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COMPLETIONS AND CO-PRODUCTS OF HEYTING ALGEBRAS

By

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A Heyting algebra is not only a lattice theoretic object, but is also related to the intuitionitic logic and a topological space and others. In this paper, we shall investigate about completions and co-products of Heyting algebras.

In §1, we shall study about a Stone space as a complete Heyting algebra, more precisely as a completion of some distributive lattice. In §2, the canonical completion of a Heyting algebra will be studied. Some proofs in §1, §2 and §6 are done intuitionistically. Those cares are necessary for §6. A co-product of Heyting algebras is defined in §3. In §4, we shall study the space of maximal ideals and Wallman-compactifications and Stone-Cech-compactifications in the Heyting algebraic view. The relationships between some properties, completions and co-products defined in the previous sections will be discussed in §5. Complete Heyting algebras in a Heyting extention will be studied in §6.

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§1. An open algebra of a Stone space

We shall use usual lattice-theoretic notations and set-theoretic ones.

DEFINITION 1.1. A lattice L is distributive, if $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$ holds.

A lattice L is bounded, if it has the least element $\mathbb{0}$ and the greatest element $\mathbb{1}$.

A lattice L is a bounded distributive lattice, if it is bounded and distributive.

A lattice L is a Heyting algebra, if it is a bounded distributive lattice and relatively pseudo-compemented. We denote the relative pseudo-complement by $a\Rightarrow b$, where $x\leq a\Rightarrow b$ if and only if $a\wedge x\leq b$.

DEFINITION 1.2. A lattice L is complete if the least upper bound for any

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subset X of L exists. We denote the least upper bound by $\bigvee X$.

DEFINITION 1.3. A lattice L is infinitely distributive, if $a\wedge\bigvee B=\bigvee_{b\in B} a\wedge b$ holds in the case that $\bigvee B$ exists.

DEFINITION 1.4. A subset F of a lattice L is a filter, if the following hold: $a\leq b$ and $a\in F\rightarrow b\in F$, $a\in F$ and $b\in F\rightarrow a\wedge b\in F$ and $\mathbb{1}\in F$ if there exists a greatest element $\mathbb{1}$ in L.

A subset I of a lattice L is an ideal, if the following hold: $a\leq b$ & $b\in I\rightarrow a\in I$, $a\in I$ & $b\in I\rightarrow a\vee b\in I$ and $\mathbb{0}\in I$ if there exists a least element $\mathbb{0}$ in L.

An ideal I is prime, if $a\wedge b\in I$ implies $a\in I$ or $b\in I$ and I is neither L nor empty.

An ideal I is maximal, if I is neither L nor empty and any ideal which includes I is I or L.

$\mathfrak{I}L$ is the set of ideals of L.

$\mathfrak{p}L$ is the set of prime ideals of L.

$\mathfrak{m}L$ is the set of maximal ideals of L.

V_a is the set of prime ideals which do not contain a, which is a basic open set for $\mathfrak{p}L$.

I_a is the principal ideal $\{x\,;x\leq a\}$.

DEFINITION 1.5. A function $\phi: L\rightarrow L'$ is a morphism, where L and L' are lattices, if ϕ preserves the operations $\vee$ and $\wedge$, i.e. $\phi(a\vee b)=\phi(a)\vee\phi(b)$ and $\phi(a\wedge b)=\phi(a)\wedge\phi(b)$ for $a, b\in L$.

A morphism ϕ is complete, if it preserves $\bigvee$, i.e. $\phi(\bigvee X)=\bigvee\phi''X$ in the case that $\bigvee X$ exists.

A morphism $\phi: L\rightarrow L'$ is a $\mathbb{0}, \mathbb{1}$-morphism, if $\phi(\mathbb{0})=\mathbb{0}$ and $\phi(\mathbb{1})=\mathbb{1}$ hold in the case $\mathbb{0}$ and $\mathbb{1}$ exist in L respectively.

A morphism $\phi: A\rightarrow A'$ is a strong Heyting morphism, where A and A' are Heyting algebras, if it is a $\mathbb{0}, \mathbb{1}$-morphism and preserves $\Rightarrow$.

We shall use abbreviations: a BDL for a bounded distributive lattice, an Ha for a Heyting algebra, a cHa for a complete Heyting algebra, a cH-morphism for a complete Heyting morphism and so on.

DEFINITION 1.6. A subset X of a complete lattice L completely generates L, if $a=\bigvee\{x\,;x\leq a$ and $x\in X\}$ holds for each $a\in L$. A complete lattice L^* is a completion of a lattice L, if there exists an injective $\mathbb{0}, \mathbb{1}$-morphism $j: L\rightarrow L^*$ such that the range of j completely generates L^*. This j is called the related morphism.

Definition 1.7. $O(X)$ is the cHa which is the set of open subsets of a topological space X, where the infinite sum and the finite intersection are the set theoretical ones. We call it an open algebra.

$R(H)$ is the set of regular elements of an Ha H, i.e. $R(H)=\{x\,;(x\Rightarrow\mathbb{0})\Rightarrow\mathbb{0}=x\}$. We denote $(x\Rightarrow\mathbb{0})\Rightarrow\mathbb{0}$ by $R(x)$.

By Def. 1.4, $\mathfrak{p}L$ can be regarded as a topological space. It is known as a Stone space.

Theorem 1.1 (Stone) Let L be a distributive lattice and $i: L\to O(\mathfrak{p}L)$ be the function such that $i(a)=V_a$. Then, $O(\mathfrak{p}L)$ is a completion of L and i is the related morphism. And if L is an *Ha*, i is a strong H-morphism.

Proof. $i(a\wedge b)=V_{a\wedge b}=V_a\cap V_b=i(a)\wedge i(b)$ and $i(a\vee b)=V_a\cup V_b=i(a)\vee i(b)$. For the injectiveness, notice that $a\nleq b$ implies the existence of a prime ideal which contains b but does not contain a. Since $\{i(a)\,; a\in L\}$ forms a topological base for $\mathfrak{p}L$, $O=\bigcup\{i(a)\,; i(a)\leq O$ and $a\in L\}$ for each $O\in O(\mathfrak{p}L)$.

Let I be a prime ideal of L. $I\in V_a\cap V_{a\Rightarrow b}\leftrightarrow a\notin I$ & $a\Rightarrow b\notin I\leftrightarrow a\wedge a\Rightarrow b\notin I\to b\notin I$. So, $V_{a\Rightarrow b}\subseteq V_a\Rightarrow V_b$. On the other hand, $I\in V_a\Rightarrow V_b$ implies that there is c such that $I\in V_c\subseteq(\mathfrak{p}L-V_a)\cup V_b$. Then, $c\wedge a\leq b$ and so $c\leq a\Rightarrow b$. Hence, $V_a\Rightarrow V_b=V_{a\Rightarrow b}$.

$i(\mathbb{0})=\phi$ and $i(\mathbb{1})=\mathfrak{p}L$.

