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THE APPROXIMATE SECTION EXTENSION PROPERTY AND HEREDITARY SHAPE EQUIVALENCES[1]

By

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Abstract. In this paper, the concept of the *approximate section extension property* (ASEP) is introduced. It is shown that hereditary shape equivalences are exactly the maps with the hereditary ASEP and every UV^{n-1}-map with n-dimentional range has the ASEP.

0. Introduction.

In this paper we will introduce the concept of the *approxsmate section extension property* (ASEP), which is a shape version of the *section extension property*, [Do]. The ASEP is defined in Section 1 to maps between metric spaces. However, using resolutions of maps, this can be extended to a general case (see Section 3). Main results of the paper are contained in Section 2. We prove the followings:

1) Hereditary shape equivalences (HSE's) are exactly the maps with the hereditary ASEP. In particular, any pull back of a HSE is also HSE.

2) Every UV^{n-1}-map with an n-dimensional range and every UV^{∞}-map between ANR's has the ASEP. If the range is a manifold, then an appropriate converse holds.

One can regard these results as a shape version of some results in the fiber homotopy theory, [Do].

Here, we list some notations to be used throughout the paper. In Sections 1 and 2, spaces are assumed to be metrizable. If A is a subset of a space X, $\bar{A}$ is the closure of A and $\mathrm{inc}(A, X)$ denotes the inclusion map $A\subset X$. $\mathrm{Cov}\,X$ is the set of all *normal* coverings of X. For $\mathcal{V}\in\mathrm{Cov}\,Y$, $\mathrm{st}\,\mathcal{V}$ is the star of $\mathcal{V}$. We say two maps f, $g: X\rightarrow Y$ are $\mathcal{V}$-near and write $(f, g)\leq\mathcal{V}$ when each $x\in X$ admits a $V\in\mathcal{V}$ such that $f(x), g(x)\in V$. An ANR is one for metric spaces. A *polyhedron* is the body $|K|$ of a simplicial complex K with the CW-topology.

We refer to [DS] for the definitions of basic terms in Shape theory, and to [A1] for relation theoretic terms. One should refer to [Do] in Sections 1, 2 and to [Ma] in Section 3.

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1. Approximate section extension property.

Let $f: X \to Y$ be a map between metric spaces. We can find closed embeddings $i: X \to M$, $j: Y \to N$ into ANR's and a map $F: M \to N$ with $Fi = jf$ ([H]). Consider a pair (U, V) of open neighborhoods (nbd's) U of $i(X)$ in M and V of $j(Y)$ in N with $F(U) \subset V$. We call such a pair an *admissible pair* for f w.r.t. F.

PROPOSITION 1.1: *Under the above notations, the following conditions* (1)-(3) *are equivalent. Furthermore if the map f satisfies the condition* (1) *(eq.,* (2), (3)), *for some i, j and F as above, then f satisfies the condition* (1) *for any such i, j and F.*

(1) *For each admissible pair* (U, V) $\mathcal{U} \in \mathrm{Cov}\, U$ *and* $\mathcal{V} \in \mathrm{Cov}\, V$, *there xeist an admissible pair* $(U_1, V_1) \geq (U, V)$ *(i.e.,* $U_1 \subset U$, $V_1 \subset V$) and $\mathcal{V}_1 \in \mathrm{Cov}\, V_1$ *such that for each closed set* A *of* V_1 *and each map* $s: A \to U_1$ *with* $(Fs, \mathrm{inc}(A, V_1)) \leq \mathcal{V}_1$, *there exist an open nbd* W *of* $j(Y)$ *in* V_1 *and a map* $S: W \to U$ *with* $(FS, \mathrm{inc}(W, V)) \leq \mathcal{V}$ *and* $(S|_{A \cap W}, s|_{A \cap W}) \leq \mathcal{U}$.

(2) *For each admissible pair* (U, V), $\mathcal{U} \in \mathrm{Cov}\, U$ *and* $\mathcal{V} \in \mathrm{Cov}\, V$, *there exist* $(U_1, V_1) \geq (U, V)$ *and* $\mathcal{V}_1 \in \mathrm{Cov}\, V_1$ *such that for each closed set* A *of* Y *and each map* $s: A \to U_1$ *with* $(Fs, \mathrm{inc}(A, V_1)) \leq \mathcal{V}_1$, *there exists a map* $S: Y \to U$ *with* $(FS, \mathrm{inc}(Y, V)) \leq \mathcal{V}$ *and* $(S|_A, s) \leq \mathcal{U}$.

(3) *For each open nbd* U *of* $i(X)$ *in* M *and* $\mathcal{V} \in \mathrm{Cov}\, N$, *there exist an open nbd* U_1 *of* $i(X)$ *in* U *and* $\mathcal{V}_1 \in \mathrm{Cov}\, N$ *such that for each closed set* A *of* Y *and each map* $s: A \to U_1$ with $(Fs, \mathrm{inc}(A, N)) \leq \mathcal{V}_1$, *there exists a map* $S: Y \to U$ *with* $(FS, \mathrm{inc}(Y, N)) \leq \mathcal{V}$ *and* $S|_A = s$.

PROOF: Note that U and U_1 are ANR's. Therefore (2)→(1) follows from the nbd extension property of ANR's and (2)→(3) follows from the *homotopy extension theorem* ([H]). (1)→(2) and (3)→(2) are obvious. For the latter statement, see Proposition 3.2, which implies the same conclusion under a more general setting.

DEFINITION 1.2: *We say the map f has the approximate section extension property (ASEP) provided f satisfies the conditions in Proposition* 1.1.

If f is a *proper* map (i.e., the inverses of compact sets are compact), then we can reduce the above conditions to a simpler one. For later use, we shall work in the setting of relation.

Suppose M and N are ANR's, Y is a closed subset of N, $R: Y \to M$ is an *(upper semi-) continuous* relation with *compact point images* (i.e., for each $y \in Y$, $R(y)$ is compact) ([A1]) and $p: N \times M \to N$ is the projection. Note that $p(R) \subset Y$.

