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ON M-RECURSIVELY SATURATED MODELS OF ARITHMETIC

By

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Introducaion.

In [6], C. Smorynski investigated the properties of models of arithmetic using the notions of recursive saturation and short recursive saturation. In this paper, we shall generalize these notions and obtain new isomorphism criteria (Theorems A and B) and embeddability criteria (Theorems D and E) for countable models of arithmetic.

Throughout, $\mathcal{PA}$ denotes Peano arithmetic with the induction schema for all formulas in some finite language $L\supseteqq\{0, ', +, \cdot\}$. Δ_0 denotes the set of all quantifier bounded formulas in L. Let M and N be countable models of $\mathcal{PA}$ with $M\subseteqq N$. We say N is *M-recursively saturated* (*M^s-recursively saturated*) if N realizes every (short) type τ which is Δ_1 on $\mathbf{HF}_M$, where τ may contain countably many parameters from M. It can be easily shown that M-recursive saturation (M^s-recursive saturation) corresponds with (short) recursive saturation, if $M=\langle\omega; 0, ', +, \cdot\rangle$. For $A\subseteqq|N|$, $Df(N, A)$ denotes the set of all elements in N which are definable in N using parameters from A. We put:

$Th_M(N)=\{\phi(c_{a_1}, \cdots, c_{a_n})\colon a_1, \cdots, a_n\in|M| \text{ and } N\models\phi(c_{a_1}, \cdots, c_{a_n})\}$,

$Th_M^{\Delta_0}(N)=\{\phi(c_{a_1}, \cdots, c_{a_n})\colon \phi\in\Delta_0, a_1, \cdots, a_n\in|M| \text{ and } N\models\phi(c_{a_1}, \cdots, c_{a_n})\}$,

$SS_M^{\Delta_0}(N)=\{X\cap|M|$: X is a subset of $|N|$ which is definable in N using a Δ_0-formula with parameters from $|N|\}$.

Our main results of this paper are as follows:

THEOREM A. *Suppose that N_1 and N_2 are M-recursively saturated countable models of $\mathcal{PA}$ such that $Th_M(N_1)=Th_M(N_2)$ and $SS_M^{\Delta_0}(N_1)=SS_M^{\Delta_0}(N_2)$. Then there is an isomorphism $f\colon N_1\rightarrow N_2$ which is identical on M.*

THEOREM B. *Suppose that N_1 and N_2 are M^s-recursively saturated models of $\mathcal{PA}$ such that $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$ and $SS_M^{\Delta_0}(N_1)=SS_M^{\Delta_0}(N_2)$. Suppose that both N_1 and N_2 are cofinal extensions of M. Then there is an isomorphism $f\colon N_1\rightarrow N_2$*

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which is identical on M.

THEOREM C. *Suppose that N_1 and N_2 are countable cofinal extensions of M with $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$. Then there are M^s-recursively saturated elementary cofinal extensions N_1^* of N_1 and N_2^* of N_2 such that $SS_M^{\Delta_0}(N_1^*)=SS_M^{\Delta_0}(N_2)$.*

THEOREM D. *Suppose that N_1 and N_2 are M-recursively saturated countable models of $\mathcal{PA}$ such that $Th_M(N_1)=Th_M(N_2)$ and $SS_M^{\Delta_0}(N_1)\subseteqq SS_M^{\Delta_0}(N_2)$. Let A be an arbitrary definable subset of $|N_2|$ such that $Df(N_2, |M|)\cap A=\emptyset$. Then there is an elementary embedding $f: N_1\to N_2$ which is identical on M and with the property $ran(f)\cap A=\emptyset$.*

THEOREM E. *Suppose that N is an M-recursively saturated countable model of $\mathcal{PA}$. Then N is an elementary extension of M if and only if for each element $b>M$, there is an elementary embedding $f: N_1\to N_2$ which is identical on M and with the property $ran(f)<b$.*

Theorem A is a generalization of C. Smorynski's result included in [6]. (See Theorem 2.7 in [6], for reference.) Theorem B is very useful and if we combine this with Theorem C, we have the following result which is closely related to the General Splitting Theorem. (See Theorem 0.17.)

COROLLARY. *Let N_1 and N_2 be countable cofinal extensions of M. Then $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$ implies $Th_M(N_1)=Th_M(N_2)$.*

Theorems A, B and C will appear in §1. In theorem D if $A=[b, d]=\{c: b\leqq c\leqq d\}$, we can choose f so that ran(f) is cofinal with N. Theorem E is an analogy of the result of [4]. Theorems D and E will appear in §3.

§0. Preliminaries.

Throughout this paper, we use the same symbol for a structure and its universe. M, N, and N_i $(i=1, 2, \cdots)$ are used to denote structures and we usually assume that M is a substructure of N or N_i $(i=1, 2, \cdots)$. Elements of M are denoted by a, a_i $(i=1, 2, \cdots)$ and elements of N or N_i $(i=1, 2, \cdots)$ are denoted by b, d, e, b_i, d_i, e_i $(i=1, 2, \cdots)$.