Theorem 1.2. Let L be a *BDL* and A be a *cHa*. And let ϕ: $L\to A$ be an $\mathbb{0}$, $\mathbb{1}$-morphism. Then, there exists a unique cH-morphism φ: $O(\mathfrak{p}L)\to A$ that satisfies the left diagram. And if A is completely generated by the range of ϕ, then φ is surjective.

$$\begin{tikzcd} O(\mathfrak{p}L) \arrow[dr, dashed, "\exists!\varphi"] & \\ L \arrow[u, "i"] \arrow[r, "\phi"'] & A \end{tikzcd}$$

Proof. Let $\varphi(a)=\bigvee\{\phi(x)\,;\ i(x)\leq a\}$ for $a\in O(\mathfrak{p}L)$. Since $O(\mathfrak{p}L)$ is a completion of L, $a=\bigvee\{i(x)\,;\ i(x)\leq a\}$ for $a\in O(\mathfrak{p}L)$. And so, the uniqueness of φ is clear.

$$\begin{aligned}\varphi(\mathbb{1})&=\{\phi(x)\,;\ i(x)\leq a\}=\phi(\mathbb{1})=\mathbb{1}.\\ \varphi(x\wedge y)&=\bigvee\{\phi(u)\,;\ i(u)\leq x\wedge y\}\\ &=\bigvee\{\phi(u)\wedge\phi(v)\,;\ i(u)\leq x,\ i(v)\leq y\}\\ &=\varphi(x)\wedge\varphi(y),\ \text{by the infinite distributive-ness of a } cHa.\end{aligned}$$

To show the preservation of the infinite sum $\bigvee$, it is sufficient to show that $\varphi(\bigvee X)=\bigvee\varphi''X$ for $X\subseteq i''L$. Suppose that $\varphi(\bigvee i''X)=\bigvee\phi''X$ does not hold for some $X\subseteq L$. Then, by the definition of φ, there exists u in L such that $i(u)\leq\bigvee i''X$ holds but $\varphi(u)\leq\bigvee\varphi''X$ does not hold. So, there exists a prime ideal I in A such that $\phi(u)\notin I$ and $\bigvee\phi''X\in I$. Let I_ϕ be the subset of L defined by the postulate: $a\in I_\phi\leftrightarrow\phi(a)\in I$. Then, I_ϕ is a prime ideal in L and contains every element of X, but does not contain u. However, this contradicts to the fact: $V_u\subseteq\bigcup_{x\in X}V_x$.

If A is completely generated by $\phi''L$, $x=\bigvee\{\phi(u)\,;\,\phi(u)\leq x\}$ for each $x\in A$. Then, $\phi(\bigvee\{i(u)\,;\,\phi(u)\leq x\})=\bigvee\{\phi(u)\,;\,\phi(u)\leq x\}=x$. Hence, ϕ is surjective.

COROLLARY 1.1. Let A be a *cHa* and a completion of a distributive lattice L. And let j be the related morphism. Suppose that for any completion A' of L with the related morphism j', there exists a *cH*-morphism ϕ that satisfies the left diagram. Then, A is isomorphic to $O(\mathfrak{p}L)$.

$$\begin{tikzcd} A \arrow[dr, dashed, "\exists\,\phi"] & \\ L \arrow[u, "j"] \arrow[r, "j'"'] & A' \end{tikzcd}$$

PROOF. If L does not contain $\mathbb{0}$ nor $\mathbb{1}$, we can add $\mathbb{0}$ or $\mathbb{1}$ and extend j and j' as the related morphisms of completions of the extended distributive lattice of L. So, we assume that L is a *BDL*.

Let A' be $O(\mathfrak{p}L)$. Then, by Th. 1, there exists a surjective *cH*-morphism $\phi':O(\mathfrak{p}L)\to A$. Now, it is easy to check that the diagram: $A\underset{\phi'}{\overset{\phi}{\rightleftarrows}}O(\mathfrak{p}L)$ commutes.

Next we shall show another representation of $O(\mathfrak{p}L)$ for a distributive lattice L.

DEFINITION 1.8. For $I,J\in L$, $I\wedge J=I\cap J$. For $\Gamma\subseteq\mathfrak{I}L$, $\bigvee\Gamma$ is the set of finite sums of elements of $\bigcup\Gamma$.

For $X\subseteq L$, $I(X)$ is the minimal ideal that contains X.

For the sake of §6, in some cases it is necessary, that the proofs are intuitionistic. So, we shall mark lemmas, theorems and corollaries by * in the case that they are proved intuitionistically.

LEMMA 1.1.* $\mathfrak{I}L$ with the operations in Def. 8 is a *cHa* and a completion of L for a distributive lattice L.

PROOF. Since $I\cap J\in\mathfrak{I}L$ holds for $I,J\in\mathfrak{I}L$, $I\wedge J$ is the maximal ideal which is included by I and J. It follows from the distributive-ness of L that $\bigvee\Gamma\in\mathfrak{I}L$ for $\Gamma\subseteq\mathfrak{I}L$.

$\bigvee\Gamma$ is the minimal ideal which includes every I in Γ. So, $\bigvee_{J\in\Gamma}I\wedge J\leq I\wedge\bigvee\Gamma$. Conversely, $x\in I\wedge\bigvee\Gamma$ implies that $x\in I$ and x is a finite sum of elements of $\bigcup\Gamma$.

So, x is a finite sum of elements of $\bigcup_{J\in\Gamma} I\wedge J$. Hence, $x\in\bigvee_{J\in\Gamma} I\wedge J$. Now we have proved that $\mathfrak{J}L$ is a *cHa*, since the infinite distributive-ness of a complete lattice implies the relatively pseudo-complemented-ness. By the way, $I\Rightarrow J=\{y\,;x\wedge y\in J$ for each $x\in I\}$.

Let j be the function such that $j(x)=I_x$ for $x\in L$. Then, $j(x)\in\mathfrak{J}L$. So, $\mathfrak{J}L$ is a completion of L and j is the related morphism.

COROLLARY 1.2. $\mathfrak{J}L$ is isomorphic to $O(\mathfrak{p}L)$.

PROOF. By Lemma 1.1 and Th. 1.2, there exists a surjective *cH*-morphism $\psi:O(\mathfrak{p}L)\to\mathfrak{J}L$. Let $\psi(O)=\psi(P)$. Then, $\bigvee\{j(x)\,;i(x)\leq O\}=\bigvee\{j(y)\,;\,i(y)\leq P\}$. So, $i(x)\leq O$ implies $x\in\bigvee\{j(y)\,;i(y)\leq P\}$. By the definition of the infinite sum, $i(x)\leq P$. This argument implies $O=P$.

§ 2. The canonical completion

In this section we shall prove the existence of the canonical completion of a Heyting algebra and its uniqueness. This has been proved by Funayama [5], and Rasiowa and Sikorski [9], but we want to prove it intuitionistically for our purpose. Our proof is on the same line of Funayama's.

LEMMA 2.1.* A Heyting algebra is infinitely distributive.

