\mathcal{FH}_B$ consisting of maps which are fiber homotopy *dominated* by some ANFR's over B. Below [*] denotes a fiber homotopy class of an appropriate f. p. map.

2.1. PROPOSITION. *Every map* $p: X\to B$ *admits an* $\mathcal{FA}_B$*-expansion* $\underline{i}: p\to \underline{p}$ *in* pro-$\mathcal{FH}_B$.

PROOF. Take an ANFR $q: E\to B$ and an f. p. closed embedding $i: X\to E$ and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open neighborhood base of $i(X)$ in E. For each $\lambda\in\Lambda$, let $i_\lambda=i: X\to U_\lambda$, $p_\lambda=q|_{U_\lambda}: U_\lambda\to B$ and for each $\lambda\leqq\lambda'$ (defined by $U_\lambda\supset U_{\lambda'}$) let $i_{\lambda\lambda'}: U_{\lambda'}\subset U_\lambda$ be the inclusion. By the same argument as in [MS, p. 50, Theorem 4], it is easily verified that $\underline{i}=\{[i_\lambda]\}: p\to\underline{p}=\{p_\lambda, [i_{\lambda\lambda'}], \Lambda\}$ satisfies the condition required in [MS, p. 20, Theorem 1].

By [MS, Ch I, § 2] we obtain a shape category $\boldsymbol{sh}(\mathcal{FH}_B, \mathcal{FA}_B)$, which we will denote by $\boldsymbol{Sh}_B$ and call the *fiber shape category* over B. Let $\boldsymbol{S}: \mathcal{FH}_B\to \boldsymbol{Sh}_B$ be the associated *shape functor*. The next proposition justifies the definition.

Assume B is a compactum (a compact metric space) and let $\boldsymbol{Sh}_B^c$ denote the full subcategory of $\boldsymbol{Sh}_B$ consisting of all maps from *compacta* to B. $\boldsymbol{M}_B$ will denote the fiber shape category over B defined in [$\mathrm{Ka}_{1,3}$].

2.2. PROPOSITION. (cf. [MS, Appendix 2]) *There exists an isomorphism* $\boldsymbol{\Omega}: \boldsymbol{M}_B\to\boldsymbol{Sh}_B^c$ *which commutes with the shape functors.*

PROOF. The proof is just an f. p. analogue of [MS, p. 332, Theorem 1]. For the sake of completeness, we will give the definition of the functor $\boldsymbol{\Omega}$.

Let Q denote the Hilbert cube, $[0, 1]^\infty$. By $\pi_1, \pi_2: Q\times Q\to Q$, we denote the projections onto the first and second factor resp. Let d be a fixed metric on Q. Fix an embedding $B\subset Q$.

Let $p: X\to B$ and $q: Y\to B$ be maps from compacta and $\underline{\phi}: p\to q$ be a morphism in $\boldsymbol{M}_B$. The corresponding morphism $\boldsymbol{\Omega}(\underline{\phi}): p\to q$ in $\boldsymbol{Sh}_B^c$ is defined as follows.

Take f. p. embeddings i of X and j of Y into $Q\times Q$ (i. e., $\pi_1 i=p$ and $\pi_1 j=q$). Since π_1 can be regarded as an extension of both p and q, by the definition of $\boldsymbol{M}_B$, $\underline{\phi}$ is represented by a *fiber fundamental sequence* $\phi_n: Q\times Q\to Q\times Q$ $(n\geqq 1)$ ([$\mathrm{Ka}_{1,3}$]). By the definition of a fiber fundamental sequence, $\{\phi_n\}$ satisfies the following:

(*) For each neighborhood V of $j(Y)$ in $Q\times Q$ and each $\varepsilon>0$ there exist a neighborhood U of $i(X)$ in $Q\times Q$ and $n_0\geqq 1$ such that for each $n\geqq n_0$ there exists a homotopy $F: U\times[0, 1]\to V$ such that $F_0=\phi_{n_0}$, $F_1=\phi_n$ and $d(\pi_1 F(y, x, t), y)<\varepsilon$ $((y, x)\in U,\ 0\leqq t\leqq 1)$.

Define $\tilde{\phi}_n: Q\times Q\to Q\times Q$ by $\tilde{\phi}_n(y, x)=(y, \pi_2\phi_n(y, x))$ and let $\psi_n=\tilde{\phi}_n|_{B\times Q}$. Then $\{\tilde{\phi}_n\}$ is also a fiber fundamental sequence which is fiber homotopic to $\{\phi_n\}$, and $\{\psi_n\}$ satisfies the following:

(**) For each neighborhood V of $j(Y)$ in $B\times Q$ there exist a neighborhood U of $i(X)$ in $B\times Q$ and $n_0\geqq 1$ such that for each $n\geqq n_0$, $\psi_n, \psi_{n_0}: U\to V$ are fiber homotopic (w. r. t. $\pi_1|_V$).