PROPOSITION 1.3: *Under the above notations, the projection $p: R \to Y$ has the ASEP iff*

(♯): *each nbd U of R in $Y \times M$ contains a nbd V of R such that*

(*): *for each closed set A of Y and each map $s: A \to V$ with $ps = \mathrm{inc}(A, Y)$, there exists a map $S: Y \to U$ with $pS = 1_Y$ and $S|_A = s$.*

Let $f: X \to Y$ be a proper map and M an ANR containing X as a closed subset. Consider the continuous relation $f^{-1} = \cup\{y \times f^{-1}(y) : y \in Y\} : Y \to M$. Since the projection $p: f^{-1} \to Y$ corresponds to f by the identification $f^{-1} \to X: (f(x), x) \mapsto x$, we get the following.

COROLLARY 1.4: *Under the above notation, f has the ASEP iff (♯) holds with R replaced by f^{-1}.*

REMARK 1.5: The map f is said to be *approximately invertible* ([A2]) if each nbd U of f^{-1} in $Y \times M$ admits a map $S: Y \to U$ such that $pS = 1_Y$.

PROOF OF 1.3: From Proposition 1.1, p has the ASEP iff

(♯♯): *for each nbd U' of R in $N \times M$ and $\mathcal{U} \in \mathrm{Cov}\, N$, there exist a nbd V' of R in U' and $\mathcal{V} \in \mathrm{Cov}\, N$ such that*

(**): *if A is a closed set of Y and $s': A \to V'$ is a map with $(ps', \mathrm{inc}(A, N)) \leq \mathcal{V}$, then there exists $S': Y \to U'$ with $(pS', \mathrm{inc}(Y, N)) \leq \mathcal{U}$ and $S'|_A = s'$.*

(♯)→(♯♯): Given U' and $\mathcal{U}$ as in (♯♯). By [A1], Lemma A-8, there exist an open nbd U_1' of R in U' and $\mathcal{U}_1 = \{U_\kappa\}_{\kappa \in K} \in \mathrm{Cov}\, N$ such that $U_\kappa \times U_1'(U_\kappa) \subset U'$ for each $\kappa \in K$ and $\mathcal{U}_1 < \mathcal{U}$. Apply (♯) to $U = U_1' \cap (Y \times M)$ and we have an open nbd V of R in U which satisfies (*). By [A1], Lemma A-8, there exist a nbd V' of R in U_1' and a $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda} \in \mathrm{Cov}\, N$ such that $(V_\lambda \times V'(V_\lambda)) \cap (Y \times M) \subset V$ for each $\lambda \in \Lambda$ and $\mathcal{V}$-near maps to N are $\mathcal{U}_1$-homotopic. Then V' and $\mathcal{V}$ satisfy (**). In fact, take s' as in (**). Since $(ps', \mathrm{inc}(A, N)) \leq \mathcal{V}$, we can define $s: A \to V$ by $s(y) = (y, \pi s'(y))$ $(y \in A)$, where $\pi: N \times M \to M$ is the projection. By (*), the map s extends to a map $S: Y \to U$ with $pS = 1_Y$, and since ps' is $\mathcal{U}_1$-homotopic to $\mathrm{inc}(A, N)$, using the homotopy extension theorem, ps' extends to a map $g: Y \to N$ which is $\mathcal{U}_1$-homotopic to $\mathrm{inc}(Y, N)$. Then the desired map $S': Y \to U'$ is defined by $S'(y) = (g(y), \pi S(y))$ $(y \in Y)$.

(♯♯)→(♯): The proof is similar and omitted.

We use 1.4 to obtain the *Uniformization Theorem* for the ASEP. Compair this with [Do], Theorem 2.7.

THEOREM 1.6: *Suppose $f: X \to Y$ is a proper map. If each $y \in Y$ admits a* *(not necessarily open) nbd V in Y such that $f|_{f^{-1}(V)}: f^{-1}(V) \to V$ has the ASEP, then f itself has the ASEP.*

PROOF: Consider the following property $\mathcal{P}(A)$ for each subset A of Y.

$\mathcal{P}(A)$: *f_A has the ASEP, where $f_A = f|_{f^{-1}(A)}: f^{-1}(A) \to A$.* In order to show $\mathcal{P}(Y)$ holds, by [Mi], Theorem 5.5, it suffices to show that $\mathcal{P}$ is an *F-hereditary* property, that is, satisfies the following conditions:

(F1) *If f_A has the ASEP and B is a closed subset of A, then f_B has the ASEP.*

(F2) *Suppose $A = A_1 \cup A_2 \subset Y$ and A_1 and A_2 are closed in Y. If $f_i = f_{A_i}$ has the ASEP $(i=1, 2)$, then f_A has the ASEP.*

(F3) *Suppose $A = \cup\{A_\lambda : \lambda \in \Lambda\} \subset Y$ and $\{A_\lambda\}_{\lambda \in \Lambda}$ is discrete in Y. If each f_{A_λ} has the ASEP, then f_A has the ASEP.*

(F1) and (F3) are easily verified, so we will prove (F2). We use the same notation as in Corollary 1.4. Note that $f_i^{-1} = f^{-1} \cap (A_i \times M)$ $(i=1, 2)$. To see the ASEP of f_A, let U be any open nbd of f_A^{-1} in $A \times M$. Since f_1 has the ASEP, the nbd $U \cap (A_1 \times M)$ of f_1^{-1} contains an open nbd V_1 of f_1^{-1} in $A_1 \times M$ which satisffes the condition (*) in Proposition 1.3 w.r.t. f_1^{-1}. In turn, by the ASEP of f_2, the open nbd $U_2 = \{V_1 \cup (U - (A_1 \times M))\} \cap (A_2 \times M)$ of f_2^{-1} in $A_2 \times M$ contains an open nbd V_2 of f_2^{-1} in $A_2 \times M$ which satisfies (*) w.r.t. f_2^{-1}. Then the nbd $V = (V_1 - (A_2 \times M)) \cup V_2$ of f^{-1} in U satisfies (*) w.r.t. f^{-1} and U.

2. The ASEP, HSE's and UV^n-maps.

We begin with a reference to [A1]. Let $f: X \to Y$ be a proper map between metric spaces, and M an ANR containing X as a closed subset.