PROOF. A usual proof is intuitionistic. See [9].

DEFINITION 2.1. An ideal I of a lattice L is closed, if the following holds:

$$\text{“}a=\bigvee\{x\,;x\in I \text{ and } x\leq a\}\text{” implies “}a\in I\text{”}.$$

$\mathfrak{J}_cL$ is the set of closed ideals of L.

LEMMA 2.2.* For any $X\subseteq L$, there exists a unique minimal closed ideal $I_c(X)$ that includes X. If L is infinitely distributive, $I_c(X)$ is the set of all elements u's such that $u=\bigvee\{v\,;v\leq u$ and $v\leq x$ for some $x\in X\}$.

PROOF. $I_c(X)$ is the intersection of all closed ideals that include X.

Let L be infinitely distributive and J be the set of all u's in the lemma. If $u\in J$ and $w\leq u$, then

$=\bigvee\{v\wedge w\,;v\leq u$ and $v\leq x$ for some $x\in X\}$

$=\bigvee\{v\,;v\leq w$ and $v\leq x$ for some $x\in X\}$ and so $w\in J$.

If $u\in J$ and $w\in J$, $u\vee w=\bigvee\{v\,;(v\leq u$ or $v\leq w)$ and $v\leq x$ for some $x\in X\}$

$$=\bigvee\{v\,;v\leq u\vee w \text{ and } v\leq x \text{ for some } x\in X\}$$

and hence $u\vee w\in J$.
Suppose that $a=\bigvee\{x\,;x\leq a$ and $x\in J\}$.
Let $A_x=\{v:v\leq x$ and $v\leq y$ for some $y\in X\}$ for $x\in J$. Then,

$$\begin{aligned} x=\bigvee A_x \text{ for } x\in J. \text{ So, } a&=\bigvee\{\bigvee A_x\,;x\leq a \text{ and } x\in J\}\\ &=\bigvee\bigcup\{A_x\,;x\leq a \text{ and } x\in J\}\\ &=\bigvee\{v\,;v\leq a \text{ and } v\in X\}, \text{ which is in } J. \end{aligned}$$

Now, we have proved that J is a closed ideal, which includes X. The minimality of J is clear.

Here we define the operations for $\mathfrak{I}_cL$, which are a little different from those for $\mathfrak{I}L$. We shall use the same notations, since no confusion will occur.

DEFINITION 2.2. For $I,J\in\mathfrak{I}_cL$, $I\wedge J=I\cap J$. For $\Gamma\subseteq\mathfrak{I}_cL$, $\bigvee\Gamma=I_c(\bigcup\Gamma)$, i.e. the minimal closed ideal that includes $\bigcup\Gamma$.

LEMMA 2.3*. Let A be an *Ha*. Then, $\mathfrak{I}_cA$ is a *cHa* and the embedding $i:A\to\mathfrak{I}_cA\,;i(x)=I_x$, is an injective strong *cH*-morphism.

PROOF. $\mathfrak{I}_cA$ is closed under the operations in Def. 2.2. $\bigvee_{J\in\Gamma}I\wedge J\leq I\wedge\bigvee\Gamma$ for Γ. Let x be an element of $I\wedge\bigvee\Gamma$. Then, $x\in I$ and $x=\bigvee\{v\,;v\leq x$ and $v\in J$ for some $J\in\Gamma\}$ by Lemma 2.2 and Def. 2.2. So, $x\in I_c(\bigcup_{J\in\Gamma}I\wedge J)=\bigvee_{J\in\Gamma}I\wedge J$. Hence, $\mathfrak{I}_cA$ is a *cHa*.

$$i(x\wedge y)=I_{x\wedge y}=I_x\cap I_y=i(x)\wedge i(y).$$

Suppose that $\bigvee X$ exists for $X\subseteq A$. Let $\Gamma=\{i(x)\,;x\in X\}$, then $\bigvee X\in I_c(\bigcup\Gamma)$. So, $i(\bigvee X)=I_c(\bigcup\Gamma)=\bigvee\Gamma=\bigvee_{x\in X}i(x)$. $i(\mathbb{1})=I_{\mathbb{1}}=A=\mathbb{1}$.

Let J be the closed ideal that satisfies the condition: $J\wedge I_x\leq I_y$. Then, $z\wedge x\leq y$ for any $z\in J$. And so, $z\leq x\Rightarrow y$. So, $J\leq I_{x\Rightarrow y}$. On the other hand, $I_{x\Rightarrow y}\subseteq I_x\Rightarrow I_y$. Hence, $I_{x\Rightarrow y}=I_x\Rightarrow I_y$.

THEOREN 2.1.* Let A and A' be an *Ha* and a *cHa* respectively. And let ϕ be a *cH*-morphism from A to A'. Then, there exists a unique *cH*-morphism ψ from $\mathfrak{I}_cA$ to A' such that the left diagram commutes.

$$\begin{tikzcd} \mathfrak{I}_cA \arrow[dr, dashed, "\exists!\psi"] & \\ A \arrow[u, "i"] \arrow[r, "\phi"'] & A' \end{tikzcd}$$

PROOF. Let $\psi(I)=\bigvee_{x\in I}\phi(x)$. By the infinite distributive-ness of A', $\psi(I)\wedge\psi(J)$ $=\bigvee_{x\in I}\phi(x)\wedge\bigvee_{y\in J}\phi(y)=\bigvee_{x\in I}\bigvee_{x\in J}\phi(x\wedge y)\leq\bigvee_{x\in I\wedge J}\phi(x)=\psi(I\wedge J)$.

Let $x\in I_c(\bigcup\Gamma)$ for $\Gamma\subseteq\mathfrak{J}_cA$. Then, $x=\bigvee\{u\,;u\leq x$ and $u\in\bigvee\Gamma\}$. $\phi(x)=\bigvee\{\phi(u)\,;$ $u\leq x$ and $u\in\bigcup\Gamma\}$, by the completeness of ϕ. $u\in\bigcup\Gamma$ implies $u\in I$ for some $I\in\Gamma$ and so $\phi(u)\leq\psi(I)$ for some $I\in\Gamma$. Hence, $\psi(\bigvee\Gamma)\leq\bigvee_{I\in\Gamma}\psi(I)$. And $\psi(\mathbb{1})=\mathbb{1}$.

The uniqueness of ψ is clear from the fact that ψ is complete.

DEFINITION 2.3. A completion of an *Ha* A is canonical, if the related morphism i is complete. We denote the canonical completion of A by $\bar{A}$.

COROLLARY 2.1.* (Funayama [5]) $\mathfrak{J}_cA$ is a canonical completion of an *Ha* A and every canonical completion of A is isomorphic to $\mathfrak{J}_cA$.

PROOF. It is sufficient to show that the morphism ψ in Th. 2.1 is injective and surjective in the case that A' is a canonical completion of A and ϕ is the related morphism.