Therefore for any decreasing open neighborhood base $\{V_n\}_{n\geqq 1}$ of $j(Y)$ in $B\times Q$ a there exist a decreasing openn eighborhood base $\{U_n\}_{n\geqq 1}$ of $i(X)$ in $B\times Q$ and a strictly increasing sequence $\{m_n\}_{n\geqq 1}$ of positive integers such that for each $n\geqq 1$ and each $m\geqq m_n$, maps $\psi_m, \psi_{m_n}: U_n\to V_n$ are fiber homotopic.

By 2.1, $\{\pi_1|_{U_n}\}$ and $\{\pi_1|_{V_n}\}$ induce $\mathcal{F}\mathcal{A}_B$-expansion of p and q resp. Define $\boldsymbol{\Omega}(\underline{\phi})$ as the morphism in $\boldsymbol{Sh}_B$ represented by a level morphism $\{[\psi_{m_n}]\}: \{\pi_1|_{U_n}\}\to\{\pi_1|_{V_n}\}$ in pro-$\mathcal{F}\mathcal{H}_B$. One can proceed in the same way as in [DS, Ch 3, § 4] or [MS, Appendix 2] to show that $\boldsymbol{\Omega}$ is well defined and is an isomorphism of categories.

The next proposition follows from [MS, p. 27, Theorem 4, Corollary 2] and implies that for ANFR's the fiber shape theory coincides with the fiber homotopy theory. $[\ ,\ ]_*$ will denote the set of morphisms in the appropriate category.

2.3. PROPOSITION. *Let $p: X\to B$ and $q: Y\to B$ be maps.*

(i) *If q is an* ANFR *then the function*

$$\boldsymbol{S}: [p, q]_{\mathcal{F}\mathcal{H}_B}\to[p, q]_{\boldsymbol{Sh}_B}$$

is bijective.

(ii) *If both p and q are* ANFR*'s then an f. p. map $f: X\to Y$ is a fiber homotopy equivalence iff $\boldsymbol{S}[f]$ is an isomorphism in $\boldsymbol{Sh}_B$.*

We will call any isomorphism in $\boldsymbol{Sh}_B$ a *fiber shape equivalence* and say that two maps p and q to B are fiber shape *equivalent* if there exists an isomorphism of p to q in $\boldsymbol{Sh}_B$. Next we will list some properties of maps which are fiber shape invariant.

2.4. PROPOSITION. ([Ka$_1$]) *A proper onto map $p: X\to B$ is a hereditary shape equivalence iff p is fiber shape equivalent to id_B.*

PROOF. Take an ANFR $q: E \to B$ and an f.p. closed embedding $i: X \to E$ and define $\underline{p}=\{p_\lambda, [i_{\lambda\lambda'}], \Lambda\}$ as in 2.1. By [A$_2$, Theorem 4.5], p is a hereditary shape equivalence iff for each neighborhood U of $i(X)$ in E there exists a neighborhood V of $i(X)$ in U, a map $g: B \to U$ and an f.p. homotopy $\phi: V \times [0, 1] \to U$ such that $qg=id_B$, $\phi_0=id_V$ and $\phi_1(V)=g(B)$.

The latter condition can be translated as follows:

(*) For each $\lambda \in \Lambda$ there exists $\lambda' \geqq \lambda$ such that $[i_{\lambda\lambda'}]$ is factored through id_B (i.e., $[i_{\lambda\lambda'}]=[g_\lambda][p_{\lambda'}]$ for some f.p. map $g_\lambda: B \to U_\lambda$).

Observing that any map $q: Y \to B$ admits a unique morphism to id_B in $\mathcal{F}\mathcal{H}_B$ (i.e., $[q]$), the above (*) is equivalent to the assertion that $\underline{p}$ is isomorphic to id_B in pro-$\mathcal{F}\mathcal{H}_B$ (cf. [MS, p. 116, Theorem 7]), which implies the conclusion.