THEOREM ([A1], Theorem 4.5): *The map f is a HSE iff the relation $f^{-1}: Y \to M$ is slice trivial, that is, satisfies the following:*

For each nbd U of f^{-1} in $Y \times M$ there exist a nbd V of f^{-1} and maps $\phi: V \times [0, 1] \to U$ and $S: Y \to U$ such that $\phi_0 = \mathrm{inc}(V, U)$, $\phi_1(y, x) = S(y)$, $p\phi(y, x, t) = y$, $pS = 1_Y$ $((y, x) \in V,\ t \in [0, 1])$, where $p: Y \times M \to Y$ is the projection.

We will use the above theorem to get another characterization of HSE's in term of the ASEP, which corresponds to [Do], Proposition 3.1.

Let $\alpha: B \to Y$ is a proper map. Define E and $f_\alpha: E \to B$, $\beta: E \to X$ by $E = \{(b, x) \in B \times X: \alpha(b) = f(x)\}$, $f_\alpha(b, x) = b$, $\beta(b, x) = x$, $((b, x) \in E)$. The map f_α is called the *map induced from f by α*. Since $E = \cup\{b \times f^{-1}\alpha(b): b \in B\} = \cup\{\alpha^{-1}(y) \times f^{-1}(y): y \in Y\}$, if we regard E as a relation from B to X, we have:

1) $E = f^{-1}\alpha$, therefore E is *continuous* and has *compact* point images.

2) $\alpha \times 1_M: B \times M \to Y \times M$ is a *closed* map and $E = (\alpha \times 1_M)^{-1}(f^{-1})$.

LEMMA 2.1: *Under the above notation, the following conditions are equivalent.*

a) f_α *has the ASEP.*

b) *Each nbd* U *of* f^{-1} *in* $Y\times M$ *contains a nbd* V *of* f^{-1} *such that if* A *is a closed set of* B *and* $\alpha_0: A\to V$ *is a map with* $p\alpha_0=\alpha|_A$, *then there exists a map* α': $B\to U$ *with* $p\alpha'=\alpha$ *and* $\alpha'|_A=\alpha_0$.

PROOF: By Proposition 1.3, a) is equivalent to 1.3 (#), with R and Y replaced by E and B. By the observation 2), $\alpha\times 1_M$ gives the correspondence between a nbd base of f^{-1} in $Y\times M$ and a nbd base of E in $B\times M$, so that the maps α_0 and α' as in b) correspond to the maps s and S as in (#) and (*). From this follows 2.1. Compair this with [Do], Proposition 3.1.

Let $\tilde{f}$ be the map induced from f by fr, i.e., $\tilde{f}=f_{fr}$, where $r: X\times[0,1]\to X$ is the projection.

THEOREM 2.2: *Let* $f: X\to Y$ *be a proper map. The following conditions are equivalent.*

a) *For each proper map* $\alpha: B\to Y$, f_α *has the ASEP.*

b) f *is approximately invertible* (see Remark 1.5) *and* $\tilde{f}$ *has the ASEP.*

c) f *is a HSE.*

d) *Each nbd* U *of* f^{-1} *in* $Y\times M$ *contains a nbd* V *of* f^{-1} *such that for any proper map* $\alpha: B\to Y$, *Lemma* 2.1, b) *holdds.*

Since $(f_\alpha)_\beta=f_{\alpha\beta}$ for any maps $C\xrightarrow{\beta}B\xrightarrow{\alpha}Y$, we have the following.

COROLLARY 2.3: *If* f *is a HSE, then* f_α *is a HSE for any proper map* α: $B\to Y$.

REMARK 2.4: By [A1], Lemma 4.6, we see the relation $E=f^{-1}\alpha: B\to M$ is slice trivial if $f^{-1}: Y\to M$ is slice trivial. This also implies 2.3.

PROOF OF 2.2: (a)$\to$(b) follows from 1.4. (b)$\to$(c): It suffices to show that each open nbd U of f^{-1} in $Y\times M$ admits a slice contraction $\phi: f^{-1}\times[0,1]\to U$ of f^{-1} in U. (See [A1], Lemma 4.3.) Applying Lemma 2.1 to $\tilde{f}$, we can find an open nbd V of f^{-1} in U such that each map $g: X\times\{0,1\}\to V$ with $pg=fr|_{X\times\{0,1\}}$ has an extension $G: X\times[0,1]\to U$ with $pG=fr$. Since f is approximately invertible, there exists a map $S: Y\to V$ with $PS=1_Y$. Then the map $g: X\times\{0,1\}\to V$ defined by $g(x,0)=(f(x),x)\in f^{-1}$ and $g(x,1)=S(f(x))\in V$ $(x\in X)$ admits an extension G as above. Define $\phi: f^{-1}\times[0,1]\to U$ by $\phi(y,x,t)=G(x,t)$ $((y,x,t)\in f^{-1}\times[0,1])$. (c)$\to$(d): See the proof of [A1], Proposition 2.2. (d)$\to$(a) follows from Lemma 2.1.

Next we study UV^n-maps with the ASEP. We recall some definitions of basic terms. For $n \geq 0$, $\boldsymbol{S}^n$ denotes the standard *n-sphere* and $\boldsymbol{B}^n$ denotes the *n-ball.* Suppose X be a metric space and $i: X \to M$ is a closed embedding into an ANR M. We say X is UV^n (or AC^n) provided each nbd U of $i(X)$ in M contains a nbd V of $i(X)$ such that every $\alpha: \boldsymbol{S}^k \to V$ $(0 \leq k \leq n)$ is null homotopic in U. The definition does *not* depend on the choice of such an embedding i. X is UV^∞ if X is UV^n for each $n \geq 0$. A UV^n-map $(0 \leq n \leq \infty)$ is an onto map each point inverse of which is UV^n. For the details, see [Dy], [K] and [L]. We will prove the following theorem.

THEOREM 2.5: *Suppose $f: X \to Y$ is a closed onto map.*

(1) *If f is a UV^{n-1}-map and $\dim Y \leq n < \infty$, then f has the ASEP.*

(2) *If f is a UV^∞-map and X and Y are ANR's, then f has the ASEP.*

The proof is based on the following *lifting* lemmas.

