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ON FUNCTION SPACES FOR GENERAL TOPOLOGICAL SPACES

By

Mitsuaki KAKIMI and Toshio YAGI

1. Introduction.

In this paper we mean by a space a topological space with no separation axiom unless otherwise specified, and we denote by $\boldsymbol{R}$ and $\boldsymbol{I}$ the real line and the closed unit interval respectively.

Given two spaces X and Y, let $F(X, Y)$ denote the set of all maps from X into Y, $C(X, Y)$ the set of all continuous maps from X into Y. In case Y is the real line $\boldsymbol{R}$, $C(X, \boldsymbol{R})$ is denoted more simply by $C(X)$. The map $\rho: F(X\times Y, T)\rightarrow F(Y, F(X, T))$ defined by the formula $[\rho(f)(y)](x)=f(x, y)$ for $f\in F(X\times Y, T)$ is bijective; this correspondence is called the *exponential map*.

A topology on $C(X, T)$ is called *proper* if for every space Y and any $f\in C(X\times Y, T)$ the map $\rho(f)$ belongs to $C(Y, C(X, T))$. Similarly, a topology on $C(X, T)$ is called *admissible* if for every space Y and any $g\in C(Y, C(X, T))$ the map $\rho^{-1}(g)$ belongs to $C(X\times Y, T)$. A topology on $C(X, T)$ that is both proper and admissible is called an *acceptable topology* (see [1], [2] and [3]).

As is well known, the compact-open topology on $C(X, T)$ is acceptable for any space T when X is locally compact Hausdorff (see [4]). Furthermore, the following theorem was proved by R. Arens [1].

THEOREM 1.1. *Let X be a Tychonoff space. Then the following conditions are equivalent.*

(1) *X is locally compact.*

(2) *There exists an acceptable topology on $C(X)$.*

In the case that X is not necessarily Tychonoff, Professor T. Ishii raised the following problem: Characterize a space X such that there exists an acceptable topology on $C(X)$.

The main purpose of this paper is to give the solution for this problem by proving the following theorem.

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THEOREM 1.2. *For a space X, the following conditions are equivalent.*

(1) *X is locally relatively w-compact (that is, every point of X has a relatively w-compact nbd (=neighborhood)).*

(2) *There exists an acceptable topology on $C(X, T)$ for any Tychonoff space T.*

(3) *There exists an acceptable topology on $C(X)$.*

The definition of relatively w-compact subsets is given in section 2. Section 3 is devoted to a study of a new topology on $C(X, T)$ which is acceptable when X is locally relatively w-compact. Theorem 1.2 is proved in section 4.

The authors wish to thank Professor T. Ishii for his valuable comments.

2. Properties of relatively w-compact subsets.

A subset P of a space X is called *τ-open* if P is a union of cozero-sets of X (see [6]). For any subset A of X, we call the intersection of all zero-sets of X containing A the *τ-closure* of A, and we denote it by $\bar{A}^{\tau}$. A subset A of X is said to be *τ-closed* if $A=\bar{A}^{\tau}$ holds.

DEFINITION 2.1. A subset A of a space X is *relatively w-compact* if for any family $\{P_\lambda \mid \lambda\in\Lambda\}$ of τ-open sets of X such that $\{A\cap P_\lambda \mid \lambda\in\Lambda\}$ has the f.i.p. (=finite intersection property), we have $\cap\{clP_\lambda \mid \lambda\in\Lambda\}\neq\phi$.

T. Ishii introduced the notion of w-compact spaces, in connection with the problem concerning a product formula for the Tychonoff functor ([6]). A space X is called *w-compact* if for any family $\{P_\lambda \mid \lambda\in\Lambda\}$ of τ-open sets of X with the f.i.p., we have $\cap\{clP_\lambda \mid \lambda\in\Lambda\}\neq\phi$. Clearly every w-compact subset of a space X is relatively w-compact.