First of all, we introduce two notions *M-recursiveness* and *M-recursive saturation.* The former is a generalization of recursivehess and first introduced by J. Barwise. The latter is a generalization of recursive saturation. To explain these notions, we need the notion of hereditarily finite set over M. (See [1], for reference.)

0.1. DEFINITION. Let A be a set. Then HF_A is the set of hereditarily finite sets over A. The explicit definition is as follows:

$\mathrm{HF}_A(0)=\emptyset$,

$\mathrm{HF}_A(n+1)=$the set of all finite subset of $\mathrm{HF}_A(n)\cup A$,

$\mathrm{HF}_A=$the union of all $\mathrm{HF}_A(n)$'s.

If A is the empty set, we omit A in the above definitions. If M is a structure, $\mathbf{HF}_M$ denotes the structure $(M;\ \mathrm{HF}_M,\ \in)$.

0.2. DEFINITION. Let $L\subseteq\mathbf{HF}$ be a finite language and M an L-structure. Then $L(M)$ is the language obtained from L by the addition of new constant $c_a=\langle a,\ \emptyset\rangle$ for each $a\in M$. i.e.,

$$L(M)=L\cup\{\langle a,\ \emptyset\rangle:\ a\in M\}.$$

0.3. DEFINITION. Let A be a subset of$\mathbf{HF}_M$. Then

i) A is *M-recursive* iff A is Δ_1 on $\mathbf{HF}_M$,

ii) A is *M-recursively enumerable* iff A is Σ_1 on $\mathbf{HF}_M$.

We denote the set of all formulas formulated in $L(M)$ by $L(M)^*$. $L(M)^*$ is clearly an M-recursive subset of $\mathbf{HF}_M$.

0.4. DEFINITION. Let M and N be structures for a finite language L such that $M\subseteq N$. Let $\tau(x,\ y_1,\ \cdots,\ y_n)$ be a subset of $L(M)^*$ and $b_1,\ \cdots,\ b_n$ elements of N. Then we say $\tau(x,\ c_{b_1},\ \cdots,\ c_{b_n})$ is an *L(M)-type* over N if it is finitely satisfiable in $(N,\ b)_{b\in N}$, i.e.,

$$(N,\ b)_{b\in N}\models\exists x\bigwedge\tau_0(x,\ c_{b_1},\ \cdots,\ c_{b_n}),$$

for every finite subset τ_0 of τ. An $L(M)$-type $\tau(x,\ c_{b_1},\ \cdots,\ c_{b_n})$ over N is said to be an *M-recursive type* (*M-recursively enumerable type*) if $\tau(x,\ y_1,\ \cdots,\ y_n)$ is M-recursive (M-recursively enumerable).

0.5. DEFINITION. Let M and N be as above. Then we say N is *M-recursively saturated* if every M-recursive type over N is realized in N.

The following theorem can be easily obtained by the elementary chain construction. (See, e.g., [6] for reference.)

0.6. THEOREM. *Let M and N be structures for a finite language L with*

$M \subseteq N$. Then there is an M-recursively saturated elementary extension N^* of N having the same cardinality as N.

The following theorem will give us some information concerning the relation between recursive saturation and M-recursive saturation.

0.7. THEOREM. *If N is M-recursively saturated, then N is recursively saturated.*

PROOF. It is clear that $\omega \subseteq \mathbf{HF}$, and a subset A of ω is recursive iff it is Δ_1 on $\mathbf{HF}$. Since $\mathbf{HF}$ is Δ_1 on $\mathbf{HF}_M$, every recursive set is Δ_1 on $\mathbf{HF}_M$. Thus every recursive type τ over N is realized in N, if N is M-recursively saturated. □

It is well-known that if N is recursively saturated, then N realizes every recursively enumerable type over N. The following theorem is a generalization of this fact and the idea of the proof is analogous to that of Theorem 4.13 of [2].

0.8. THEOREM. *If N is M-recursively saturated, then N realizes every M-recursively enumerable type over N.*

PROOF. Let $c_x = \langle x, \emptyset \rangle$ be an M-recursive function which gives a new constant of $L(M)^*$ corresponding to $x \in M$. Let $\mathrm{sub}(x)$ be an M-recursive function defined on $L(M)^*$ which gives the set of subformulas of x. Using these functions, we define three formulas $\mathrm{Eq}(x)$, $\mathrm{And}(x)$ and $H\text{-}\mathrm{And}(x)$ by:

$$\mathrm{Eq}(x) = \text{“ } x \text{ is a sentence of the form } c_y = c_y \text{ for some } y \text{ ”},$$
$$\mathrm{And}(x) = \mathrm{Eq}(x) \vee \text{“ } x \text{ is the sentence } \forall v_0 (v_0 = v_0) \text{ ”}$$
$$\vee \exists y, z \in \mathrm{sub}(x) (\neg \mathrm{Eq}(y) \wedge \text{“ } x \text{ is the sentence } y \wedge z \text{ ”}),$$
$$H\text{-}\mathrm{And}(x) = \mathrm{And}(x) \wedge \forall y \in \mathrm{sub}(x) H\text{-}\mathrm{And}(y).$$