For any $x\in A'$, $x=\bigvee\{\phi(a)\,;\phi(a)\leq x$ and $a\in A\}$. $\psi(\bigvee\{i(a)\,;\phi(a)\leq x$ and $a\in A\})=x$ holds and so ψ is surjective. Suppose that $\bigvee_{x\in I}\phi(x)=\bigvee_{y\in J}\phi(y)$. $\phi(x)\leq\bigvee_{y\in J}\phi(y)$ implies $\phi(x)=\bigvee_{y\in J}\phi(x\wedge y)$. Let z be an element such that $x\wedge y\leq z$ for any $y\in J$. Then, $\phi(x\wedge y)\leq\phi(z)$ for any $y\in J$. So, $\phi(x)=\bigvee_{y\in J}\phi(x\wedge y)\leq\phi(z)$. And so, $\phi(x)=\phi(x)\wedge\phi(z)=$ $\phi(x\wedge z)$. By the injective-ness of ϕ, $x=x\wedge z$ and hence $x\leq z$. So, $x=\bigvee_{y\in J}x\wedge y\in J$. These imply $I=J$.

From now on, we shall assume that i in Def. 2.3 is the inclusion map.

§3. A co-product of Heyting algebras

We shall define a co-product of bounded distributive lattices, the existence of which have been well-known. Our object is a co-product of Heyting algebras as bounded distributive lattices. The fact that it forms a Heyting algebra has perhaps already known, but we shall prove it to check additional properties for the following chapters.

DEFINITION 3.1. For distributive lattices L_α $(\alpha\in\Lambda)$.[i)] the co-product $\bigotimes_{\alpha\in\Lambda}L_\alpha$ is the sublattice of $O(\prod_{\alpha\in\Lambda}\mathfrak{p}L_\alpha)$ finitely generated by $\{p_\alpha^{-1}V_a\,;a\in L_\alpha,\,\alpha\in\Lambda\}$, where p_α is the projection from $\prod_{\alpha\in\Lambda}\mathfrak{p}L_\alpha$ to $\mathfrak{p}L_\alpha$ for each $\alpha\in\Lambda$. The embedding $i_\alpha: L_\alpha\to\bigotimes_{\alpha\in\Lambda}L_\alpha$ is defined by the postulate: $i_\alpha(a)=p_\alpha^{-1}V_a$ for $\alpha\in\Lambda$.

i) To avoid the triviality and for the simplicity, we assume that L_α has at least two elements, when we treat the co-products $\otimes$.

LEMMA 3.1. Any element x of $\bigotimes_{\alpha\in\Lambda} L_\alpha$ can be represented by the following two forms.

$$x=\bigvee_{k<n}\bigwedge_{\alpha\in F_k} i_\alpha(a_k^\alpha)=\bigwedge_{j<m}\bigvee_{\alpha\in G_i} i_\alpha(b_j^\alpha),\ \text{where } F_k \text{ and } G_j$$

are finite subsets of Λ.

The proof can be done by the induction on the construction of $\bigotimes_{\alpha\in\Lambda} L_\alpha$. It is a routine, so we omit it.

THEOREM 3.1. (Sikorski [10]) Let L_α and L be distributive lattices and h_α be a $\mathbb{0}, \mathbb{1}$-morphism from L_α to L for each $\alpha\in\Lambda$. Then, there exists a unique $\mathbb{0}, \mathbb{1}$-morphism ϕ that makes the left diagram commutative. And ϕ is injective if and only if $\bigwedge_{\alpha\in F} h_\alpha(a^\alpha)\leq\bigvee_{\alpha\in G} h_\alpha(b^\alpha)$ implies that $a^\alpha\leq b^\alpha$ for some $\alpha\in F\cap G$, $a^\alpha=\mathbb{0}$ for some $\alpha\in F$ or $b^\alpha=\mathbb{1}$ for some $\alpha\in G$, where F and G are finite.

$$\begin{tikzcd} & L_\alpha \arrow[dl, "i_\alpha"'] \arrow[dr, "h_\alpha"] & \\ \bigotimes_{\alpha\in\Lambda} L_\alpha \arrow[rr, dashed, "\phi"'] & & L_\alpha \end{tikzcd}$$

PROOF. For $x\in\bigotimes_{\alpha\in\Lambda} L_\alpha$, x can be represented as $\bigvee_{k<n}\bigwedge_{\alpha\in F_k} i_\alpha(a_k^\alpha)$. Let ϕ be the function such that $\phi(x)=\bigvee_{k<n}\bigwedge_{\alpha\in F_k} h_\alpha(a_k^\alpha)$. Suppose that $\bigwedge_{\alpha\in F} i_\alpha(a^\alpha)\leq x$. Then, $x=\bigwedge_{f\in\prod_{k<n} F_k}\bigvee i_{f(k)}(a_k^{f(k)})$ by the distributive-ness. Remind that i_α is the inverse of the projection and finite sums and intersections are the set theoretical ones. Then, $i_\alpha(a^\alpha)\leq\bigvee_{f(k)=\alpha} i_\alpha(a_k^\alpha)=i_\alpha(\bigvee_{f(k)=\alpha} a_k^\alpha)$ holds for some $\alpha\in F$, $a^\alpha=\mathbb{0}$ for some $\alpha\in F$, or $i_\alpha(\bigvee_{f(k)=\alpha} a_k^\alpha)=\mathbb{1}$ holds for some $\alpha\notin F$. So, $\bigwedge_{\alpha\in F} h_\alpha(a^\alpha)\leq\bigvee_{k<n} h_{f(k)}(a_k^{f(k)})$ for each $f\in\prod_{k<n} F_k$. Hence, $\bigwedge_{\alpha\in F} h_\alpha(a^\alpha)\leq\bigwedge_{f\in\prod_{k<n} F_k}\bigvee_{k<n} h_{f(k)}(a_k^{f(k)})=\bigvee_{k<n}\bigwedge_{\alpha\in F_k} h_\alpha(a_k^\alpha)$.

These above implies the well-defined-ness of ϕ. By the definition of ϕ, $\vee$, $\wedge$, $\mathbb{0}$ and $\mathbb{1}$ are preserved under ϕ.

Suppose that ϕ is injective. Then, $\bigwedge_{\alpha\in F} h_\alpha(a^\alpha)\leq\bigvee_{\alpha\in G} h_\alpha(b^\alpha)$ implies $\bigwedge_{\alpha\in F} i_\alpha(x^\alpha)\leq\bigvee_{\alpha\in G} i_\alpha(b^\alpha)$ and so $a^\alpha\leq b^\alpha$ for some $\alpha\in F\cup G$, $a^\alpha=\mathbb{0}$ for some $\alpha\in F$ or $b^\alpha=\mathbb{1}$ for some $\alpha\in G$. The converse is similar by Lemma 3.1.

Lemma 3.2. $\mathfrak{p}\bigotimes_{\alpha\in\Lambda} L_\alpha$ is homeomorphic to $\prod_{\alpha\in\Lambda}\mathfrak{p}(L_\alpha)$.