PROPOSITION 2.2. *Let A be a subset of a space X. Then the following conditions are equivalent.*

(1) *A is relatively w-compact.*

(2) *For any collection $\{A_\alpha \mid \alpha\in\Omega\}$ of closed sets of X such that it is closed under the finite intersection and each A_α contains a cozero-set of X which meets A, we have $\cap\{A_\alpha \mid \alpha\in\Omega\}\neq\emptyset$.*

(3) *For every open cover $\{U_\alpha \mid \alpha\in\Omega\}$ of X there exists a finite set $\{\alpha(1), \cdots, \alpha(n)\}\subset\Omega$ such that $A\subset\cup\{\bar{U}^{\tau}_{\alpha(i)} \mid i=1, \cdots, n\}$.*

(4) *For every family $\{U_\alpha \mid \alpha\in\Omega\}$ of open sets of X such that $\bar{A}^{\tau}\subset\cup\{U_\alpha \mid \alpha\in\Omega\}$ there exists a finite set $\{\alpha(1), \cdots, \alpha(n)\}\subset\Omega$ such that $A\subset\cup\{\bar{U}^{\tau}_{\alpha(i)} \mid i=1, \cdots, n\}$.*

PROOF. The equivalences of (1) and (2) and of (1) and (3) are clear. And the implication (4)⇒(3) is obvious. We prove only (3)⇒(4).

Let $\{U_\alpha \mid \alpha \in \Omega\}$ be a family of open sets of X with $\overline{A}^\tau \subset \cup \{U_\alpha \mid \alpha \in \Omega\}$. For each $x \in X - \overline{A}^\tau$ we can take a zero-set nbd of x which misses $\overline{A}^\tau$ since $X - \overline{A}^\tau$ is a union of cozero-sets of X. Thus we obtain an open nbd V_x of x with $\overline{V}_x^\tau \cap \overline{A}^\tau = \emptyset$. Then $\{U_\alpha \mid \alpha \in \Omega\} \cup \{V_x \mid x \in X - \overline{A}^\tau\}$ is an open cover of X. From this fact (4) follows from (3).

As is easily seen, a relatively w-compact subset of a Tychonoff space is relatively compact. Hence, if a Tychonoff space is locally relatively w-compact, then it is locally compact. However, there exists a regular T_1-space on which every continuous real-valued function is constant (for instance, see [5]). This example shows that there exists a locally relatively w-compact regular T_1-space that is not locally compact. Hence, Theorem 1.2 shows that in case X is not Tychonoff, the local compactness of X is unessential in Theorem 1.1.

Let X and Y be two spaces and A a subset of X. The projection $\pi_Y : A \times Y \to Y$ is called a *relative Z-map* if $\pi_Y((A \times Y) \cap Z)$ is closed in Y for any zero-set Z of $X \times Y$ (see [10]).

The following proposition is a generalization of [6, Proposition 2.7].

PROPOSITION 2.3. *Let A be a subset of a space X. Then the following conditions are equivalent.*

(1) *A is relatively w-compact.*

(2) *For any space Y the set $\pi_Y((\overline{A}^\tau \times Y) \cap F)$ is closed in Y for every τ-closed subset F of $X \times Y$, where π_Y is the projection: $\overline{A}^\tau \times Y \to Y$.*

(3) *The projection $\pi_Y : \overline{A}^\tau \times Y \to Y$ is a relative Z-map for any space Y.*

(4) *The projection $\pi_Y : \overline{A}^\tau \times Y \to Y$ is a relative Z-map for any paracompact Hausdorff space Y.*