LEMMA 1 ([Dy], Lemma 8.3): *Under the same notation as in Proposition* 1.1, *suppose the map f is a closed UV^{n-1}-map $(0 \leq n < \infty)$. Then for each admissible pair (U, V) for f w.r.t. F and each $\mathcal{V}' \in \mathrm{Cov}\, V$, there exist $(U_1, V_1) \geq (U, V)$ and $\mathcal{V}_1 \in \mathrm{Cov}\, V_1$ such that*

(*) *if (P, Q) is a polyhedral pair with $\dim p \leq n$ and $g: P \to V_1$, $h: Q \to U_1$ are maps with $(g|_Q, Fh) \leq \mathcal{V}_1$, then there exists a map $g': P \to U$ with $g'|_Q = h$ and $(Fg', g) \leq \mathcal{V}'$.*

LEMMA 2 ([K], Theorem 1, Part II): *Let $f: X \to Y$ be an LC^∞-dense map. Then for each locally finite open covering $\mathcal{V}$ of Y, polyhedral pair (P, Q) and maps $g: P \to Y$, $h: Q \to X$ with $g|_Q \simeq fh$ ($\mathcal{V}$-homotopic), there exists a map $g': P \to X$ with $g'|_Q = h$ and $fg' \simeq g$ (st $\mathcal{V}$-homotopic).*

REMARK 2.6: 1) Among *closed onto maps with ANR domains,* LC^∞-(dense) maps coincide with UV^∞-maps.

2) Theorem 2.5, (2) holds even if X is an *approximate polyhedron* (AP) ([Ma]) and f is an LC^∞-dense map.

However, it will be shown that Taylor's map $F: X \to Q$ does not have the ASEP (Example 3.7). Hence we can't omit the assumption on X if $\dim Y = \infty$.

PROOF OF 2.5: (1) Under the same notation as in Proposition 1.1, we will show that f satisfies the condition (2) in 1.1. Given any admissible pair (U, V), $\mathcal{U} \in \mathrm{Cov}\, U$ and $\mathcal{V} \in \mathrm{Cov}\, V$. Let $\mathcal{V}'$ be a star refinement of $\mathcal{V}$. By Lemma 1, we get $(U_1, V_1) \geq (U, V)$ and $\mathcal{V}_1 \in \mathrm{Cov}\, V_1$ which satisfy (*). We must verify that

(U_1, V_1), $\mathcal{V}_1$ satisfy the required condition in Proposition 1.1, (2) w.r.t. (U, V), $\mathcal{U}$ and $\mathcal{V}$. Let A be a closed set of Y, W an open nbd of A in V_1 and $s:W\to U_1$ a map with $(Fs, \operatorname{inc}(W, V_1))\leq\mathcal{V}_1$. Take an open nbd W_1 of A with $\bar{W}_1\subset W$ and a common refinement $\mathcal{W}\in\operatorname{Cov}V_1$ of coverings $\mathcal{V}'|_{V_1}$, $\{V_1-A, W_1\}$ and $s^{-1}(\mathcal{U}|_{U_1})\cup\{V_1-A\}$. Since $\dim Y\leq n$ and V_1 is an ANR, there exist a polyhedron P and maps $Y\xrightarrow{i}P\xrightarrow{r}V_1$ such that $\dim P\leq n$ and $(ri, \operatorname{inc}(Y, V_1))\leq\mathcal{W}$ ([H], Themrem 6.1). Choose a triangulation K of P such that $\{|\sigma|:\sigma\in K\}$ refines $\{r^{-1}(W), P-r^{-1}(\bar{W}_1)\}$, and put $Q=\cup\{|\sigma|:\sigma\in K, |\sigma|\cap r^{-1}(\bar{W}_1)\neq\emptyset\}$. Then Q is a subpolyhedron of P and $i(A)\subset Q$, $r(Q)\subset W$. Since $(Fsr|_Q, r|_Q)\leq\mathcal{V}_1$, by (*), we obtain a map $r':P\to U$ with $(Fr', r)\leq\mathcal{V}'$ and $r'|_Q=sr|_Q$. Put $S=r'i:Y\to U$, then it is easy to see that $(FS, \operatorname{inc}(U, V))\leq\mathcal{V}$ and $(S|_A, s|_A)\leq\mathcal{U}$. This completes the proof of (1).

(2) is verified by the same argument as in (1), using Lemma 2 instead of Lemma 1.

The next theorem is a partial converse of the above theorem.

THEOREM 2.7: *Let $f:X\to Y$ be a proper (onto) map with the ASEP and $y\in Y$. If each nbd V of y in Y contains an n-sphere contractible in V, then $f^{-1}(y)$ is* UV^n.

PROOF. Take a closed embedding $X\hookrightarrow M$ into an ANR M. To show $f^{-1}(y)$ is UV^n (in M), let U_1 be any open nbd of $f^{-1}(y)$ in M. We must find a nbd U_2 of $f^{-1}(y)$ in U_1 as in the definition of UV^n-property. Take an open nbd V_1, V_2 of y in Y with $f^{-1}(V_1)\subset U_1$, $\bar{V}_2\subset V_1$ and let $\tilde{U}=V_1\times U_1\cup(Y-\bar{V}_2)\times M$. Then by Corollary 1.4, we get an open nbd $\tilde{V}$ of f^{-1} in $\tilde{U}$ which satisfies Proposition 1.3, (*). Take a nbd V_3 of y in V_2 and a nbd U_2 of $f^{-1}(y)$ in U_1 such that $V_3\times U_2\subset\tilde{V}$. By the assumption, V_3 contains an n-sphere $\boldsymbol{S}^n$ contractible in V_3. Now for $0\leq i\leq n$, consider $\boldsymbol{S}^i\subset\boldsymbol{S}^n$ and given any map $\alpha:\boldsymbol{S}^i\to U_2$. Then the map $s:\boldsymbol{S}^i\to\tilde{V}$ defined by $s(x)=(x, \alpha(x))$ $(x\in\boldsymbol{S}^i)$ can be extended to a map $S:Y\to\tilde{U}$ with $pS=1_Y$. Since $\boldsymbol{S}^i\simeq 0$ in V_3, we have a map $h:\boldsymbol{B}^{i+1}\to V_3$ such that $h|_{\boldsymbol{S}^i}=\operatorname{inc}(\boldsymbol{S}^i, Y)$. Then $\pi Sh:\boldsymbol{B}^{i+1}\to U_1$ is an extension of α, where $\pi:Y\times M\to M$ is the projection. This completes the proof.