Clearly, they are Δ_1-formulas of set theory. Next we define a function f by:

$$f(c_a = c_a) = a,$$
$$f(\forall v_0 (v_0 = v_0)) = \emptyset,$$
$$f(x \wedge y) = f(x) \cup \{f(y)\}.$$

Then $\mathrm{ran}(f) = \mathbf{HF}_M$, and f can be expressed by a Δ_1-formula. Let $\tau(x, c_{b_1}, \cdots, c_{b_n})$ be an M-recursively enumerable type over N and let $\exists z D(y, z)$ be a Σ_1-formula which defines $\tau(x, y_1, \cdots, y_n)$ on $\mathbf{HF}_M$. Let $D^*(x)$ denote the following Δ_1-formula:

$$\exists y, z \in \mathrm{sub}(x) (D(y, f(z)) \wedge \text{“ } x \text{ is the sentence } y \wedge z \text{ ”} \wedge H\text{-}\mathrm{And}(z)).$$

Then D^* defines a certain set $\tau^*(x, y_1, \cdots, y_n) \subseteq L(M)^*$ on $\mathbf{HF}_M$. Evidently, $\tau^*(x, c_{b_1}, \cdots, c_{b_n})$ is an M-recursive type over N and it is realized by some $b \in N$. Now it is clear this b also realizes $\tau(x, c_{b_1}, \cdots, c_{b_n})$ in N. $\square$

In the remainder of this paper, we shall concentrate on the study of countable models of $\mathcal{PA}$. We fix some finite language $L \subseteq \mathbf{HF}$ which contains 0, 1, $+$, $\cdot$. We assume that $\mathcal{PA}$ is formulated in L, i.e., $\mathcal{PA}$ is a 1-st order Peano arithmetic with the induction schema for every formula of L. M, N and N_i ($i=1, 2, \cdots$) are used to denote countable models of $\mathcal{PA}$. We usually identify $(N, b)_{b \in N}$ with N itself.

0.9. Definition. Let $M \subseteq N$. Then we say:

i) N is an *end extension* of M ($M \subseteq_e N$) iff every element of N which is less than some element of M actually belongs to M,

ii) N is a *confinal extension* of M ($M \subseteq_c N$) iff for each element $b \in N$, there is an element $a \in M$ such that $b < a$.

$M \prec_e N$ and $M \prec_c N$ mean the elementary end extension and the elementary cofinal extension, respectively.

0.10. Definition. Let $M \subseteq N$. Then:

i) N is *M-short* iff there is an element $b \in N$ such that every element $d \in N$ is less than some $e \in N$ which is definable with parameters from $M \cup \{b\}$. N is M-tall iff N is not M-short.

ii) An $L(M)$-type $\tau(x, c_{b_1}, \cdots, c_{b_n})$ over N is *short* iff τ contains a formula of the form $x < c_b$ for some $b \in M \cup \{b_1, \cdots, b_n\}$.

iii) N is *M^s-recursively saturated* iff N realizes every short M-recursive type.

0.11. Definition. A function $\ulcorner * \urcorner^M : L(M) \to M$ is called a *coding function* of $L(M)^*$ if it suffices the following conditions:

i) $\ulcorner * \urcorner^M$ is one-one and M-recursive,

ii) $\ulcorner \phi \urcorner^M > \ulcorner \phi_0 \urcorner^M$ for every $\phi \in L(M)^*$ and every subformula ϕ_0 of ϕ.

Coding functions do exist. Moreover, if $M \subseteq N$, $\ulcorner * \urcorner^M$ can be taken as a restriction of $\ulcorner * \urcorner^N$ to $L(M)^*$. In this context, we usually write $\ulcorner * \urcorner$ instead of $\ulcorner * \urcorner^M$.

0.12. Definition. Let $M \subseteq N$, $A \subseteq N$ and $\Gamma \subseteq L(M)^*$. Then $SS_M^\Gamma(N, A)$ is the set which is defined by:

$SS_M^\Gamma(N, A)=\{X\cap M$: X is a subset of N which is definable in N using a Γ-formula with parameters from $A\}$. If $\Gamma=L(M)^*$ we omit it, if $A=N$ we omit it, if $A=N$ we omit it. If $Y\subseteq L(M)^*$, we use the expression $Y\in{}^*SS_M(N)$ to denote the relation: There is a set $Y^*\in SS_M(N)$ such that $Y=\{\phi: \ulcorner\phi\urcorner\in Y^*\}$.

It is clear that if there is an element $b>M$, $SS_M(N)$ and $SS_M^{\Delta_0}(N)$ determine the same set.

0.13. DEFINITIONN. Let $M\subseteq N$ and $\Gamma\subseteq L(M)^*$. Then we put:

i) $Th_M(N)=\{\phi\in L(M)^*: N\models\phi\}$,

ii) $Th_M^\Gamma(N)=\{\phi(c_{a_1}, \cdots, c_{a_n}): a_1, \cdots, a_n\in M,\ \phi\in\Gamma$ and $N\models\phi(c_{a_1}, \cdots, c_{a_n})\}$.