PROOF. (1)⇒(2). Let A be a relatively w-compact subset of X and F a τ-closed set of $X \times Y$. Take a point $y_0 \in Y - \pi_Y((\overline{A}^\tau \times Y) \cap F)$. Since $(\overline{A}^\tau \times \{y_0\}) \cap F = \emptyset$ and $X \times Y - F$ is τ-open, for each $x \in \overline{A}^\tau$ the point (x, y_0) has an open nbd of the form $U_x \times V_x$ such that $(\overline{U}_x^\tau \times V_x) \cap F = \emptyset$. Clearly $\overline{A}^\tau \subset \cup \{U_x \mid x \in \overline{A}^\tau\}$, so that by Proposition 2.2, there exists a finite set $\{x(1), \cdots, x(n)\} \subset \overline{A}^\tau$ such that $A \subset \cup \{\overline{U}_{x(i)}^\tau \mid i = 1, \cdots, n\}$. Put $V = \cap \{V_{x(i)} \mid i = 1, \cdots, n\}$. Then V is an open nbd of y_0 and $V \cap \pi_Y((\overline{A}^\tau \times Y) \cap F) = \emptyset$. It follows that $\pi_Y((\overline{A}^\tau \times Y) \cap F)$ is closed in Y.

(2)⇒(3) and (3)⇒(4) are obvious and (4)⇒(1) is easily verified by making use of the method of [6, Proposition 2.6]. Thus the proof is completed.

Let A be a relatively w-compact subset of a space X. Then, by Proposition

2.2, we clearly have the following facts:

(1) $\bar{A}^{\tau}$ is also relatively w-compact.

(2) If $B \subset A$, then B is also relatively w-compact.

Let X and Y be two spaces. Then $\overline{A \times B}^{\tau} = \bar{A}^{\tau} \times \bar{B}^{\tau}$ holds for $A \subset X$ and $B \subset Y$. Thus, the following proposition is easy to prove.

PROPOSITION 2.4. *Let X and Y be two spaces. If A and B are relatively w-compact subsets of X and Y respectively, then $A \times B$ is also a relatively w-compact subset of $X \times Y$.*

The following proposition is also clear.

PROPOSITION 2.5. *Let X and Y be two spaces and f a map in $C(X, Y)$. If A is a relatively w-compact subset of X, then $f(A)$ is a relatively w-compact subset of Y.*

3. A topology on function spaces.

In this section, we consider a new topology on function spaces. Let X be a space and T a Tychonoff space. For $A \subset X$ and $B \subset T$ we denote by $M(A, B)$ the totality of maps f in $C(X, T)$ for which $f(A) \subset B$. We consider the topology on $C(X, T)$ generated by the base consisting of all sets $\cap \{M(\bar{A}_i^{\tau}, U_i) \mid i=1, \cdots, n\}$, where A_i is a relatively w-compact subset of X and U_i is an open subset of T for $i=1, \cdots, n$. Throughout this section, a topology on function spaces is assumed to be the topology defined above.

The following proposition easily follows from Proposition 2.3.

PROPOSITION 3.1. *Let X be a space and T a Tychonoff space. Then the topology on $C(X, T)$ is proper.*

The following lemma is also clear.

LEMMA 3.2. *Let X be a locally relatively w-compact space. Then for each point x of X and for any τ-open nbd G of x there exists a relatively w-compact nbd U of x such that $\bar{U}^{\tau} \subset G$.*

PROPOSITION 3.3. *Let X be a locally relatively w-compact space and T a Tychonoff space. Then the topology on $C(X, T)$ is acceptable.*

PROOF. By Proposition 3.1, it suffices to prove that the topology on $C(X, T)$

is admissible. Let Y be a space and g a map in $C(Y, C(X, T))$. We shall show that $\rho^{-1}(g)$ is continuous. Take a point (x_0, y_0) in $X\times Y$ and an open nbd U of $\rho^{-1}(g)(x_0, y_0)$ in T. Here, notice $\rho^{-1}(g)(x_0, y_0)=[g(y_0)](x_0)$. Since T is a Tychonoff space and $g(y_0)$ is continuous, $g(y_0)^{-1}(U)$ is a τ-open nbd of x_0. By Lemma 3.2, there exists a relatively w-compact nbd V of x_0 such that $\overline{V}^{\tau}\subset g(y_0)^{-1}(U)$. Hence we have $g(y_0)\in M(\overline{V}^{\tau}, U)$. Because of the continuity of g there exists a nbd W of y_0 such that $g(W)\subset M(\overline{V}^{\tau}, U)$. Therefore we obtain a nbd $V\times W$ of (x_0, y_0) such that $\rho^{-1}(g)(V\times W)\subset U$. Hence $\rho^{-1}(g)$ is continuous. This completes the proof.