PROOF. Let I_α be a prime ideal for each $\alpha\in\Lambda$ and $\bigotimes_{\alpha\in\Lambda} I_\alpha$ be the subset of $\bigotimes_{\alpha\in\Lambda} L_\alpha$ defined by the following: $x\in\bigotimes_{\alpha\in\Lambda} I_\alpha$ if and only if there exist some finite $F\subseteq\Lambda$ and $a^\alpha\in I_\alpha$ for each $\alpha\in F$ such that $x\leq\bigvee_{\alpha\in F} i_\alpha(a^\alpha)$. Then, $\bigotimes_{\alpha\in F} I_\alpha$ is clearly an ideal. Suppose that $x\wedge y\in\bigotimes_{\alpha\in\Lambda} I_\alpha$. By Lemma 3.1, x and y have the representations; $\bigvee_{k<m}\bigwedge_{\alpha\in F_k} i_\alpha(a_k^\alpha)$ and $\bigvee_{j<n}\bigwedge_{\alpha\in G_j} i_\alpha(b_j^\alpha)$ respectively. By the definition of $\bigotimes_{\alpha\in\Lambda} I_\alpha$, $x\wedge y\leq\bigvee_{\alpha\in F} i_\alpha(a^\alpha)$ for some

finite F and some $a^{\alpha}\in I_{\alpha}$ for each $\alpha\in F$. Without any loss of generality, we may assume that $F_k=G_j=F$ for any k,j. So, $\bigwedge_{\alpha\in F} i_{\alpha}(a_k^{\alpha})\wedge\bigwedge_{\alpha\in F} i_{\alpha}(b_j^{\alpha})\leq\bigvee_{\alpha\in F} i_{\alpha}(a^{\alpha})$ for any k,j. And so, $a_k^{\alpha}\in I_{\alpha}$ or $b_j^{\alpha}\in I_{\alpha}$ for some $\alpha\in F$. For each $\alpha\in F$, let A_{α} and B_{α} be the subsets of m and n respectively such that $k\in A_{\alpha}\leftrightarrow a_k^{\alpha}\in I_{\alpha}$ and $j\in B_{\alpha}\leftrightarrow b_j^{\alpha}\in I_{\alpha}$. Suppose that $\bigcup_{\alpha\in F} A_{\alpha}=m$, then $x=\bigvee_{k<m}\bigwedge_{\alpha\in F} i_{\alpha}(a_k^{\alpha})\in\bigotimes_{\alpha\in A} I_{\alpha}$. Otherwise, there exists $k<m$; $k\notin A_{\alpha}$ for any $\alpha\in F$. Then, there is α in F such that $b_j^{\alpha}\in I_{\alpha}$ for any $j<n$. Hence, $\bigcup_{\alpha\in F} B_{\alpha}=n$ and $y=\bigvee_{j<n}\bigwedge_{\alpha\in F} i_{\alpha}(b_j^{\alpha})\in\bigotimes_{\alpha\in A} I_{\alpha}$. Now, we have proved that $\bigotimes_{\alpha\in A} I_{\alpha}$ is a prime ideal.

Let $(I)_{\alpha}$ be the subset of L_{α} for a prime ideal I and each $\alpha\in A$ such that $x\in(I)_{\alpha}\leftrightarrow i_{\alpha}(x)\in I$. Then, $(I)_{\alpha}$ is a prime ideal and $(\bigotimes_{\alpha\in A} I_{\alpha})_{\alpha}=I_{\alpha}$ for each $\alpha\in A$. And $I=\bigotimes_{\alpha\in A}(I)_{\alpha}$ holds. Let $c=\bigvee_{k<m}\bigwedge_{\alpha\in F_k} i_{\alpha}(a_k^{\alpha})$. Then,

$$\begin{aligned} I\in V_c &\leftrightarrow \bigwedge_{\alpha\in F_k} i_{\alpha}(a_k^{\alpha})\notin I \quad \text{for some } k<m \\ &\leftrightarrow a_k^{\alpha}\notin(I)_{\alpha} \text{ for each } \alpha\in F_k \text{ for each } k<m. \end{aligned}$$

These above imply the lemma.

THEOREM 3.2. Let A_{α} be an Ha for each $\alpha\in A$, then $\bigotimes_{\alpha\in A} A_{\alpha}$ is an Ha and i_{α} is a strong cH-morphism for each $\alpha\in A$.

PROOF. Let p_{α} be the projection in Def. 3.1. p_{α} is an open continuous map and hence an easy calculation shows that i_{α} is a strong cH-morphism from A_{α} to $\bigotimes_{\alpha\in A} A_{\alpha}$.

By Th. 3.1, what we must prove is that $\bigotimes_{\alpha\in A} A_{\alpha}$ is relatively pseudo-complemented. By Th. 1.1 and Lemma 3.2, it is sufficient to show that $x\Rightarrow y$ is in $\bigotimes_{\alpha\in A} A_{\alpha}$ for each $x,y\in\bigotimes_{\alpha\in A} A_{\alpha}$, where $\Rightarrow$ is in the sense $O(\prod_{\alpha\in A}\mathfrak{p}A_{\alpha})$. Let $x=\bigvee_{k<m}\bigwedge_{\alpha\in F_k} i_{\alpha}(a_k^{\alpha})$ and $y=\bigwedge_{j<n}\bigvee_{\alpha\in G_j} i_{\alpha}(b_j^{\alpha})$. We can assume $F_k=G_j=F$ for each $k<m$ and $j<n$. Then,

$$\begin{aligned} x\Rightarrow y &= \bigwedge_{k<m}\Big(\bigwedge_{\alpha\in F} i_{\alpha}(a_k^{\alpha})\Rightarrow\bigwedge_{j<n}\bigvee_{\alpha\in F} i_{\alpha}(b_j^{\alpha})\Big) \\ &= \bigwedge_{k<m}\bigwedge_{j<n}\Big(\bigwedge_{\alpha\in F} i_{\alpha}(a_k^{\alpha})\Rightarrow\bigvee_{\alpha\in F} i_{\alpha}(b_j^{\alpha})\Big) \\ &= \bigwedge_{k<m}\bigwedge_{j<n}\Big(\bigvee_{\alpha\in F} i_{\alpha}(a_k^{\alpha}\Rightarrow b_j^{\alpha})\Big) \end{aligned}$$

Hence, $x\Rightarrow y\in\bigotimes_{\alpha\in A} A_{\alpha}$ and $\bigotimes_{\alpha\in A} A_{\alpha}$ is an Ha.

In some cases, a co-product of open algebras $O(X_{\alpha})$'s in the category of the cHa's is isomorphic to $O(\prod_{\alpha\in A} X_{\alpha})$. (cf. Isbell [8]) We next show that $O(\mathfrak{p}L)$'s are such cHa's.