0.14. DEFINITION. Let Γ be a subset of $L(M)^*$. A formula $Tr_\Gamma(x, y)$ is said to be a truth definition for Γ in M if for each $\phi(x_1, \cdots, x_n)\in\Gamma$ and each $a\in M$,

$$M\models Tr_\Gamma(\ulcorner\phi\urcorner, c_a)\leftrightarrow\phi((c_a)_1, \cdots, (c_a)_n),$$

where $(x)_y$ is the y-th index of the binary expansion of x.

$\Delta_0(M)$ is the set of formulas in $L(M)^*$ which have only bounded quantifiers. $\Sigma_n(M)$ is the set of formulas in $L(M)^*$ which have form $\exists x_1\forall x_2\exists x_3\forall x_4\cdots Q_n x_n\phi$, where ϕ is a formula in $\Delta_0(M)$. It can be easily shown that there is a truth definition for $\Sigma_n(M)$ in M for each $n\in\omega$. The reader who is not familiar with the properties of truth definitions can refer to § 3 of Chapter 1 in [8].

0.15. DEFINITION. Let M be a common submodel of N_1 and N_2. Then:

i) A partial function $f: N_1\to N_2$ is said to be *M-identical* iff f is identical on M,

ii) A partial function $f: N_1\to N_2$ is said to be a *partial elementary embedding* iff f is a restriction of some elementary embedding $g: N_1\to N_2$,

iii) Emb$(N_1, N_2; M)$=the set of M-identical elementary embeddings of N_1 into N_2. P-Emb$(N_1, N_2; M)$=the set of partial M-identical elementary embeddings of N_1 into N_2 such that dom(f)-M is finite. Isom$(N_1, N_2; M)$=the set of M-identical isomorphisms of N_1 onto N_2.

Before beginning the study of models of arithmetic, we must state the Elementary Splitting Theorem and the General Splitting Theorem. The latter is, of course, an extension of the former.

0.16. THEOREM. (ELEMENTARY SPLITTING THEOREM) Let $M\prec N$. Then there is another model M^* such that $M\prec_c M^*\prec_e N$.

0.17. THEOREM. (GENERAL SPLITTING THEOREM BY GAIFMAN) Let N be a Δ_0-elementary extension of M. Then there is another model M^* such that $M \prec_c M^* \subseteq_e N$.

§1. Isomorphism conditions.

In §0 we showed that if N is M-recursively saturated, it realizes every M-recursively enumerable type. In case N is a model of arithmetic, the following more useful result holds. The proof is almost the same as that of Theorem 1.12 of [6].

1.1. PROPOSITION. *Suppose that N is M-recursively saturated (M^s-recursively saturated). Then every (short) $L(M)$-type $\tau(x, c_{b_1}, \cdots, c_{b_n}) \in {}^*SS_M(N)$ is realized in N.*

This proposition will be used freely without any mention. The following theorem is also a generalization of C. Smorynski's result included in [6].

1.2. THEOREM. *Suppose that N_1 and N_2 are M-recursively saturated. Then the following three conditions are equivalent:*

i) $\mathrm{Isom}(N_1, N_2; M) \neq \emptyset$,

ii) $Th_M(N_1) = Th_M(N_2)$ *and* $SS_M(N_1) = SS_M(N_2)$,

iii) $Th_M(N_1) = Th_M(N_2)$ *and* $SS_M^{\Delta_0}(N_1) = SS_M^{\Delta_0}(N_2)$.

The following example shows that Theorem 1.2 fails if we assume only the M^s-recursive saturation of N_1 and N_2.

1.3. REMARK. Let M be an arbitrary model of $\mathcal{PA}$. Then there are non-isomorphic elementary extensions N_1 and N_2 of M such that

i) N_1 and N_2 are M^s-recursively saturated, and

ii) $SS_M(N_1) = SS_M(N_2)$.

The existence of such N_1 and N_2 can be shown by the method similar to the one used in the proof of Theorem 3.9 in [5]. In spite of Remark 1.3, the following form of isomorphism conditions hold.

1.4. THEOREM. *Suppose that N_1 and N_2 are cofinal extensions of M. If N_1 and N_2 are M^s-recursively saturated, the following three conditions are equivalent:*

i) $\mathrm{Isom}(N_1, N_2; M) \neq \emptyset$,

ii) $Th_M(N_1) = Th_M(N)$ *and* $SS_M(N_1) = SS_M(N_2)$,

iii) $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$ *and* $SS_M^{\Delta_0}(N_1)=SS_M^{\Delta_0}(N_2)$.

RROOF. The implications i)$\Rightarrow$ii) and i)$\Rightarrow$iii) are immediate. The proofs of the implications ii)$\Rightarrow$i) and iii)$\Rightarrow$i) are similar and so we prove only the implication iii)$\Rightarrow$i). Let $N_1-M=\{b_i\}_{i\in\omega}$, $N_2-M=\{d_i\}_{i\in\omega}$. We construct partial isomorphisms f_n by induction so that for all $n\in\omega$,

a) $f_n\subseteq f_{n+1}$, $\mathrm{dom}(f_n)\supseteq M\cup\{b_i\}_{i<n}$ and $\mathrm{ran}(f_n)\supseteq M\cup\{d_i\}_{i<n}$,

b) for every $\phi(x_1, \cdots, x_m)\in\Delta_0(M)$ and every $e_1, \cdots, e_m\in\mathrm{dom}(f_n)$, if $N_1\models\phi(c_{e_1}, \cdots, c_{e_m})$ then $N_2\models\phi(c_{f_n(e_1)}, \cdots, c_{f_n(e_m)})$.