COROLLARY 2.8: *Let $f:X\to Y$ be a proper map.*

(1) *Suppose* $\dim Y\leq n$ $(0\leq n<\infty)$ *and each non-empty open set U of Y contains an $(n-1)$-sphere contractible in U. Then the map f has the ASEP iff f is a* UV^{n-1}-*map*

(2) *Suppose X and Y are ANR's and each non-empty open set U of Y contains an n-sphere for each $n\geq 0$. Then f has the ASEP iff f is a* UV^{∞}-*map.*

REMARK 2.9: If $\dim Y=n$ $(0\leq n<\infty)$ and Y contains a *dense open set which* *is an n-manifold*, then Y satisfies the condition in (1). If Y is a $\boldsymbol{Q}$ (or $\mathcal{L}_2$)-*manifold*, then Y satisfies the condition in (2).

3. Generalization.

In this section, we extend the definition of the ASEP to the general case. The manner of the generalization has been established in [Ma], where the concept of shape fibrations is generalized using (*AP*-) *resolutions* of maps. We shall show the same method works for the ASEP as well as shape fibrations.

We use the same notations as in [Ma]. In addition, $A(\underline{p})$ denotes the set of all *admissible* pairs of *a map $\underline{p}$ of systems*. In this section, spaces are not assumed to be metrizable. Let B be a space and $A\subset V$ subsets of B. By definition, V is a *halo* of A in B if there is a map $\tau: B\to[0,1]$ with $A\subset\tau^{-1}(1)$ are $B-V\subset\tau^{-1}(0)$.

DEFINITION 3.1: *Let $p: E\to B$ be a map of systems, where $\underline{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)$, $\underline{B}=(B_\mu, r_{\mu\mu_1}, M)$ and $\underline{p}=(p_\mu, \phi)$. We say $\underline{p}$ has the ASEP provided the following holds.*

For each $(\lambda,\mu)\in A(\underline{p})$ and each $\mathcal{U}_\lambda\in\mathrm{Cov}\,E_\lambda$, $\mathcal{V}_\mu\in\mathrm{Cov}\,B_\mu$, there exist $(\lambda_1,\mu_1)\geq(\lambda,\mu)$ in $A(\underline{p})$ and $\mathcal{V}_{\mu_1}\in\mathrm{Cov}\,B_{\mu_1}$ which satisfies the following:

(*) *Suppose $\mu_2\geq\mu_1$, V is a halo of a subset A in B_{μ_2} and $s: V\to E_{\lambda_1}$ is a map with $(p_{\mu_1\lambda_1}s, r_{\mu_1\mu_2}|_V)\leq\mathcal{V}_{\mu_1}$. Then there exist $\mu_3\geq\mu_2$ and a map $S: B_{\mu_3}\to E_\lambda$ with $(p_{\mu\lambda}S, r_{\mu\mu_3})\leq\mathcal{V}_\mu$ and $(S|_{r^{-1}_{\mu_2\mu_3}(A)}, q_{\lambda\lambda_1}sr_{\mu_2\mu_3}|_{r^{-1}_{\mu_2\mu_3}(A)})\leq\mathcal{U}_\lambda$.*

PROPOSITION 3.2: *Let $(\underline{q},\underline{r},\underline{p})$ and $(\underline{q}',\underline{r}',\underline{p}')$ be two* AP-*resolutions of a map $p: E\to B$. If $\underline{p}$ has the ASEP, then so does $\underline{p}'$.*

Note that under the notation of Proposition 1.1, all admissible pairs form an ANR (hence *AP*)-resolution of the map f. Therefore by Proposition 1.1, the follhwing definition extends Definition 1.2.

DEFINITION 3.3: *Let $p: E\to B$ be a map. We say the map p has the ASEP provided some* (eq., *any*) AP-*resolution of p has the ASEP.*

We can extend the concept of approximate invertibility ([A2]) by the same method.

PROPOSITITION AND DEFINITION 3.4: *Suppose $p: E\to B$ be a map and $(\underline{q},\underline{r},\underline{p})$ is an* AP-*resolution of p, where $\underline{q}=(q_\lambda): E\to\underline{E}=(E_\lambda, q_{\lambda\lambda_1}, \Lambda)$, $\underline{r}=(r_\mu): B\to\underline{B}=(B_\mu, r_{\mu\mu_1}, M)$ and $\underline{p}=(p_\mu):\underline{E}\to\underline{B}$. Then the following conditions are equivalent and depend only on the map p.*

(1) *For each $(\lambda,\mu)\in A(\underline{p})$ and each $\mathcal{V}_\mu\in\mathrm{Cov}\,B_\mu$, there exist $\mu_1\geq\mu$ in M and a*

map $S: B_{\mu_1} \to E_\lambda$ *with* $(p_{\mu\lambda}S, r_{\mu\mu_1}) \leq \mathcal{V}_\mu$.

(2) *For each* $(\lambda, \mu) \in A(\underline{p})$ *and each* $\mathcal{V}_\mu \in \mathrm{Cov}\, B_\mu$, *there exists* $S: B \to E_\lambda$ *with* $(p_{\mu\lambda}S, r_\mu) \leq \mathcal{V}_\mu$.

We say the map p *has approximate sections when the above conditions are satisfied.*

REMARK 3.5: T. Watanabe also introduced ([W], Section 24) the concept of *weak approximative dominations*, which is essentially the same as ours.

REMMARK 3.6: 1) Every ANR-resolution of a space X induces an HPol-*expansion* of X in the category *pro*-HTop. Therefore by the definition, if a map $p: E \to B$ has approximate sections then p induces a *weak domination* in Shape category.