Let X be a space, T a Tychonoff space, and f a map in $C(X, T)$. Then it is easily shown that $f(\overline{A}^{\tau})$ is compact for any relatively w-compact subset A of X. From this fact, one can easily prove the following lemma and proposition (see [3, 3.4.14 and 3.4.15]).

LEMMA 3.4. *Let X be a space and A a relatively w-compact subset of X. Assigning to each $f\in C(X, \boldsymbol{I})$ the number $\xi(f)=\sup\{f(x)\mid x\in\overline{A}^{\tau}\}$ defines a continuous function $\xi: C(X, \boldsymbol{I})\to\boldsymbol{I}$.*

PROPOSITION 3.5 *Let X be a space and T a Tychonoff space. Then $C(X, T)$ is also a Tychonoff space.*

LEMMA 3.6. *Let X be a space, T a Tychonoff space, and $\mathcal{B}$ a subbase for T. Then the sets $M(\overline{A}^{\tau}, U)$, where A is a relatively w-compact subset of X and $U\in\mathcal{B}$, form a subbase for the space $C(X, T)$.*

PROOF. Let A be a relatively w-compact subset of X, U an open set of T, and f a map in $M(\overline{A}^{\tau}, U)$. For each $x\in\overline{A}^{\tau}$ we can take sets $U_1^x, \cdots, U_{n(x)}^x\in\mathcal{B}$ with $x\in W_x=\cap\{f^{-1}(U_j^x)\mid j=1, \cdots, n(x)\}$ and $\cap\{U_j^x\mid j=1, \cdots, n(x)\}\subset U$. Since W_x is a τ-open nbd of x, we can take an open nbd V_x of x such that $\overline{V}_x^{\tau}\subset W_x$. By Proposition 2.2, there exists a finite set $\{x(1), \cdots, x(k)\}\subset\overline{A}^{\tau}$ such that $\overline{A}^{\tau}\subset\cup\{\overline{V}_{x(i)}^{\tau}\mid i=1, \cdots, k\}$. Put $A_i=\overline{A}^{\tau}\cap\overline{V}_{x(i)}^{\tau}$. Clearly, A_i is relatively w-compact, and we have $\overline{A}^{\tau}=\cup\{\overline{A}_i^{\tau}\mid i=1, \cdots, k\}$ and $\overline{A}_i^{\tau}\subset\cap\{f^{-1}(U_j^i)\mid j=1, \cdots, n(i)\}$, where $U_j^i=U_j^{x(i)}$ and $n(i)=n(x(i))$. Therefore, $f\in\cap\{\cap\{M(\overline{A}_i^{\tau}, U_j^i)\mid j=1, \cdots, n(i)\}\mid i=1, \cdots, k\}\subset M(\overline{A}^{\tau}, U)$. Thus the proof is completed.

THEOREM 3.7. *Let X and Y be two spaces and T a Tychonoff space. Then the exponential map $\rho: C(X\times Y, T)\to C(Y, C(X, T))$ is a homeomorphic embedding.*

PROOF. We first notice that $\rho(C(X\times Y, T))\subset C(Y, C(X, T))$ by Proposition 3.1.

Let A and B be relatively w-compact subsets of X and Y respectively, and U an open set of T. Then we clearly have $\rho^{-1}[M(\bar{B}^{\tau}, M(\bar{A}^{\tau}, U))]=M(\bar{A}^{\tau}\times\bar{B}^{\tau}, U)$.

Since $\bar{A}^{\tau}\times\bar{B}^{\tau}$ is relatively w-compact by Proposition 2.4, the last lemma implies that ρ is continuous.