We put $f_0=id_M$. Then f_0 satisfies the condition b) by the assumption $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$. We assume that f_n is already defined. We shall specify the image of b_n and the inverse image of d_n. Let $\tau(x, y_1, \cdots, y_m)=\{\phi\in\Delta_0(M): N_1\models\phi(c_{b_n}, c_{b'_1}, \cdots, c_{b'_m})\}$, where $\{b'_1, \cdots, b'_m\}=\mathrm{dom}(f_n)-M$. Choose a from M with $a>b_n$ and $\phi_0, \cdots, \phi_p\in\tau$. Then $\exists x<c_a \bigwedge_{i\leq p}\phi_i(x, c_{b'_1}, \cdots, c_{b'_m})$ holds in N_1, so by the induction hypothesis, $\exists x<c_a \bigwedge_{i\leq p}\phi_i(x, c_{f_n(b'_1)}, \cdots, c_{f_n(b'_m)})$ holds in N_2. This shows that $\tau(x, c_{f_n(b'_1)}, \cdots, c_{f_n(b'_m)})$ is a short $L(M)$-type over N_2. Since there is truth definition for $\Delta_0(M)$-formulas, $\tau(x, c_{f_n(b'_1)}, \cdots, c_{f_n(b'_m)})\in{}^*SS_M^{\Delta_0}(N_2)$. So that this type is realized by some $d^*\in N_2$. In the same way we choose $b^*\in N_1$ corresponding to d_n. Finally, we put $f_{n+1}=f_n\cup\{\langle b_n, d^*\rangle, \langle b^*, d_n\rangle\}$. Then $f=\bigcup_{n\in\omega} f_n$ is the desired isomorphism. □

1.5. COROLLARY. *Suppose that* N_1 *and* N_2 *are* M^s*-recursively saturated extensions of* M *with* $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$ *and* $M\subseteq_c N_1$. *If* N_1 *and* N_2 *satisfy the condition:*

$$SS_M^{\Delta_0}(N_1)=SS_M^{\Delta_0}(N_2, N_2^*),$$

then there is an M*-identical isomorphism* f *of* N_1 *onto* N_2^*, *where* $N_2^*=\{b\in N_2: b$ *is less than some* $a\in M\}$.

As mentioned earlier, if $M\subseteq N$, then N can be elementarily extendable to an M-recursively saturated model. Now we state some theorems concerning about extendability.

1.6. THEOREM. *Suppose that* N *is* M*-tall* (M*-short*). *Then there is an elementary cofinal extension* N^* *of* N *such that* N^* *is* M*-recursively satrated (respectively,* M^s*-recursively saturated).*

PROOF. Let N' be an M-recursively saturated elementary extension of N.

Then, using the Elementary Splitting Theorem, we have N^* such that $N \prec_c N^* \prec_e N'$. This N^* is the desired one. □

1.7. THEOREM. *Let $N_1, N_2 \supseteq M$. Theu there are M-recursively saturated elementary extensions N_1^* of N_1 and N_2^* of N_2 such that $SS_M(N_1^*)=SS_M(N_2^*)$.*

1.8. THEOREM. *Suppose that N_1 and N_2 are cofinal extensions of M with $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$. Then there are M^s-recursively saturated elementary cofinal extensions N_1^* of N_1 and N_2^* of N_2 such that $SS_M^{\Delta_0}(N_1^*)=SS_M^{\Delta_0}(N_2^*)$.*

The proofs of the above two theorems are similar. So we give a proof only for the one that seems more difficult.