THEOREM 3.3. Let L_{α} be a distributive lattice and i_{α} be the embedding such that $\bar{i}_\alpha : O(\mathfrak{p}L_\alpha)\to O(\prod_{\alpha\in\Lambda}\mathfrak{p}L_\alpha)$ and $\bar{i}_\alpha(x)=p_\alpha^{-1}x$, where $p_\alpha : \prod_{\alpha\in\Lambda}\mathfrak{p}L_\alpha\to\mathfrak{p}L_\alpha$ is the projection, for each $\alpha\in\Lambda$.

$$\begin{tikzcd} & O(\mathfrak{p}L_\alpha) \arrow[dl, "\bar{i}_\alpha"'] \arrow[dr, "\bar{h}_\alpha"] & \\ O(\prod_{\alpha\in\Lambda}\mathfrak{p}L_\alpha) \arrow[rr, dashed, "\exists!\bar{\phi}"'] & & A \end{tikzcd}$$

If A is a cHa and $\bar{h}_\alpha$ is a cH-morphism from $O(\mathfrak{p}L_\alpha)$ to A for each $\alpha\in\Lambda$, then there exists a unique cH-morphism $\bar{\phi}$ such that the left diagram commutes.

Proof. By Th. 1.1, there exists a unique $\mathbb{0},\mathbb{1}$-morphism h_α such that the diagram (1) commutes, where $j_\alpha : L_\alpha\to O(\mathfrak{p}L_\alpha)$ is the morphism in Th. 1.1 for each $\alpha\in\Lambda$. By Th. 3.1, there exists a unique $\mathbb{0},\mathbb{1}$-morphism ϕ such that the diagram (2) commutes. Now, by Th. 1.2, there exists a unique cH-morphism $\bar{\phi}$ such that the diagram (3) commutes. By Lemma 3.2, $O(\mathfrak{p}\bigotimes_{\alpha\in\Lambda}L_\alpha)$ and $O(\prod_{\alpha\in\Lambda}\mathfrak{p}L_\alpha)$ are isomorphic to each other and so we regard them as the same thing.

(1)

$$\begin{tikzcd} O(\mathfrak{p}L_\alpha) \arrow[drr, "\bar{h}_\alpha"] & & \\ L_\alpha \arrow[u, "j_\alpha"] \arrow[rr, dashed, "\exists!h_\alpha"'] & & A \end{tikzcd}$$

(2)

$$\begin{tikzcd} & L_\alpha \arrow[dl, "i_\alpha"'] \arrow[dr, "h_\alpha"] & \\ \bigotimes_{\alpha\in\Lambda}L_\alpha \arrow[rr, dashed, "\exists!\phi"'] & & A \end{tikzcd}$$

(3)

$$\begin{tikzcd} O(\mathfrak{p}\bigotimes_{\alpha\in\Lambda}L_\alpha) \arrow[drr, dashed, "\exists!\bar{\phi}"] & & \\ \bigotimes_{\alpha\in\Lambda}L_\alpha \arrow[u, "j"] \arrow[rr, "\phi"'] & & A \end{tikzcd}$$

Now, the only thing we must prove is $\bar{\phi}\cdot\bar{i}_\alpha=\bar{h}_\alpha$ for each $\alpha\in\Lambda$. Let $P=\bigcup\{V_a ; V_a\subseteq P \text{ and } a\in L_\alpha\}\in O(\mathfrak{p}L_\alpha)$. $\bar{i}_\alpha(P)=\bigcup\{p_\alpha^{-1}V_a ; V_a\subseteq P \text{ and } a\in L_\alpha\}=\bigcup\{i_\alpha(a) ; V_a\subseteq P \text{ and } a\in L_\alpha\}$. So,

$$\begin{aligned}\bar{\phi}\cdot\bar{i}_\alpha(P)&=\bigvee\{\bar{\phi}\cdot i_\alpha(a) ; V_a\subseteq P \text{ and } a\in L_\alpha\}\\ &=\bigvee\{\phi\cdot i_\alpha(a) ; V_a\subseteq P \text{ and } a\in L_\alpha\}\\ &=\bigvee\{h_\alpha(a) ; V_a\subseteq P \text{ and } a\in L_\alpha\}=\bar{h}_\alpha(P).\end{aligned}$$

§ 4. A space of maximal ideals

We have studied about the open algebras of the spaces of prime ideals. In this section we shall investigate the open algebras of the spaces of maximal ideals.

Definition 4.1. For distributive lattice L, the topology of $\mathfrak{m}L$ is the subspace-topology of $\mathfrak{p}L$. (See Def. 1.4)

Definition 4.2. *A BDL L* has the *T*-property, if $a\nleq b$ implies that there is an element c ; $a\vee c=\mathbb{1}$ and $b\vee c\neq\mathbb{1}$.

A BDL L is normal, if $a \vee b = \mathbb{1}$ implies that there are elements u and v such that $u \wedge v = \mathbb{0}$ and $a \vee u = b \vee v = \mathbb{1}$.

A BDL L is compact, if $\bigvee X = \mathbb{1}$ implies the existence of a finite subset F of X such that $\bigvee F = \mathbb{1}$.

An element x of a *BDL* L is a co-atom, if there exists no element between x and $\mathbb{1}$ and x is not $\mathbb{1}$.

By the definitions, the following are immediate.

PROP. Let X be a topological space. If X is a T_1-space, then $O(X)$ has the T-property. X is normal,(i) if and only if $O(X)$ is normal.

PROP. Let L be a *BDL*. L is compact if and only if every maximal ideal contains a co-atom.

PROP. If L is a *BDL* with the T-property, then $O(\mathfrak{m}L)$ is a completion of L and the related morphism is e, where $e(a) = V_a \cap \mathfrak{m}A$ for $a \in A$.

PROP. Let A be an *Ha*. A is compact if and only if $\bar{A}$ is compact.

PROP. $O(\mathfrak{m}L)$ has the T-property.

LEMMA 4.1. Let L be a *BDL* and A' be a compact *cHa* with the T-property. And let $j : L \to A'$ be an $\mathbb{0}, \mathbb{1}$-morphism such that $j(x) = \mathbb{1}$ implies $x = \mathbb{1}$ and $j''L$ completely generates A'. Then, there exists a unique *cH*-morphism ϕ such that the left diagram commutes. And ϕ is surjective.

$$\begin{tikzcd} O(\mathfrak{m}L) \arrow[dr, dashed, "\exists ! \phi"] & \\ L \arrow[u, "e"] \arrow[r, "j"'] & A' \end{tikzcd}$$

PROOF. Let $\phi(\bigcup_{a \in A} e(a)) = \bigvee_{a \in A} j(a)$. We prove the well-defined-ness. Suppose that $e(b) \subseteq \bigcup_{a \in A} e(a)$ and $j(b) \not\leq \bigvee_{a \in A} j(a)$. By the conditions, there exists c; $j(b) \vee j(c) = \mathbb{1}$ and $\bigvee_{a \in A} j(a) \vee j(c) \neq \mathbb{1}$. Let I be a maximal ideal which includes $A \cup \{c\}$. Then, b does not belong to I, since $b \vee c = \mathbb{1}$. But, this contradicts to $e(b) \subseteq \bigcup_{a \in A} e(a)$.

Now, it is easy to prove the lemma.