2.5. PROPOSITION. *Let $p: X \to B$ and $q: Y \to B$ be two proper maps.*

(i) *If there exists a morphism from p to q in $\boldsymbol{Sh}_B$ and p is approximately invertible, then so is q.*

(ii) *If there exists an epimorphism from p to q in $\boldsymbol{Sh}_B$ and p is a hereditary shape equivalence, then so is q.*

(iii) ([Ka$_2$]) *If p weakly dominates q and p is a shape fibration* (*or p has the approximate section extension property* (ASEP)), *then so is q.*

PROOF. (ii) By 2.4 there exists an epimorphism $\phi: id_B \to q$. As noted in the proof of 2.4, every map $r: Z \to B$ admits a unique morphism to id_B in $\mathcal{F}\mathcal{H}_B$ and hence in $\boldsymbol{Sh}_B$ (see 2.3 (i)). Therefore $\boldsymbol{S}[q]\phi=1_{id_B}$. Since ϕ is an epimorphism and $\phi \boldsymbol{S}[q]\phi=\phi$, $\phi \boldsymbol{S}[q]=1_q$. Hence ϕ is an isomorphism and by 2.4, q is a hereditary shape equivalence.

For the proof of (i) and (iii), we need a lemma.

2.6. LEMMA. (I) *Let $p: X \to B$ be a proper map, $\tilde{p}: E \to B$ be an* ANFR *and $i: X \to E$ be an f.p. closed embedding.*

(i) ([A$_1$]) *p is approximately invertible iff each neighborhood U of $i(X)$ in E admits a map $s: B \to U$ with $\tilde{p}s=id_B$.*

(ii) ([Y$_1$, Proposition 1.3]) *p has the* ASEP *iff for each neighborhood U of $i(X)$ in E, there exists a neighborhood U_1 of $i(X)$ in U such that any map $s: C \to U_1$ from a closed subset of B to U_1 with $\tilde{p}s=id_C$ admits an extension $\tilde{s}: B \to U$ with $\tilde{p}\tilde{s}=id_B$.*

(II) *Let $p: X \to B$ be a proper map, $\tilde{p}: M \to L$ an* ANFR *between* ANR's *and $i: X \to M$, $j: B \to L$ be closed embeddings such that $\tilde{p}i=jp$.*

(iii) *p is a shape fibration iff the following holds:*

(*) *For each neighborhood U' of $i(X)$ in M there exist neighborhoods U_1' of $i(X)$ in U' and W of B in L such that for any maps $h: Z \to U_1'$ and $H: Z \times [0, 1] \to W$ with $\tilde{p}h = H_0$, there exists a map $H': Z \times [0, 1] \to U'$ with $H_0' = h$ and $\tilde{p}H' = H$.*

PROOF OF 2.6. (iii) By [Ma] and [MR_1, Proposition 2] p is a shape fibration iff the following holds:

(**) For each neighborhood U'' of $i(X)$ in M and each open cover $\mathcal{V}$ of L there exist neighborhoods U_1' of $i(X)$ in U'' and W of B in L such that for any maps $h: Z \to U_1'$ and $H: Z \times [0, 1] \to W$ with $\tilde{p}h = H_0$ there exists a map $H'': Z \times [0, 1] \to U''$ such that $H_0'' = h$ and $(\tilde{p}H'', H) \leqq \mathcal{V}$.

We must show (**)$\to$(*). First consider the special case that q is the projection $\pi_L: L \times M \to L$, where M and L are ANR's containing X and B as a closed subset resp. Let U' be given. Since p is proper, if we choose U'' so small and $\mathcal{V}$ so fine, then we can adjust the map $H'': Z \times [0, 1] \to U''$ given by (**) to the desired $H': Z \times [0, 1] \to U'$ by defining

$$H'(z, t) = (H(z, t), \pi_M H''(z, t)).$$

We have shown that for any ANR L and some ANFR $q: M \to L$ between ANR's, the shape fibration p satisfies (*). It remains to show that if p satisfies (*) for some ANFR $\tilde{p}: M \to L$, then so does p for any such ANFR over L. This follows from the proof of 2.5 (iii) (see below), considering the identity fiber shape morphism on p.

(i) and (ii) are also known in the special case that q is the projection $\pi_B: B \times M \to B$, with M an ANR. The general case follows from the proof of 2.5.

We return to the proof of 2.5.

(i) Take ANFR's $E \xrightarrow{\tilde{p}} B \xleftarrow{\tilde{q}} F$ and f.p. closed embeddings $X \xrightarrow{i} E$ and $Y \xrightarrow{j} F$ (i.e., $\tilde{p}i = p$ and $\tilde{q}j = q$). The existence of a morphism from p to q implies that for each neighborhood V of $j(Y)$ in F there exist a neighborhood U of $i(X)$ in E and an f.p. map $f: U \to V$ (i.e., $\tilde{q}f = \tilde{p}|_U$). Then for a section $s: B \to U$, fs gives the section required in 2.6 (i).