PROOF OF THEOREM 1.8. Let $SS_M^{\Delta_0}(N_1)=\{X_i\}_{i\in\omega}$ and $C=\{c_i\}_{i\in\omega}$ (a set of new constants). For each X_i, choose $\phi_i\in\Delta_0(M)$, $b_i\in N_1$ and $a_i\in M$ such that $X_i=\{a\in M: N_1\models Tr_0(\ulcorner\phi\urcorner, \langle b_i, c_a\rangle)\}$ and $\langle\ulcorner\phi_i\urcorner, b_i\rangle<a_i$, where Tr_0 is a truth definition for $\Delta_0(M)$-formulas. We put $T=Th_{N_2}(N_2)\cup\bigcup_{i\in\omega}\{Tr_0((c_i)_1, \langle(c_i)_2, c_a\rangle): a\in X_i\} \cup\bigcup_{i\in\omega}\{\neg Tr_0((c_i)_1, \langle(c_i)_2, c_a\rangle): a\notin X_i\}\cup\bigcup_{i\in\omega}\{c_i<c_{a_i}\}$. Clearly T is a consistent theory. Let N_2' be a model of T. Then N_2' is an elementary extension of N_2 with $SS_M^{\Delta_0}(N_1)\subseteq SS_M^{\Delta_0}(N_2')$. By the Elementary Splitting Theorem, there is another model N_2'' such that $N_2\prec_c N_2''$ and $SS_M^{\Delta_0}(N_1)\subseteq SS_M^{\Delta_0}(N_2'')$. Now we extend N_2'' to an M^s-recursively saturated model N_2^0 so that $N_2\prec_c N_2^0$ and $SS_M^{\Delta_0}(N_1)\subseteq SS_M^{\Delta_0}(N_2^0)$. Next we construct an M^s-recursively saturated extension N_1^0 of N_1 so that $N_1\prec_c N_1^0$ and $SS_M^{\Delta_0}(N_2^0)\subseteq SS_M^{\Delta_0}(N_1^0)$. Iterating these constructions, we obtain elementary chains $\{N_1^i\}_{i\in\omega}$ and $\{N_2^i\}_{i\in\omega}$ such that for each $n\in\omega$,

a) N_1^n and N_2^n are M^s-recursively saturated,
b) $N_1^n\prec_c N_1^{n+1}$ and $N_2^n\prec_c N_2^{n+1}$,
c) $SS_M^{\Delta_0}(N_1^n)\subseteq SS_M^{\Delta_0}(N_2^{n+1})$ and $SS_M^{\Delta_0}(N_2^n)\subseteq SS_M^{\Delta_0}(N_1^{n+1})$.

Finally we put $N_1^*=\bigcup_{i\in\omega}N_1^i$ and $N_2^*=\bigcup_{i\in\omega}N_2^i$. It is a routine to check that N_1^* and N_2^* have the desired properties. □

Now we apply our results.

1.9. THEOREM. *Let N_1 and N_2 be cofinal extensions of M. Then $Th_M^{\Delta_0}(N_1)=Th_M^{\Delta_0}(N_2)$ implies $Th_M(N_1)=Th_M(N_2)$.*

PROOF. Applying Theorem 1.8, we can construct two models N_1^* and N_2^* such that

a) N_1^* and N_2^* are M^s-recursively saturated,
b) $N_1 \prec_c N_1^*$ and $N_2 \prec_c N_2^*$,
c) $SS_M^{\Delta_0}(N_1^*) = SS_M^{\Delta_0}(N_2^*)$.

Then, by Theorem 1.4, we have

d) $\mathrm{Isom}(N_1^*, N_2^*; M) \neq \emptyset$.

From this and b), it follows that $Th_M(N_1) = Th_M(N_2)$. □

1.10. COROLLARY. *Let N_1 and N_2 be extensions of M such that $Th_M^{\Delta_0}(N_1) = Th_M^{\Delta_0}(N_2)$. Let $N_i^* = \{b \in N_i : b$ is less than some $a \in M\}$ ($i=1, 2$). Suppose that N_1^* and N_2^* are models of PA. Then $Th_M(N_1^*) = Th_M(N_2^*)$.*

PROOF. Since $N_i^* \subseteq_e N_i$ ($i=1, 2$), we have $Th_M^{\Delta_0}(N_i) = Th_M^{\Delta_0}(N_i^*)$ ($i=1, 2$). Hence $Th_M^{\Delta_0}(N_1^*) = Th_M^{\Delta_0}(N_2^*)$. By the above theorem, we have the desired property. □

The reader should note that we used only the Elementary Splitting Theorem to prove Theorem 1.9 and Corollary 1.10. Corollary 1.10 is closely related to the General Splitting Theorem. But Corollary 1.10 is neither stronger nor weaker than the General Splitting Theorem.

§ 2. Embeddability Conditions.

In this section, we shall give some theorems concerning embeddability. The main tool of this section is again the back and forth method and so we usually omit the details of the proofs

2.1. DEFINITION. Let A be a structure and B a subset of A. Then $\mathrm{Df}(A, B)$ is the set defined by:

$\mathrm{Df}(A, B) = \{a \in A : a$ is definable in A with parameters from $B\}$.

2.2. PROPOSITION. *Let A and B as above. Then:*

i) $B \subseteq \mathrm{Df}(A, B)$,
ii) $\mathrm{Df}(A, B) = \mathrm{Df}(A, \mathrm{Df}(A, B))$.

First we state a useful lemma, which is interesting of itself.

2.3. LEMMA. *Let N_1 and N_2 be M-recursively saturated elementary extensions of M with $SS_M(N_1) \subseteq SS_M(N_2)$. Suppose that A is a definable subset of N_2 such that $A \subseteq N_2 - M$. Then for each $f \in P\text{-}\mathrm{Emb}(N_1, N_2; M)$ with* $\mathrm{Df}(N_2, \mathrm{ran}(f))$

$\cap A=\emptyset$, *and each* $b \in N_1$, *there is an extension* $f^* \in P\text{-Emb}(N_1, N_2; M)$ *of* f *such that* $\mathrm{dom}(f^*)=\mathrm{dom}(f)\cup\{b\}$ *and* $\mathrm{Df}(N_2, \mathrm{ran}(f^*))\cap A=\emptyset$.