The above equality implies that

$$\rho(M(\bar{A}^{\tau}\times\bar{B}^{\tau}, U))=M(\bar{B}^{\tau}, M(\bar{A}^{\tau}, U))\cap\rho(C(X\times Y, T));$$

hence—ρ being a one-to-one map—it suffices to show that the sets $M(\bar{A}^{\tau}\times\bar{B}^{\tau}, U)$, where A and B are relatively w-compact subsets of X and Y respectively and U is open in T, form a subbase for $C(X\times Y, T)$.

Take a relatively w-compact subset C of $X\times Y$, an open set U of T and a map $f\in M(\bar{C}^{\tau}, U)$. Since C is relatively w-compact and $f^{-1}(U)$ is τ-open, there exist open sets $V_1, \cdots, V_n\subset X$ and $W_1, \cdots, W_n\subset Y$ with $\bar{C}^{\tau}\subset\cup\{\bar{V}_i^{\tau}\times\bar{W}_i^{\tau}|i=1, \cdots, n\}\subset f^{-1}(U)$. Let $A_i=\pi_X(\bar{C}^{\tau})\cap\bar{V}_i^{\tau}$ and $B_i=\pi_Y(\bar{C}^{\tau})\cap\bar{W}_i^{\tau}$ for $i=1, \cdots, n$, where $\pi_X: X\times Y\to X$ and $\pi_Y: X\times Y\to Y$ are the projections. Then A_i and B_i are relatively w-compact subsets of X and Y respectively for $i=1, \cdots, n$. Moreover, we have $\bar{A}_i^{\tau}\subset\bar{V}_i^{\tau}$, $\bar{B}_i^{\tau}\subset\bar{W}_i^{\tau}$ and $\bar{C}^{\tau}\subset\cup\{\bar{A}_i^{\tau}\times\bar{B}_i^{\tau}|i=1, \cdots, n\}$. Hence, we have $f\in\cap\{M(\bar{A}_i^{\tau}\times\bar{B}_i^{\tau}, U)|i=1, \cdots, n\}\subset M(\bar{C}^{\tau}, U)$, and this completes the proof.

Proposition 3.3 and Theorem 3.7 imply

THEOREM 3.8. *Let X be a locally relatively w-compact space, Y a space and T a Tychonoff space. Then the exponential map $\rho: C(X\times Y, T)\to C(Y, C(X, T))$ is a homeomorphism.*

4. Proof of Theorem 1.2.

The following two lemmas are due to R. Arens and J. Dugundji [2].

LEMMA 4.1. *For spaces X and T, a topology on $C(X, T)$ is admissible if and only if the evaluation mapping $\omega(f, x)=f(x)$ of $C(X, T)\times X$ into T is continuous.*

LEMMA 4.2. *Let X and T be spaces and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on $C(X, T)$. If $\mathcal{T}_1$ is proper and $\mathcal{T}_2$ is admissible, then $\mathcal{T}_1\subset\mathcal{T}_2$.*

We are now in a position to prove the main theorem. We modify the proof by R. Arens ([1, Theorem 3]).

PROOF OF THEOREM 1.2. Since the implication (1)⇒(2) is a direct consequence of Proposition 3.3 and (2)⇒(3) is obvious, we prove only (3)⇒(1).

Suppose that there exists an acceptable topology on $C(X)$ and denote it by $\mathcal{T}_{ac}$. Let g be the element of $C(X)$ such that $g(x)=0$ for each $x\in X$. Since $\omega: C(X)\times X\to \boldsymbol{R}$ is continuous with respect to the topology $\mathcal{T}_{ac}$ on $C(X)$ by Lemma 4.1, for any $x_0\in X$ there exist an open nbd V of x_0 and an element W of $\mathcal{T}_{ac}$ such that $g\in W$ and $W\times V\subset\omega^{-1}((-1, 1))$.