(iii) Take maps $\begin{array}{ccc} X & \xrightarrow{i} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ B & \subset & L \end{array}$ and $\begin{array}{ccc} Y & \xrightarrow{j} & N \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \tilde{q}} \\ B & \subset & L \end{array}$, where $\tilde{p}$, $\tilde{q}$ are ANFR's between ANR's, L contains B as a closed subset and i, j are f.p. closed embeddings. Let $M|_B = \tilde{p}^{-1}(B)$ and $N|_B = \tilde{q}^{-1}(B)$. By 1.3. (i) the restrictions $M|_B \xrightarrow{\tilde{p}} B \xleftarrow{\tilde{q}} N|_B$

are also ANFR's.

The weak domination condition implies the following (see [Dy_1, §2]):

(a) For each open neighborhood V of $j(Y)$ in $N|_B$ there exist a neighborhood U of $i(X)$ in $M|_B$ and an f.p. map $f: U \to V$ such that for any neighborhood U_1 of $i(X)$ in U there exist a neighborhood V_1 of $j(Y)$ in V and an f.p. map $g: V_1 \to U_1$ such that $fg: V_1 \to V$ is f.p. homotopic to the inclusion $V_1 \subset V$.

Suppose p is a shape fibration. To see that q is a shape fibration, let V' be any open neighborhood of $j(Y)$ in N. We must find neighborhoods V_1' of $j(Y)$ in V' and W of B in L as in 2.6 (iii) for V'. By (a) and 1.1. (ii) we have:

(b) a neighborhood U' of $i(X)$ in M and an f.p. map $f': U' \to V'$,

(c) neighborhoods U_1' of $i(X)$ in U' and W of B in L as in 2.6 (iii) for U',

(d) a neighborhood V_1' of $j(Y)$ in V' and an f.p. map $g': V_1' \to U_1'$ such that $f'g': V_1' \to V'$ is f.p. homotopic to the inclusion $V_1' \subset V'$.

$$\begin{tikzcd}
N & \supset & V' \arrow[d, "\tilde{q}"'] & U' \arrow[l, "f'"'] \arrow[d, "\tilde{p}"'] & \supset & U_1' \arrow[d] & V_1' \arrow[l, "g'"'] \arrow[d, "\tilde{q}"] & Z \arrow[l, "h"'] \arrow[d, phantom, "\cap\times 0"] \\
 & & L \arrow[r, equal] & L & \supset & W \arrow[r, equal] & W & Z\times[0,1] \arrow[l, "H"] \arrow[ulllll, "H'"']
\end{tikzcd}$$

To see that V_1' and W satisfy the required condition, let $h: Z \to V_1'$ and $H: Z\times[0, 1] \to W$ be maps with $H_0 = \tilde{q}h$. By (c) we have a map $G: Z\times[0, 1] \to U'$ with $\tilde{p}G = H$ and $G_0 = g'h$. Define $H' = f'G$. Then $\tilde{q}H' = H$ and $H_0' = f'g'h$ is f.p. homotopic to h. Using 1.1 (iv), H' can be adjusted so that $H_0' = h$.

Using 2.6 (ii), the same argument shows that the ASEP is preserved by any weak domination.

Finally, we will be concerned with inverse limits (cf. [MS, Ch I, §5]). Let $p: X \to B$ be a map between compacta. Suppose that X is the inverse limit of an inverse sequence $\underline{X} = \{X_i, f_{ij}\}$ of compacta, together with the projections $f_i: X \to X_i$ $(i \geqq 1)$ $(f_{ij}f_j = f_i,\ i \leqq j)$ and that p is induced from a level map $\underline{p} = \{p_i: X_i \to B\}$, that is, $p_i f_{ij} = p_j$ and $p_i f_i = p$ $(j \geqq i \geqq 1)$. The following proposition shows that the level map $\underline{p}$ reflects the fiber shape of the inverse limit p.

2.7. PROPOSITION. (cf. [MS, p. 65, Theorem 9]) *Under the above notations, the induced morphism* $\underline{f} = \{[f_i]\}: p \to \underline{p} = \{p_i, [f_{ij}]\}$ *in* pro-$\mathcal{F}\mathcal{H}_B$ *is an* $\mathcal{F}\mathcal{H}_B$*-expansion of* p.