PROOF. Suppose that $\mathrm{dom}(f)=M\cup\{b_1, \cdots, b_n\}$ and $\tau(x, x_1, \cdots, x_n)=\{\phi(x, x_1, \cdots, x_n)\in L(M)^*: N_1 \models \phi(c_b, c_{b_1}, \cdots, c_{b_n})\}$. Then for each finite subset τ_0 of τ, the sentence $\exists x \bigwedge\tau_0(x, c_{f(b_1)}, \cdots, c_{f(b_n)})$ holds in N_2. Moreover, if $F_1, \cdots, F_m$ are $L(M)$-Skolem functions, the following sentence also holds in N_2:

$$\exists x(\bigwedge_i \neg\alpha(F_i(x, c_{f(b_1)}, \cdots, c_{f(b_n)}))\wedge\bigwedge\tau_0(x, c_{f(b_1)}, \cdots, c_{f(b_n)})),$$

where α is the defining formula of A in N_2. Hence the set $\tau(x, c_{f(b_1)}, \cdots, c_{f(b_n)}) \cup\{\neg\alpha(F(x, c_{f(b_1)}, \cdots, c_{f(b_n)})): F \text{ an } L(M)\text{-Skolem function}\}$ is an $L(M)$-type over N_2 and is realized by some $d \in N_2$. If we put $f^*=f\cup\{\langle b, d\rangle\}$, then f^* is the desired partial elementary embedding. □

2.4. THEOREM. *Let* N_1, N_2, M *and* A *be as in the above lemma. Then there is an elementary embedding* $f \in \mathrm{Emb}(N_1, N_2; M)$ *such that* $\mathrm{Df}(N_2, \mathrm{ran}(f))\cap A=\emptyset$.

PROOF. Let $N_2-M=\{b_i\}_{i\in\omega}$. We construct partial elementary embeddings $f_n \in P\text{-Emb}(N_1, N_2; M)$ by induction so that for all $n\in\omega$,

a) $\mathrm{dom}(f_n)=M\cup\{b_i\}_{i<n}$,
b) $\mathrm{Df}(N_2, \mathrm{ran}(f_n))\cap A=\emptyset$.

We put $f_0=id_M$ and assume that f_n is already defined. Using the above lemma, we take $f_{n+1}\in P\text{-Emb}(N_1, N_2; M)$ so that $\mathrm{dom}(f_{n+1})=\mathrm{dom}(f_n)\cup\{b_n\}$. Then $f=\bigcup_{n\in\omega} f_n$ is the desired elementary embedding. □

2.5. COROLLARY. *Let* N *be a* Δ_0*-elementary extension of* M *and suppose that* N *is* M*-recursively saturated. Then the following* i) *and* ii) *are equivalent*:

i) N *is an elementary extension of* M,

ii) *For each* $b>M$, *there is an elementary embedding* $f\in\mathrm{Emb}(N, N; M)$ *such that* $\mathrm{ran}(f)<b$.

PROOF. The implication i)⇒ii) is immediate by the above theorem. We shall prove only the implication ii)⇒i). Suppose that N is not an elementary extension of M. We only have to show that there is an element $b\in\mathrm{Df}(N, M)$ with $b>M$. By way of contradiction, we assume that there is no such element. Then, by the General Splitting Theorem, $\mathrm{Df}(N, M)$ must be an elementary cofinal extension of M. Since $\mathrm{Df}(N, M)\prec N$ is clear, we have $M\prec N$. This is a contradiction. □

In Corollary 2.5 we assumed that N is a Δ_0-elementary extension of M. The author doesn't know whether this assumption can be eliminated or not.

2.6. THEOREM. *Let N be M-recursively saturated and suppose that $M<b<d$. If $\mathrm{Df}(N, M)\cap[b, d]=\emptyset$, then there is an elementary embedding $f\in\mathrm{Emb}(N, N; M)$ such that:*

i) $\mathrm{ran}(f)\cap[b, d]=\emptyset$,
ii) $\mathrm{ran}(f)$ *is cofinal with N.*

To prove Theorem 2.6, we need the following lemma.

2.7. LEMMA. *Let N be an M-recursively saturated extension of M. Suppose that $M<b<d$ and $\mathrm{Df}(N, M\cup\{e_1, \cdots, e_n\})\cap[b, d]=\emptyset$. Then there is an arbitrarily large element e such that $\mathrm{Df}(N, M\cup\{e, e_1, \cdots, e_n\})\cap[b, d]=\emptyset$.*

PROOF. Let $e^*\in N$ be an arbitrary element. Define the set $\tau(x)$ by:

$$\tau(x)=\{\neg(b<F(x, e_1, \cdots, e_n)<d): F \text{ an } L(M)\text{-Skolem function}\}\cup\{x>e^*\}.$$