Now we shall prove that V is relatively w-compact. Let $\{G_\lambda\mid\lambda\in\Lambda\}$ be a family of open sets of X such that $\overline{V}^\tau\subset\cup\{G_\lambda\mid\lambda\in\Lambda\}$ and $\mathcal{T}$ the topology on $C(X)$ with its subbase consisting of the sets of the form $M(\overline{A}^\tau, U)$, where A is a subset of X such that A is contained in some element of $\{G_\lambda\mid\lambda\in\Lambda\}$ or $\overline{A}^\tau\cap\overline{V}^\tau=\emptyset$, and U is open in $\boldsymbol{R}$. Then $\mathcal{T}$ is admissible. To see this, it suffices to show that $\omega: C(X)\times X\to\boldsymbol{R}$ is continuous with respect to $\mathcal{T}$ by Lemma 4.1. Let U be an open set of $\boldsymbol{R}$, and take a point $(f, x)\in\omega^{-1}(U)$. Then $f^{-1}(U)$ is a cozero-set nbd of x in X. Consequently, there exists an open nbd V_1 of x such that $f(\overline{V}_1^\tau)\subset U$.

Case (a). If $x\in\overline{V}^\tau$, then there exists $\lambda\in\Lambda$ such that $x\in G_\lambda$. Put $A=V_1\cap G_\lambda$, then A is an open nbd of x such that $A\subset G_\lambda$.

Case (b). If $x\notin\overline{V}^\tau$, then there is an open nbd V_2 of x such that $\overline{V}_2^\tau\cap\overline{V}^\tau=\emptyset$. Put $A=V_1\cap V_2$, then $\overline{A}^\tau\cap\overline{V}^\tau=\emptyset$.

Hence each of the two cases above implies that there exists an open nbd A of x such that $f(\overline{A}^\tau)\subset U$ and $M(\overline{A}^\tau, U)\in\mathcal{T}$. Furthermore $(f, x)\in M(\overline{A}^\tau, U)\times A\subset\omega^{-1}(U)$. Thus ω is continuous.

This implies $\mathcal{T}_{ac}\subset\mathcal{T}$ by Lemma 4.2, so that there exist subsets $A_1, \cdots, A_n$ of X and open subsets $U_1, \cdots, U_n$ of $\boldsymbol{R}$ such that $g\in\cap\{M(\overline{A}_i^\tau, U_i)\mid i=1, \cdots, n\}\subset W$. Here, notice $0\in U_i$ for each i.

Finally, we shall show that $V\subset\cup\{\overline{A}_i^\tau\mid i=1, \cdots, n\}$. Assume that there exists a point $x_1\in V$ such that $x_1\notin\cup\{\overline{A}_i^\tau\mid i=1, \cdots, n\}$. Then there exists a continuous map $h: X\to\boldsymbol{I}$ such that $h(x_1)=1$ and $h(x)=0$ for $x\in\cup\{\overline{A}_i^\tau\mid i=1, \cdots, n\}$, and so $h\in\cap\{M(\overline{A}_i^\tau, U_i)\mid i=1, \cdots, n\}$. Hence $h\in W$. Since $x_1\in V$ and $W\times V\subset\omega^{-1}((-1, 1))$, we have $h(x_1)\in(-1, 1)$. This is a contradiction. This implies $V\subset\cup\{\overline{A}_i^\tau\mid i=1, \cdots, n\}$. Thus V is contained in the union of finitely many members of $\{\overline{G}_\lambda^\tau\mid\lambda\in\Lambda\}$. Hence V is relatively w-compact by Proposition 2.2, and this completes the proof of Theorem 1.2.

5. An example.

A space X is called *locally cozero-set w-compact* if for each point x of X there exists a cozero-set nbd G of x such that clG is w-compact ([6] and [7]). The following theorem, where we denote by τ the Tychonoff functor, was proved by T. Ishii in [6].

THEOREM 5.1. *A space X is locally cozero-set w-compact if and only if $\tau(X\times Y)=\tau(X)\times\tau(Y)$ for any space Y.*

Clearly every locally cozero-set w-compact space is locally relatively w-compact. But the converse is false. We construct such an example. A space X is called *τ-compact* if $\tau(X)$ is compact ([7]). The following lemma is easy to prove.