PROOF. Let $q: E \to B$ be an ANFR. We must show the followings ([MS, p. 20, Theorem 1]):

(i) For each f. p. map $g: X \to E$ there exist $i \geqq 1$ and an f. p. map $g_i: X_i \to E$ such that $g_i f_i$ is f. p. homotopic to g.

(ii) for each $i \geqq 1$ and any f. p. maps $g_0, g_1: X_i \to E$ such that $g_0 f_i$ and $g_1 f_i$ are f. p. homotopic, there exists $j \geqq i$ such that $g_0 f_{ij}$ and $g_1 f_{ij}$ are f. p. homotopic.

The simplest way to verify (i) and (ii) may be an f. p. analogue of [DS, Ch 4, § 1]. Let $\tilde{X}$ be a compactum defined as follows: The underlying set of $\tilde{X}$ is the disjoint union of $\{X_i\}_{i \geqq 1}$ and X. The topology of $\tilde{X}$ is given by the open basis consisting of all subsets of the form U_i or $f_i^{-1}(U_i) \cup (\cup \{f_{ij}^{-1}(U_i): j \geqq i\})$, where $i \geqq 1$ and U_i is an open set of X_i. Note that each neighborhood U of X in $\tilde{X}$ contains almost all X_i (finitely many exceptions). Define $\tilde{p}: \tilde{X} \to B$ by $\tilde{p}|_X = p$ and $\tilde{p}|_{X_i} = p_i$ $(i \geqq 1)$.

Now (i) and (ii) are verified as follows.

(i) By the f. p. neighborhood extension property, g admits an extension $\tilde{g}: U \to E$ to a neighborhood U of X in $\tilde{X}$ with $q\tilde{g} = \tilde{p}|_U$. If we choose $i \geqq 1$ sufficiently large, then $X_i \subset U$ and $\tilde{g} f_i$, g are so close that they are f. p. homotopic (recall 1.1 (i)). Define $g_i = \tilde{g}|_{X_i}$.

(ii) Let $X' = X \times [0, 1] \cup (\cup \{X_j : j \geqq i\} \times \{0, 1\}) \subset \tilde{X} \times [0, 1]$ and define a map $G: X' \to E$ by $G|_{X \times [0,1]} =$ an f. p. homotopy from $g_0 f_i$ to $g_1 f_i$ and $G|_{X_j \times \{k\}} = g_k f_{ij}$ $(j \geqq i,\ k = 0, 1)$.

Then G extends to a map $\tilde{G}: V \to E$ from a neighborhood V of X' in $\tilde{X} \times [0, 1]$ with $q\tilde{G}(x, t) = \tilde{p}(x)$ $((x, t) \in V)$. Take $j \geqq i$ with $X_j \times [0, 1] \subset V$. Then $\tilde{G}|_{X_j \times [0,1]}$ is an f. p. homotopy from $g_0 f_{ij}$ to $g_1 f_{ij}$.

§ 3. Complements of maps.

In this section we will prove Chapman's complement theorem in the fiber shape theory and give some applications.

All spaces below are assumed to be separable. $Q = [0, 1]^\infty$ (the Hilbert cube). A closed set X of $B \times Q$ is *a sliced Z-set* ([Fe$_2$]) if for each open cover $\mathcal{U}$ of $B \times Q$ there exists an f. p. map $f: B \times Q \to B \times Q - X$ with $(f, id_{B \times Q}) \leqq \mathcal{U}$, where f. p. means that $\pi_B f = \pi_B$.

3.1. Complement Theorem. *Let X and Y be sliced Z-sets in $B \times Q$. Then the projections $\pi_B|_X$ and $\pi_B|_Y$ are fiber shape equivalent iff there exists an f. p. homeomorphism*

$$h: B \times Q - X \to B \times Q - Y.$$

Using the description of the fiber shape theory given in § 2 and some well known results of *Q-manifold bundles*, the proof of 3.1 is an f. p. analogue of the one in [DS, Ch 3, §5]). First we will recall some results on Q-manifold bundles. Note that every proper map $p: X \to B$ admits an f. p. closed embedding $i: X \to B \times Q$ since X is separable.

3.2. LEMMA. ([Fe$_2$], [Sa]) *Let $p: X \to B$ be a proper map.*

(i) *Every f. p. map $f: X \to B \times Q$ can be approximated arbitrarily closely by a sliced Z-embedding (i. e., an f. p. embedding whose image is a sliced Z-set) which is f. p. homotopic to f by a small homotopy.*

(ii) *If maps $f, g: X \to B \times Q$ are sliced Z-embeddings and f. p. homotopic in an open subset U of $B \times Q$, then there exists an f. p. ambient isotopy $f_t: B \times Q \to B \times Q$ $(0 \leqq t \leqq 1)$ such that $f_0 = id$, $f_1 f = g$ and $f_t|_{B \times Q - U} = id$ $(0 \leqq t \leqq 1)$.*

The next lemma is an f. p. analogue of the main part of the proof of the Complement theorem.