2) Let $f: X \to Y$ be a map between metric spaces. If the map f has approximate sections, then f has a *dense* image.

The proof of 3.2 is similar to those of [Ma], Theorem 4, or [MR], Theorem 1. For the sake of completeness, we give a full detail of the proof. As for notations, let $\underline{q}=(q_\lambda): E \to \underline{E}=(E_\lambda, q_{\lambda\lambda_1}, \Lambda)$, $\underline{r}=(r_\mu): B \to \underline{B}=(B_\mu, r_{\mu\mu_1}, M)$, $\underline{p}=(p_\mu): \underline{E} \to \underline{B}$, $\underline{q}'=(q'_\kappa): E \to \underline{F}=(F_\kappa, q'_{\kappa\kappa_1}, K)$, $\underline{r}'=(r'_\nu): B \to \underline{C}=(C_\nu, r'_{\nu\nu_1}, N)$, $\underline{p}'=(p'_\nu): \underline{F} \to \underline{C}$. We need the following lemma. See [Ma], Definition 3 and [MR], Theorem 1.

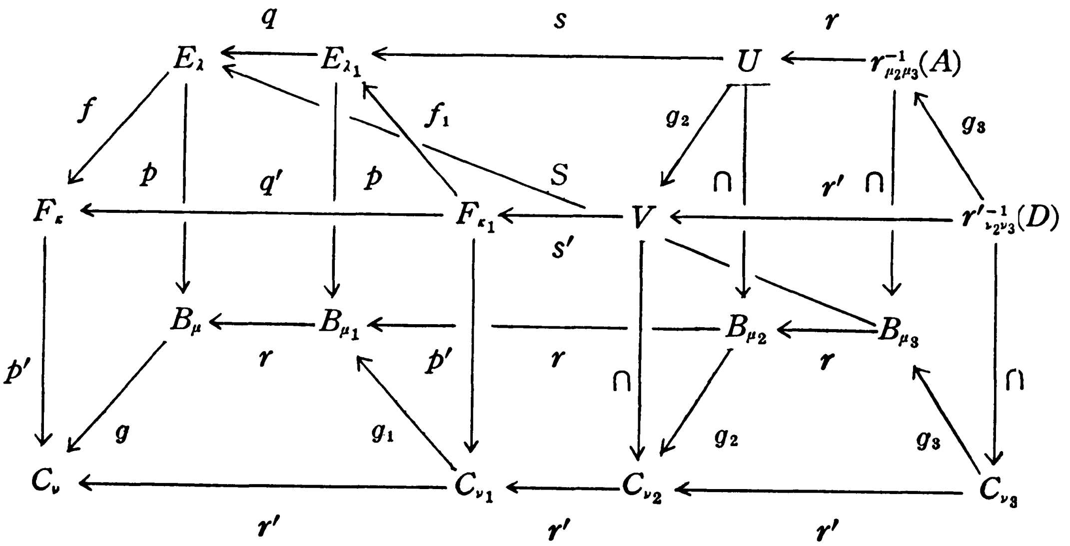
LEMMA 3.7: *Under the above notation:*

(1) *For each* $(\kappa, \nu) \in A(\underline{p}')$, $\mathcal{W} \in \mathrm{Cov}\, F_\kappa$ *and* $\mathcal{Z} \in \mathrm{Cov}\, C_\nu$, *there exist* $(\lambda, \mu) \in A(\underline{p})$ *and maps* $f: E_\lambda \to F_\kappa$, $g: B_\mu \to C_\nu$ *with* $(fq_\lambda, q'_\kappa) \leq \mathcal{W}$, $(gr_\mu, r'_\nu) \leq \mathcal{Z}$ *and* $(p'_{\nu\kappa}f, gp_{\mu\lambda}) \leq \mathcal{Z}$.

(2) *For each* $(\kappa, \nu) \in A(\underline{p}')$, $\mathcal{W} \in \mathrm{Cov}\, F_\kappa$ *and* $\mathcal{Z} \in \mathrm{Cov}\, C_\nu$, *there exist* $\mathcal{W}' \in \mathrm{Cov}\, F_\kappa$ *and* $\mathcal{Z}' \in \mathrm{Cov}\, C_\nu$ *which refine* $\mathcal{W}$ *and* $\mathcal{Z}$ *resp. and satisfy the following:*

For each $(\lambda, \mu) \in A(\underline{p})$ *and maps* $f: E_\lambda \to F_\kappa$, $g: B_\lambda \to C_\nu$ *with* $(fq_\lambda, q'_\kappa) \leq \mathcal{W}'$, $(gr_\mu, r'_\nu) \leq \mathcal{Z}'$ *and* $(p'_{\nu\kappa}f, gp_{\mu\lambda}) \leq \mathcal{Z}'$ *and for each* $(\lambda_1, \mu_1) \geq (\lambda, \mu)$ *in* $A(\underline{p})$ *and* $\mathcal{U}_1 \in \mathrm{Cov}\, E_{\lambda_1}$, $\mathcal{V}_1 \in \mathrm{Cov}\, B_{\mu_1}$, *there exist* $(\kappa_1, \nu_1) \geq (\kappa, \nu)$ *in* $A(\underline{p}')$ *and two maps* $f_1: F_{\kappa_1} \to E_{\lambda_1}$, $g_1: C_{\nu_1} \to B_{\mu_1}$ *with* $(f_1 q'_{\kappa_1}, q_{\lambda_1}) \leq \mathcal{U}_1$, $(g_1 r'_{\nu_1}, r_{\mu_1}) \leq \mathcal{V}_1$, $(p_{\mu_1\lambda_1} f_1, g_1 p'_{\nu_1\kappa_1}) \leq \mathcal{V}_1$ *and* $(fq_{\lambda\lambda_1} f_1, q'_{\kappa\kappa_1}) \leq \mathcal{W}$, $(gr_{u\mu_1} g_1, r'_{\nu\nu_1}) \leq \mathcal{Z}$.

PROOF OF 3.2: The proof consists of the construction of the following diagram by Lemma 3.7 and the ASEP of $\underline{p}$.