Notice that the condition "$j(x) = \mathbb{1} \to x = \mathbb{1}$" is equivalent to the injective-ness for a *BDL A* with the T-property.

(i) This "normal" has the usual topological context.

Lemma 4.2. Let L be a BDL. And let L' be a compact completion of L and j be the related morphism. Then, there exists a unique cH-morphism $\phi: L' \to O(\mathfrak{m}L)$ such that the left diagram commutes.

$$\begin{tikzcd} O(\mathfrak{m}L) & \\ L \arrow[u, "e"] \arrow[r, "j"'] & L' \arrow[ul, dashed, "\exists!\phi"'] \end{tikzcd}$$

Proof. Let $\phi(x) = \bigcup\{e(a); j(a) \leq x, a \in L\}$. Then, the unique-ness of ϕ and the fact $\phi \cdot j = e$ are clear.

$I \in \phi(x)$ implies that $j(a) \leq x$ and $a \notin I$ for some a. So, there is b such that $b \in I$ and $a \vee b = \mathbb{1}$. Hence, $x \vee \bigvee j''I = \mathbb{1}$. Conversely, $x \vee \bigvee j''I = \mathbb{1}$ implies the existence of a and b such that $j(a) \leq x$, $b \in I$ and $j(a) \vee j(b) = \mathbb{1}$, by the compactness of A'. By the property of j, $a \vee b = \mathbb{1}$, and so $a \notin I$. So, $\phi(x) = \{I; x \vee \bigvee j''I = \mathbb{1}\}$.

Now, it is easy to check that ϕ is a cH-morphism.

Theorem 4.1. Let L be a BDL with the T-property. Then, every compact completion of L with the T-property is isomorphic to $O(\mathfrak{m}L)$.

Proof. By Lemma 4.2, $\phi: O(\mathfrak{m}L) \to A'$ in Lemma 4.1 is injective.

The next corollary is one characterization of the Wallman-compactification of a T_1-space.

Corollary 4.1. (Wallman) Let X be a T_1-space. Then, $O(\mathfrak{m}O(X))$ is the unique compact completion of $O(X)$ with the T-property up to an isomorphism.

Corollary 4.2. If L is a compact BDL, $e: L \to O(\mathfrak{m}L)$ is complete.

Proof. Let j be the related morphism of $\bar{A}$. Then, j is complete. $\bar{A}$ is compact since A is compact. So, e is complete by Lemma 4.2.

Corollary 4.3. Let A be a compact cHa with the T-property. Then, $O(\mathfrak{m}A)$ is isomorphic to A.

Next, we shall study about a normal cHa.

Definition 4.3. Let A be a cHa. U_x is the set $\{u; x \vee v = \mathbb{1}$ and $u \wedge v = \mathbb{0}$ for some $v\}$ for $x \in A$.

T: $A \to A$ is the function such that $T(x) = \bigvee U_x$.

A^* is the range of T.

LEMMA 4.3. Let A be a normal *cHa*. Then, T is a $\mathbb{0}$, $\mathbb{1}$-morphism.

PROOF. $U_x \subseteq U_y$ for $x \leq y$ and so $T(x) \leq T(y)$. For $u \in U_x$ and $u' \in U_y$, there exist v and v' such that $x \vee v = y \vee v' = \mathbb{1}$ and $u \wedge v = u' \wedge v' = \mathbb{0}$, by the definition. Then, $(x \wedge y) \vee (v \vee v') = \mathbb{1}$ and $(u \wedge u') \wedge (v \vee v') = \mathbb{0}$. So, $u \wedge u' \in U_{x \wedge y}$. Hence, $T(x) \wedge T(y) = T(x \wedge y)$.

Let u^* be an element of $U_{x \vee y}$. Then, there exists v^* such that $x \vee y \vee v^* = 1$ and $u^* \wedge v^* = \mathbb{0}$. By the normality, there exist u_0 and v_0 such that $x \vee v_0 = y \vee v^* \vee u_0 = \mathbb{1}$ and $u_0 \wedge v_0 = \mathbb{0}$. Again by the normality, there exist u_1 and v_1 such that $y \vee v_1 = v^* \vee u_0 \vee u_1 = \mathbb{1}$ and $u_1 \wedge v_1 = \mathbb{0}$. Then, $u_0 \in U_x$ and $u_1 \in U_y$ and $u^* \leq u_0 \vee u_1$. So, $T(x \vee y) \leq T(x) \vee T(y)$ and so $T(x \vee y) = T(x) \vee T(y)$. Clearly, $T(\mathbb{0}) = \mathbb{0}$ and $T(\mathbb{1}) = \mathbb{1}$.

LEMMA 4.4. Let A be a normal *cHa*. Then, $U_{T(x)} = U_x$ and so $T(T(x)) = T(x)$. And A^* is a *cHa*, where the infinite sum in A^* is as same as that in A.

PROOF. $T(x) \leq x$ and so $U_{T(x)} \subseteq U_x$. Let u be an element of U_x. Then, $x \vee v = \mathbb{1}$ and $u \wedge v = \mathbb{0}$ for some v. By the normality, there exist w_0 and w_1 such that $x \vee w_0 = v \vee w_1 = \mathbb{1}$ and $w_0 \wedge w_1 = \mathbb{0}$. So, $w_1 \in U_x$ and $u \in U_{w_1}$. Hence, $u \in U_{T(x)}$.

By Lemma 4.3, A^* is a *BDL*. Let X be a subset of A^* and $y = \bigvee X$. Since $T(x) = x$ for $x \in X$, $y = \bigvee X \leq T(y)$. So, $T(y) = y$ and hence $y \in A^*$.

So, A^* is a *cHa* and the infinite sum in A^* is as same as that in A.

In the next lemma we need to discern the operations of A and those of A^* and so we shall do it fixing A, A^* or $*$ to the operations.

LEMMA 4.5. Let A be a normal *cHa*. Then, $T^*(x) = x$ for $x \in A^*$.

PROOF. Let $u \in U_x^A$ and $x \in A^*$. Then, $x \vee v = \mathbb{1}$ and $u \wedge v = \mathbb{0}$ for some v. By Lemma 4.1, $T(x) \vee T(v) = \mathbb{1}$ and $T(u) \wedge T(v) = \mathbb{0}$. So, $T(u) \in U_x^{A^*}$. Hence, $T''U_x^A \subseteq U_x^{A^*} \subseteq U_x^A$. As indicated before, for $u \in U_x^A$, there is $w \in U_x^A$ such that $u \leq T(w)$.

These above show $\bigvee U_x^{A^*} = \bigvee U_x^A$. So, $x = \bigvee U_x^A = U_x^{A^*} = \bigvee^* U_x^{A^*}$. Hence, $T^*(x) = x$.

LEMMA 4.6. Let A be a *cHa*. If T is the identity, A has the T-property. If A is normal and has the T-property, T is the identity.