Let U be an open set in $B \times Q$ and let X and Y be sliced Z-sets in $B \times Q$ contained in U. Suppose there exists an isomorphism $\underline{\phi}: \pi_B|_X \to \pi_B|_Y$ in $\boldsymbol{Sh_B}$ such that $\boldsymbol{S}[i(Y, U)]\underline{\phi} = \boldsymbol{S}[i(X, U)]$, where $i(X, U)$ denotes the inclusion $X \subset U$ and $\boldsymbol{S}[i(X, U)]$ is the morphism in $\boldsymbol{Sh_B}$ induced from $[i(X, U)]: \pi_B|_X \to \pi_B|_U$. $\boldsymbol{S}[i(Y, U)]$ is defined similarly. In this case we say that $\pi_B|_X$ and $\pi_B|_Y$ are fiber shape equivalent in U.

3.3. LEMMA. (cf. [DS, 3.5.6, Claim 1]) *Under the above notations, for each neighborhood V of Y in U there exists a neighborhood U_0 of X in U such that for each neighborhood U_1 of X in U_0 there exists an f. p. ambient isotopy $h_t: B \times Q \to B \times Q$ such that $h_0 = id$, $h_1(U_0) \subset V$, $h_1(U_1) \supset Y$, $h_t|_{B \times Q - U} = id$ $(0 \leqq t \leqq 1)$ and $\pi_B|_{h_1(X)}$, $\pi_B|_Y$ are fiber shape equivalent in $h_1(U_1)$.*

PROOF. Since $\pi_B|_V$ is an ANFR, by 2.3 (i), there exists an f. p. map $f: X \to V$ such that $\boldsymbol{S}[f] = \boldsymbol{S}[i(Y, V)]\underline{\phi}$. By 3.2 (i) we may assume f is a sliced Z-embedding. Since $\boldsymbol{S}[i(V, U)]\boldsymbol{S}[f] = \boldsymbol{S}[i(X, U)]$, by 2.3 (i) $i(V, U)f$ is f. p. homotopic to $i(X, U)$. By 3.2 (ii) there exists an f. p. ambient isotopy $f_t: B \times Q \to B \times Q$ such that $f_0 = id_{B \times Q}$, $f_1|_X = f$ and $f_t|_{B \times Q - U} = id$ $(0 \leqq t \leqq 1)$. Take a neighborhood U_0 of X such that $f_1(U_0) \subset V$.

Let U_1 be any neighborhood of X in U_0. Applying the same argument to the fiber shape equivalence $\boldsymbol{S}[f](\underline{\phi})^{-1}: \pi_B|_Y \to \pi_B|_{f(X)}$ in V and the neighborhood $f_1(U_0)$ of $f(X)$, we obtain an f. p. ambient isotopy $g_t: B \times Q \to B \times Q$ such that $g_0 = id_{B \times Q}$, $g_1(Y) \subset f_1(U_1)$ and $g_t|_{B \times Q - V} = id$ $(0 \leqq t \leqq 1)$. Define $h_t = g_t^{-1} f_t$ $(0 \leqq t \leqq 1)$.

PROOF OF 3.1. Suppose there exists an isomorphism $\underline{\phi}: \pi_B|_X \to \pi_B|_Y$ in $\boldsymbol{Sh_B}$.

Note that $S[i(Y, B\times Q)]\underline{\phi}=S[i(X, B\times Q)]$, since π_B is isomorphic to id_B in Sh_B. Applying 3.3 inductively, we can find:

(i) open neighborhoods U_i $(i\geqq 1)$ of X and V_i $(i\geqq 1)$ of Y in $B\times Q$ such that $U_{i+1}\subset U_i\subset N(X, 1/i)$ (the $1/i$-neighborhood of X in $B\times Q$) and $V_{i+1}\subset V_i\subset N(Y, 1/i)$ $(i\geqq 1)$,

(ii) f.p. homeomorphisms $h_i: B\times Q\to B\times Q$ $(i\geqq 1)$ such that $V_i\supset h_i\cdots h_1(U_i)\supset V_{i+1}$ and $h_{i+1}|_{B\times Q-h_i\cdots h_1(U_i)}=id$ $(i\geqq 1)$.