It is sufficient to prove that $\tau(x)$ is an M-recursive type over N. The M-recursiveness of $\tau(x)$ is clear and so we prove that $\tau(x)$ is finitely satisfiable in N. By way contradiction, assume that there are $L(M)$-Skolem functions $F_1, \cdots, F_n$ such that

$$N\models\forall x>e^*(\bigvee_k(b<F_k(x, e_1, \cdots, e_n)<d))\,.$$

Now define $\tau^*(u, v)$ by:

$$\begin{aligned}\tau^*(u, v)=&\{a<u<v<b: a\in M\}\\ &\cup\{\neg(u<F(e_1, \cdots, e_n)<v): F \text{ an } L(M)\text{-Skolem function}\}\\ &\cup\{\exists y\forall x>y(\bigwedge_k(u<F_k(x, e_1, \cdots, e_n)<v)\}\end{aligned}$$

It is a routine to check that this $\tau^*(u, v)$ is an M-recursive type over N. Suppose that a pair $\langle b_1, d_1\rangle$ realizes $\tau^*(u, v)$. Then the following hold:

a) $M<b_1<d_1<b$ and $\mathrm{Df}(N, M\cup\{e_1, \cdots, e_n\})\cap[b_1, d_1]=\emptyset$,
b) $N\models\exists y\forall x>y(\bigwedge_k(b_1<F_k(x, e_1, \cdots, e_n)<d_1))$.

Continueing all these, finally we have a sequence $\{\langle b_i, d_i\rangle\}_{i\in\omega}$ such that for each $i\in\omega$,

c) $M<b_{i+1}<d_{i+1}<b_i$,
d) $N\models\exists y\forall x>y(\bigwedge_k(b_i<F_k(x, e_1, \cdots, e_n)<d_i))\,.$

But this is impossible and so we conclude that $\tau(x)$ is finitely satisfiable in N. $\square$

PROOF OF THEOREM. Let $N-M=\{b_i\}_{i\in\omega}$. We construct partial elementary embeddings $f_n \in P\text{-Emb}(N, N; M)$ so that for each $n\in\omega$,

a) $f_n \subseteqq f_{n+1}$ and $\mathrm{dom}(f_{2n}) \supseteqq M\cup\{b_i\}_{i<n}$,

b) $\mathrm{Df}(N, \mathrm{ran}(f_n))\cap[b, d]=\emptyset$

c) There is an element $e\in\mathrm{ran}(f_{2n+1})$ such that $e>b_n$.

The construction of f_{2n} is the same one that is shown in Theorem 2.4. We shall show only the construction of f_{2n+1}. Suppose that f_{2n} is already constructed. By the above lemma, there is an element $e>b_n$ such that $\mathrm{Df}(N, \mathrm{ran}(f_{2n})\cup\{e\})\cap[b, d]=\emptyset$. Then choose an element e^* so that $f_{2n}\cup\{\langle e^*, e\rangle\}$ will become a partial elementary embedding. Let f_{2n+1} be this partial elementary embedding. It is clear that $f=\bigcup_{n\in\omega} f_n$ is the required one. $\square$

QUESTIONS. We state some open questions:

i) In [7], C. Smorynski and J. Stavi proved that recursive saturation is preserved under elementary cofinal extensions. Is M-recursive saturation also preserved under elementary cofinal extensions ?

ii) Let N be an extension of M. N is said to be *M-short legged* if there is an element $b>M$ such that $\mathrm{Df}(N, M\cup\{b\})-M$ is downward cofinal with $N-M$. Is there any model N which is M-recursively saturated and M-short legged ?

iii) Let N_1 and N_2 be not M-short legged and suppose that $Th_M(N_1)=Th_M(N_2)$. Is it always possible to find another model N and elementary embeddings $f: N_i\rightarrow M$ $(i=1, 2)$ so that $\mathrm{ran}(f_1)-M$ and $\mathrm{ran}(f_2)-M$ are downward cofinal with each other ?

References

[1] Barwise, J., Admissible sets and structures, Springer-Verlag, Berlin (1975).

[2] Feferman, S., Arithmetization of metamathematics in a general setting, Fundamenta Mathematicae, **49** (1960), 35-92.

[3] Friedman, H., Countable models of set theories, Cambridge Summer School in Mathematical Logic, Lecture Notes in Mathematics, Springer-Verlag, Berlin (1972), 539-537.

[4] Lipshitz, D., Diophantine correct models of arithmetic, Proceedings of the American Mathematical Society, **73** (1979), 107-108.

[5] Smorynski, C., Cofinal extensions of nonstandard models of Arithmetic, Notre Dame Journal of Formal Logic, **22** (1981), 133-144.

[6] Smorynski, C., Recursively saturated nonstandard models of arithmetic, The Journal of Symbolic Logic, **46** (1981), 259-286.

[7] Smorynski, C. and J. Stavi, Cofinal extension preserves recursive saturation, Model Theory of Algebra and Arithmetic, Lecture Notes in Mathematics, Springer-Verlag, Berlin (1981), 338–345.

[8] Troelstra, A. S., ed., Mathematical Investigation of Intuitionistic Arithmetic and Analysis, Lecture Notes in Mathematics, Springer-Verlag, Berlin (1972).

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