To show p' has the ASEP, given $(\kappa,\nu)\in A(p')$, $\mathcal{W}_\kappa\in\operatorname{Cov}F_\kappa$ and $\mathcal{Z}_\nu\in\operatorname{Cov}C_\nu$. Take $\mathcal{W}'\in\operatorname{Cov}F_\kappa$ and $\mathcal{Z}'\in\operatorname{Cov}C_\nu$ with $\operatorname{st}\mathcal{W}'<\mathcal{W}_\kappa$, $\operatorname{st}^2\mathcal{Z}'<\mathcal{Z}_\nu$. Apply 3.7, (2) to (κ,ν), $\mathcal{W}'$, $\mathcal{Z}'$ and we obtain

(a): $\mathcal{W}'_\kappa\in\operatorname{Cov}F_\kappa$ and $\mathcal{Z}'_\nu\in\operatorname{Cov}C_\nu$ as in 3.7, (2).

By 3.7, (1), there exist $(\lambda,\mu)\in A(p)$ and maps $f:E_\lambda\to F_\kappa$, $g:B_\mu\to C_\nu$ with $(fq_\lambda, q'_\kappa)\le\mathcal{W}'_\kappa$, $(gr_\mu, r'_\nu)\le\mathcal{Z}'_\nu$, $(p'_{\nu\kappa}f, gp_{\mu\lambda})\le\mathcal{Z}'_\nu$. Since p has the ASEP, there exist

(b): $(\lambda_1,\mu_1)\ge(\lambda,\mu)$ and $\mathcal{V}_{\mu_1}\in\operatorname{Cov}B_{\mu_1}$ which satisfy 3.1, (*) w. r. t. (λ,μ), $f^{-1}(\mathcal{W}')\in\operatorname{Cov}E_\lambda$ and $g^{-1}(\mathcal{Z}')\in\operatorname{Cov}B_\mu$.

Take $\mathcal{V}'_{\mu_1}\in\operatorname{Cov}B_{\mu_1}$ with $\operatorname{st}\mathcal{V}'_{\mu_1}<\mathcal{V}_{\mu_1}\wedge(gr_{\mu\mu_1})^{-1}(\mathcal{Z}')$. By 3.7, (2) (in this case, we apply the lemma only to q and q'), there exists

(c): $\mathcal{V}''_{\mu_1}\in\operatorname{Cov}B_{\mu_1}$ as in 3.7, (2) w. r. t. μ_1 and $\mathcal{V}'_{\mu_1}$.

We apply (a) to the data (λ,μ), f, g, (λ_1,μ_1) and $\mathcal{V}''_{\mu_1}\in\operatorname{Cov}B_{\mu_1}$ to get $(\kappa_1,\nu_1)\ge(\kappa,\nu)$ in $A(p)$ and maps $f_1:F_{\kappa_1}\to E_{\lambda_1}$ and $g_1:C_{\nu_1}\to B_{\mu_1}$ with $(g_1r'_{\nu_1}, r_{\mu_1})\le\mathcal{V}''_{\mu_1}$, $(p_{\mu_1\lambda_1}f_1, g_1p'_{\nu_1\kappa_1})\le\mathcal{V}''_{\mu_1}$ and $(fq_{\lambda\lambda_1}f_1, q'_{\kappa\kappa_1})\le\mathcal{W}'$, $(gr_{\mu\mu_1}g_1, r'_{\nu\nu_1})\le\mathcal{Z}'$.

Claim: (κ_1,ν_1) *and* $g_1^{-1}(\mathcal{V}'_{\mu_1})\in\operatorname{Cov}C_{\nu_1}$ *satisfy* 3.1, (*) *w. r. t.* (κ,ν), $\mathcal{W}_\kappa$ *and* $\mathcal{Z}_\nu$.

To see this, suppose $\nu_2\ge\nu_1$ in N, V is an open halo of a subset D in C_{ν_2} and $s':V\to F_{\kappa_1}$ is a map with $(p'_{\nu_1\kappa_1}s', r'_{\nu_1\nu_2}|_V)\le g_1^{-1}(\mathcal{V}'_{\mu_1})$. Take a *haloing function* $\tau:C_{\nu_2}\to[0,1]$ with $D\subset\tau^{-1}(1)$ and $C_{\nu_2}-V\subset\tau^{-1}(0)$. Let $V_1=\tau^{-1}(1/2,1]$. Then V is a halo of V_1, $\bar{D}\subset V_1$, and $r'^{-1}_{\nu\nu_2}(\mathcal{Z}')$, $\{C_{\nu_2}-\bar{D}, V_1\}$, $(q'_{\kappa\kappa_1}s')^{-1}(\mathcal{W}')\cup\{C_{\nu_2}-\bar{D}\}\in\operatorname{Cov}C_{\nu_2}$. Take $\mathcal{Z}_{\nu_2}\in\operatorname{Cov}C_{\nu_2}$ which refines the above three coverings. Apply 3.7, (2) to ν_2 and $\mathcal{Z}_{\nu_2}$, we obtain

(d): $\mathcal{Z}'_{\nu_2}\in\operatorname{Cov}C_{\nu_2}$ as in 3.7, (2).

We apply (c) to the data ν_1, g_1, ν_2, $\mathcal{Z}'_{\nu_2}$ and obtain $\mu_2\ge\mu_1$ and a map $g_2:B_{\mu_2}\to C_{\nu_2}$ with $(g_2r_{\mu_2}, r'_{\nu_2})\le\mathcal{Z}'_{\nu_2}$ and $(g_1r'_{\nu_1\nu_2}g_2, r_{\mu_1\mu_2})\le\mathcal{V}'_{\mu_1}$. Let $U=g_2^{-1}(V)$, $A=g_2^{-1}(V_1)$ and $s=f_1s'g_2|_U:U\to E_{\lambda_1}$. Then U is a halo of A in B_{μ_2} and $(p_{\mu_1\lambda_1}s, r_{\mu_1\mu_2}|_U)\le\mathcal{V}_{\mu_1}$.