The desired f.p. homeomorphism $h: B\times Q-X\to B\times Q-Y$ is defined by $h|_{B\times Q-U_i}=h_i\cdots h_1|_{B\times Q-U_i}$ $(i\geqq 1)$.

Conversely suppose there exists an f.p. homeomorphism h as above. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open neighborhood base of X in $B\times Q$. Define $V_\lambda=h(U_\lambda-X)\cup Y$ $(\lambda\in\Lambda)$. Then V_λ is open in $B\times Q$. To see this, let $(b, q)\in V_\lambda$. Since $\{b\}\times Q-V_\lambda=h(\{b\}\times Q-U_\lambda)$ is compact, there exist open neighborhoods U of q and V of $\pi_Q(\{b\}\times Q-V_\lambda)$ in Q such that $U\cap V=\phi$. Note that $\pi_B|_{B\times Q-V_\lambda}$ is a closed map since $\pi_B|_{B\times Q-U_\lambda}$ is a closed map and $\pi_B|_{B\times Q-V_\lambda}=\pi_B|_{B\times Q-U_\lambda}(h^{-1})|_{B\times Q-V_\lambda}$. Therefore there exists a neighborhood W of b in B such that $W\times Q-V_\lambda\subset B\times V$. Then $W\times U$ is a neighborhood of (b, q) in $B\times Q$ contained in V_λ. Therefore V_λ is open and $\{V_\lambda\}$ is an open neighborhood base of Y in $B\times Q$.

To see that $\pi_B|_X$ and $\pi_B|_Y$ are isomorphic in Sh_B, by 2.1, it suffices to show that the ANFR-neighborhood systems $\{\pi_B|_{U_\lambda}, [i_{\lambda\lambda'}]\}$ and $\{\pi_B|_{V_\lambda}, [j_{\lambda\lambda'}]\}$ are isomorphic in pro-$\mathcal{FH}_B$. Note that $i_\lambda: U_\lambda-X\subset U_\lambda$ is a fiber homotopy equivalence. In fact, since X is a sliced Z-set, by [Fe$_2$, §4] there exists an f.p. homotopy $f_t: B\times Q\to B\times Q$ $(0\leqq t\leqq 1)$ such that $f_0=id$, $f_t(B\times Q)\subset B\times Q-X$ $(0<t\leqq 1)$ and $f_t(U_\lambda)\subset U_\lambda$ $(0\leqq t\leqq 1)$. Then $f_1: U_\lambda\to U_\lambda-X$ is a fiber homotopy inverse of i_λ since $f_t: U_\lambda-X\to U_\lambda-X: id\simeq f_1 i_\lambda$ and $f_t: U_\lambda\to U_\lambda: id\simeq i_\lambda f_1$. Similarly the inclusion $j_\lambda: V_\lambda-Y\subset V_\lambda$ is a fiber homotopy equivalence. Therefore we have isomorphisms

$$\{\pi_B|_{U_\lambda}\} \underset{\approx}{\xleftarrow{\{[i_\lambda]\}}} \{\pi_B|_{U_\lambda-X}\} \underset{\approx}{\xrightarrow{\{[h]\}}} \{\pi_B|_{V_\lambda-Y}\} \underset{\approx}{\xrightarrow{\{[j_\lambda]\}}} \{\pi_B|_{V_\lambda}\}.$$

This completes the proof of 3.1.

By the construction of $\mathcal{FA}_B$-expansions in 2.1, one can easily show that the notion of movability defined in [Y$_2$] (see the definition before 1.4) coincides with the one in the shape category Sh_B ([MS, Ch II, §6]). Therefore the movability of maps is preserved by any *weak domination*. Once we have obtained the Complement theorem 3.1, by the same argument as in [Dy$_2$, Lemma 2], we can show that the strong movability is also a fiber shape invariant.

3.4. COROLLARY. *If proper maps* $p: X \to B$ *and* $q: Y \to B$ *are fiber shape equivalent and* p *is strongly movable then so is* q.

Finally we will characterize hereditary shape equivalences and approximate fibrations by their *complements*. Below $p: X \to B$ will denote a proper onto map.

3.5. COROLLARY. *Let* $i: X \to B \times Q$ *be a sliced Z-embedding. Then the map* p *is a hereditary shape equivalence iff the projection* $\pi_B: B \times Q - i(X) \to B$ *is f.p. homeomorphic to the projection* $B \times Q \times [0, 1) \to B$.