LEMMA 5.2 *Every τ-compact locally cozero-set w-compact space is w-compact.*

EXAMPLE 5.3. Let ω_1 be the first uncountable ordinal and let us put $S=W(\omega_1+1)\times W(\omega_1+1)-\{(\omega_1, \omega_1)\}$, where $W(\omega_1+1)$ is the space of all ordinals less than ω_1+1 with the usual interval topology. Now let X be a space obtained by adding a new point ξ to S and introducing the topology in X as follows: the base at ξ is given by the totality of the sets $U_\beta(\xi)=\{(\alpha, \alpha)\mid\alpha$ is a non-limit ordinal and $\beta<\alpha\}\cup\{\xi\}$, $\beta<\omega_1$ and the base at $x\neq\xi$ is the same as in S. Then X has the following properties:

(1) X is Hausdorff but not regular.
(2) X is τ-compact.
(3) X is not w-compact.
(4) X is locally relatively w-compact.

Indeed (1) is obvious and (2) follows from the fact that any cozero-set of X containing ξ has to contain a set of the form $\{\xi\}\cup T_\alpha$ for some $\alpha<\omega_1$, where $T_\alpha=\{(\lambda, \mu)\mid\lambda, \mu>\alpha\}\subset S$, and (3) follows from the fact that $\{T_\alpha-U_\alpha(\xi)\mid\alpha<\omega_1\}$ is a family of closed sets of X such that $T_\alpha-U_\alpha(\xi)$ contains isolated points of X and $\cap\{T_\alpha-U_\alpha(\xi)\mid\alpha<\omega_1\}=\emptyset$ (see [6, p. 175]). Furthermore, each point $x\in S$ has a compact nbd, and $A_\beta=\{(\alpha, \alpha)\mid\beta<\alpha\}\cup\{\xi\}$ for $\beta<\omega_1$ is a relatively w-compact nbd of ξ. To show this, take an open cover $\{G_\lambda\mid\lambda\in\Lambda\}$ of X and a $\theta\in\Lambda$ such that $\xi\in G_\theta$. Then there exists a $\gamma<\omega_1$ such that $U_\gamma(\xi)\subset G_\theta$. It is easily seen that $clU_\gamma(\xi)$ contains the set A_γ and $A_\beta-A_\gamma$ is compact. Hence, A_β is contained in the union of finitely many members of $\{clG_\lambda\mid\lambda\in\Lambda\}$, which implies that A_β is relatively w-compact. Thus, (4) is proved.

Lemma 5.2 and properties (2) and (3) imply that X is not locally cozero-set w-compact.

REMARK. Recently, T. Ishii has obtained the following result ([8]): Let X be a space. Then $\tau(X\times Y)=\tau(X)\times\tau(Y)$ holds for any Tychonoff space Y if and only if X satisfies the following property (*):

(*) Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be a family of τ-open sets of X such that there exists a point x_0 of X such that $\{P_\lambda \mid \lambda \in \Lambda\} \cup \{U\}$ has the f.i.p. for any cozero-set nbd U of x_0. Then we have $\cap \{clP_\lambda \mid \lambda \in \Lambda\} \neq \emptyset$.

He also showed in [8] that if a space X satisfies the property (*) and $\tau(X)$ is locally compact, then X is locally cozero-set w-compact.

From this fact and Example 5.3, it follows that there exists a locally relatively w-compact space which does not satisfy the property (*).

After submitting this paper, the authors observed the following fact: If a space X is locally relatively w-compact and the equality $\tau(X \times Y) = \tau(X) \times \tau(Y)$ holds for any Tychonoff space Y, then X is locally cozero-set w-compact. See the proof of Lemma 1.4 in: K. Morita, Čech cohomology and covering dimension for topological spaces, Fund. Math. **87** (1975), 31-52.

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Shizuoka Gakuen High School
226-3, Hijiriishiki, Shizuoka
422 Japan

Department of Mathematics
Faculty of Science
Shizuoka University
Ohya, Shizuoka
422 Japan