Therefore (b) gives $\mu_3 \geq \mu_2$ and a map $S: B_{\mu_3} \to E_\lambda$ with $(p_{\mu\lambda}S, r_{\mu\mu_3}) \leq g^{-1}(\mathcal{Z}')$ and $(S|_{r^{-1}_{\mu_2\mu_3}(A)}, q_{\lambda\lambda_1}sr_{\mu_2\mu_3}|_{r^{-1}_{\mu_2\mu_3}(A)}) \leq f^{-1}(\mathcal{W}')$. Apply (d) to μ_2, g_2, μ_3, and we get $\nu_3 \geq \nu_2$ and a map $g_3: C_{\nu_3} \to B_{\mu_3}$ with $(g_2 r_{\mu_2\mu_3} g_3, r'_{\nu_2\nu_3}) \leq \mathcal{Z}_{\nu_2}$. Let $S' = fSg_3: C_{\nu_3} \to F_\kappa$. Then adjacent maps of the followings are $\mathcal{Z}'$-near:

$$p'_{\nu\kappa}S',\ gp_{\mu\lambda}Sg_3,\ gr_{\mu\mu_3}g_3,\ gr_{\mu\mu_1}g_1r'_{\nu_1\nu_2}g_2r_{\mu_2\mu_3}g_3,\ r'_{\nu\nu_2}g_2r_{\mu_2\mu_3}g_3,\ r'_{\nu\nu_3}.$$

This implies $(p'_{\nu\kappa}S', r'_{\nu\nu_3}) \leq \mathcal{Z}_\nu$.

It remains only to show $(q'_{\kappa\kappa_1}s'r'_{\nu_2\nu_3}|_{r'^{-1}_{\nu_2\nu_3}(D)}, S'|_{r'^{-1}_{\nu_2\nu_3}(D)}) \leq \mathcal{W}_\kappa$. First note that $g_3(r'^{-1}_{\nu_2\nu_3}(D)) \subset r^{-1}_{\mu_2\mu_3}(A)$ and that, since $(g_2r_{\mu_2\mu_3}g_3, r'_{\nu_2\nu_3}) \leq (q'_{\kappa\kappa_1}s')^{-1}(\mathcal{W}') \cup \{C_{\nu_2} - \bar{D}\}$, maps $g_2r_{\mu_2\mu_3}g_3, r'_{\nu_2\nu_3}: r'^{-1}_{\nu_2\nu_3}(D) \to V$ are $(q'_{\kappa\kappa_1}s')^{-1}(\mathcal{W}')$-near. Thus adjacent maps of the followings are $\mathcal{W}'$-near on $r'^{-1}_{\nu_2\nu_3}(D)$:

$$q'_{\kappa\kappa_1}s'r'_{\nu_2\nu_3},\ q'_{\kappa\kappa_1}s'g_2r_{\mu_2\mu_3}g_3,\ fq_{\lambda\lambda_1}sr_{\mu_2\mu_3}g_3,\ S'.$$

Since $\mathrm{st}\,\mathcal{W}' < \mathcal{W}_\kappa$, we have the conclusion.

We conclude the section, inspecting Taylor's map $F: X \to \boldsymbol{Q}$ ([DW], [T]).

EXAMPLE 3.8 (Taylor's map): The map F is obtained as the *inverse limit* of the following *level* map $\underline{f} = \{f_n\}$:

$$\begin{array}{ccccccccccc}
L & \xleftarrow{\beta} & \Sigma^1 L & \xleftarrow{\Sigma\beta} & \Sigma^2 L & \xleftarrow{\Sigma^2\beta} & -- & \xleftarrow{\Sigma^{n-1}\beta} & \Sigma^n L & \xleftarrow{\Sigma^n\beta} & \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \\
* & \xleftarrow[p_0]{} & I^1 & \xleftarrow[p_1]{} & I^2 & \xleftarrow[p_2]{} & -- & \xleftarrow[p_{n-1}]{} & I^n & \xleftarrow[p_n]{} &
\end{array}$$

where $I = [-1, 1]^r$ (r is a fixed positive integer), $\Sigma^n L$ is the nr-th *suspension* of a compact polyhedron L and β is a map for which the composition $\beta \circ \Sigma\beta \circ \cdots \circ \Sigma^n\beta \not\simeq 0$ for each $n \geq 0$, f_n is induced from the projection $L \times I^n \to I^n$ and p_n is the projection. Note that, by [Ma], Theorem 8, $\underline{f}$ is an ANR-resolution of F. The map F is an example of CE-maps which are *not* shape equivalences ([T]), and moreover, F is approximately invertible ([A2]). Note that each f_n has a section.

We now show that F does *not* have the ASEP. To see this, on the contrary, suppose F, hence $\underline{f}$ has the ASEP. Then there exist $n \geq 0$ and $\varepsilon > 0$ which satisfy 3.1, (*) w.r.t. the index 0 in $\underline{f}$. Since $\Sigma^n L$ is a finite dimensional compact metric space, we can find an embedding $i: \Sigma^n L \to I^m$ for some $m \geq 0$. Define an embedding $j: \Sigma^n L \to I^{n+m} = I^n \times I^m$ by $j(x) = (f_n(x), i(x))$ $(x \in \Sigma^n L)$ and let $A = j(\Sigma^n L)$, $s = j^{-1}: A \to \Sigma^n L$. Since $f_n s = p_{n\,n+m}|_A$, by the choice of n, there exist $k \geq n+m$ and a map S: $I^k \to L$ such that S coincides with the composition $\beta \circ \cdots \circ \Sigma^{n-1}\beta \circ s \circ p_{n+m,k}$ on $p^{-1}_{n+m,k}(A)$ $= A \times I^{k-n-m}$. Since $S \simeq 0$ and $sp_{n+m,k}: A \times I^{k-n-m} \to \Sigma^n L$ is a homotopy equivalence,

$\beta\circ\cdots\circ\Sigma^{n-1}\beta\simeq 0$. This contradicts the choice of β.

One can embeds X into $\boldsymbol{Q}$ and extend F to the CE-map $G:\boldsymbol{Q}\to\boldsymbol{Q}\cup_F\boldsymbol{Q}$. It is known that $\boldsymbol{Q}\cup_F\boldsymbol{Q}$ does *not* have the *trivial* shape ([DS]), hence G is not a weak domination. This implies that G is *not* approximately invertible.

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