PROOF. Consider the sliced Z-embedding $B \approx B \times \{q\} \subset B \times Q$, where $q \in Q$ is fixed. Note that $Q - \{q\} \approx Q \times [0, 1)$ ([Ch_2, 12.2]). Then 3.5 follows from 2.4 and 3.1.

The map p is said to be *locally shape trivial* provided each $b \in B$ admits a closed neighborhood V for which p_V is fiber shape equivalent to the projection $\pi_V: V \times p^{-1}(b) \to V$. The space B is said to be *semi-locally contractible* if each $b \in B$ admits a neighborhood V which contracts in B.

3.6. PROPOSITION. *Suppose* B *is locally compact and semi-locally contractible and that each fiber of* p *is an* FANR. *Then the following assertions are equivalent:*

(i) p *is a shape fibration*

(ii) p *is locally shape trivial*

(iii) p *is strongly movable.*

Moreover if B *is finite dimensional, then* (i)-(iii) *is equivalent to the following:*

(iv) p *is completely movable.*

PROOF. (i)→(ii). Let $b \in B$ and let K be a compact neighborhood which contracts in B. By the same argument as in [Ka_2, Proposition 1.3] (cf. [Sp, p. 102, Theorem 14]) it is seen that p_K is fiber shape equivalent to the projection $K \times p^{-1}(b) \to K$.

(ii)→(iii). Let $b \in B$ and let V be a neighborhood of b for which p_V is fiber shape equivalent to $\pi_V: V \times p^{-1}(b) \to V$. Since $p^{-1}(b)$ is an FANR, by [Y_2, Example 3.4, (3)], π_V is strongly movable. Then by 3.4, so is p_V. By [Y_2, Proposition 3.5], p is strongly movable.

(iii)→(i). This follows from [Y_2, Theorem 1.1].

As for (iii)↔(iv) under the assumption $\dim B < \infty$, see [Y_2, Remark 5.3, Theorem 1.3].

3.7. COROLLARY. *Suppose B is locally compact and locally contractible. If p is a local shape fibration (i.e., each $b \in B$ admits a (closed) neighborhood V for which p_V is a shape fibration) and each fiber of p is an* FANR, *then p is a shape fibration.*

PROOF. Let $b \in B$. Take compact neighborhoods $K \subset L$ of b such that p_L is a shape fibration and $K \simeq *$ in L. As in the proof of 3.6 (i)→(ii), p_K is fiber shape equivalent to the projection $\pi_K : K \times p^{-1}(b) \to K$. Therefore p is locally shape trivial and then by 3.6 p is a shape fibration.

3.8. COROLLARY. *Let $i : X \to B \times Q$ be a sliced Z-embedding.*

(i) *p is locally shape trivial iff the projection $\pi_B : B \times Q - i(X) \to B$ is a bundle map.*

(ii) *Suppose B is a locally compact* ANR *and each fiber of p is an* FANR. *Then p is a shape fibration iff $\pi_B : B \times Q - i(X) \to B$ is a bundle map.*

(iii) *Suppose B and X are locally compact* ANR's. *Then p is an approximate fibration iff $\pi_B : B \times Q - i(X) \to B$ is a bundle map.*

PROOF. (i) Let $b \in B$ and V be a neighborhood of b in B. We may assume $p^{-1}(b)$ is Z-embedded into Q. If p_V is fiber shape equivalent to $\pi_V : V \times p^{-1}(b) \to V$, then by 3.1, $\pi_B^{-1}(V) = V \times Q - i(p^{-1}(V))$ is f.p. homeomorphic to $V \times (Q - p^{-1}(b))$. This implies π_B is trivial over V.

Conversely if $\pi_B^{-1}(V)$ is f.p. homeomorphic to a product $V \times F$, then since $F \approx Q - p^{-1}(b)$, by 3.1 p_V is fiber shape equivalent to π_V.

(ii) This follows from (i) and 3.6.

(iii) By [Ka$_2$, Theorem 1.4], p is an approximate fibration iff p is locally shape trivial. Then (iii) follows from (i).

3.9. REMARK. (i) In 3.6, in general, (iv) does not imply (i), since the *Taylor* map ([T]) is not a shape fibration ([MR$_1$, Example 6]).

(ii) In 3.6, if each fiber of p is cell-like, then by [Y$_2$, Theorem 1.2], the conditions (i)-(iii) are equivalent to the condition that p is a hereditary shape equivalence (cf. [Ka$_2$, Theorem 2.5]).

(iii) In 3.7 we cannot omit the assumption that each fiber of p is an FANR (even if each fiber is movable). See [Ru, Example 1].

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