PROOF. Suppose that $x \vee c = \mathbb{1}$ implies $y \vee c = \mathbb{1}$ for any c. Then, $U_x \subseteq U_y$ and so $T(x) \leq T(y)$. By these reasoning the first proposition is obvious.

Suppose that $T(x) < x$ for some x. The T-property of A implies that $T(x) \vee c \neq \mathbb{1}$ and $x \vee c = \mathbb{1}$ for some c. By the normality of A, there exist u and v such that $x \vee v = c \vee u = \mathbb{1}$ and $u \wedge v = \mathbb{0}$. So, $u \in U_x$ and $u \leq T(x)$. Hence, $c \vee T(x) = \mathbb{1}$, which is a contradiction.

THEOREM 4.2. Let A be a normal cHa. Then, A^* is a normal cHa with the T-property.

PROOF. Let x and y be elements of A^* such that $x\vee^* y=\mathbb{1}$. Then, $x\vee y=\mathbb{1}$ holds in A. By the normality of A, there are u and v such that $x\vee u=y\vee v=\mathbb{1}$ and $u\wedge v=\mathbb{0}$. By Lemma 4.3, $T(x)\vee T(u)=T(y)\vee T(v)=\mathbb{1}$ and $T(u)\wedge T(v)=\mathbb{0}$. Hence, A^* is normal and so A^* is a normal cHa with the T-property by Lemma 4.4, 4.5 and 4.6.

LEMMA 4.7. Let A be a normal cHa. Then, $e=e\cdot T$.

PROOF. Clearly, $e\cdot T(x)\subseteq e(x)$. Let $I\in e(x)$, then $x\vee y=\mathbb{1}$ for some $y\in I$. By the normality, $x\vee w_0=y\vee w_1=\mathbb{1}$ and $w_0\wedge w_1=\mathbb{0}$ for some w_0 and w_1. Since $w_1\in U_x$, $w_1\leq T(x)$ and so $y\vee T(x)=\mathbb{1}$. Hence, $I\in e\cdot T(x)$.

LEMMA 4.8. Let A be a normal cHa. Then, $\mathfrak{m}A$ and $\mathfrak{m}A^*$ are homeomorphic to each other.

PROOF. For $I\in\mathfrak{m}A$, let $I^*=I\cap A^*$. Clearly, I^* is an ideal. Suppose that $x\notin I^*$ and $x\in A^*$. Then, $x\notin I$. So, $x\vee y=\mathbb{1}$ for some $y\in I$. Since $T(y)\leq y$, $T(y)\in I^*$. By Lemma 4.3, $x\vee T(y)=\mathbb{1}$ in A^*. Hence, $I^*\in\mathfrak{m}A^*$.

Suppose that $I^*=J^*$. By Lemma 4.7, $a\in I\leftrightarrow T(a)\in I\leftrightarrow T(a)\in I^*\leftrightarrow T(a)\in J^*\leftrightarrow T(a)\in J\leftrightarrow a\in J$. So, $I=J$.

Any ideal of A^* can be extended to a maximal ideal of A and so the above correspondence is injective and surjective.

By Lemma 4.7, $I\in e(x)\leftrightarrow I\in e\cdot T(x)\leftrightarrow I^*\in e^*\cdot T(x)$.

COROLLARY 4.4. Let A be a normal compact cHa. Then, $O(\mathfrak{m}A)$ is isomorphic to A^*.

PROOF. The compactness of A implies that of A^*. By Th. 4.2, Lemma 4.8 and Cor. 4.3, the corollary is clear.

LEMMA 4.9. Let A be a normal compact cHa and L be a sublattice of A which completely generates A and contains $\mathbb{0}$ and $\mathbb{1}$. And let ϕ be a $\mathbb{0},\mathbb{1}$-morphism from L to L', where L' is a BDL.

For $I\in\mathfrak{m}L$, there exists a unique maximal ideal of A that includes $I^\phi=\{x;\phi(x)\in I\}$.

PROOF. Since $\mathbb{1}\notin I^\phi$, there is a maximal ideal which includes I^ϕ. Suppose that there exist such different ideals, then there are different co-atoms a and b such that $\bigvee I^{\phi}\leq a\wedge b$. Then, $a\vee b=\mathbb{1}$. By the normality of A, $a\vee u=b\vee v=(u\Rightarrow\mathbb{0})\vee(v\Rightarrow\mathbb{0})=\mathbb{1}$ for some u and v. Since L completely generates A and A is compact, there are elements u_0, u_1, v_0 and v_1 of L such that $a\vee u_0=b\vee v_0=u_1\vee v_1=\mathbb{1}$ and $u_0\wedge u_1=v_0\wedge v_1=\mathbb{0}$. Then, u_0 and v_0 do not belong to I^{ϕ}. So, $\phi(u_1)$ and $\phi(v_1)$ are elements of I. But, $\mathbb{1}=\phi(u_1\vee v_1)=\phi(u_1)\vee\phi(v_1)\notin I$, which is absurd.

LEMMA 4.10. For $I\in\mathfrak{m}L'$, let $f(I)$ be the maximal ideal of A which is determined by Lemma 4.9. Then, $f:\mathfrak{m}L'\to\mathfrak{m}A$ is a continuous function.

In addition if $\phi(x)=\mathbb{0}$ implies $x=\mathbb{0}$ in the condition of Lemma 4.9, f is surjective.

PROOF. Let I be an element of $f^{-1}e(x)$. Then, $\bigvee f(I)\vee x=\mathbb{1}$ and so $\bigvee f(I)\vee u=x\vee v=\mathbb{1}$ and $u\wedge v=\mathbb{0}$ for some u and v in L. So, $I\in V_{\phi(u)}$. Let J be an element of $V_{\phi(u)}$. Then, $\phi(v)\in J$. So, $v\in J^{\phi}$ and hence $f(J)\in e(x)$. So, f is continuous.

Let a be a co-atom of A and I be the subset of L' defined by the following: $x\in I\leftrightarrow u\vee a=\mathbb{1}$ and $u\wedge v=\mathbb{0}$ and $x\leq\phi(v)$ for some $v\in L$ and some u. Then, I is an ideal of L' and $\mathbb{1}\notin I$ by the condition. Let $\bar{I}$ be the maximal ideal that contains I. Suppose that $a\neq\bigvee f(\bar{I})$. Then, $a\vee\bigvee f(\bar{I})=\mathbb{1}$. By the property of A and L, $a\vee u=\bigvee f(\bar{I})\vee v=\mathbb{1}$ and $u\wedge v=\mathbb{0}$ for some u and v which belong to L. By the definition of I, $\phi(v)\in I$ and so $v\in f(\bar{I})$, which contradicts to the fact $\bigvee f(\bar{I})\vee v=\mathbb{1}$. Hence, f is surjective.

Let C be the conjunction of the following conditions:

1) A is a normal *cHa* and is completely generated by a sublattice L with $\mathbb{0}$ and $\mathbb{1}$.
2) A' is a *cHa* and is completely generated by a sublattice L'.
3) ϕ is a *cH*-morphism from A to A' and $\phi''L\subseteq L'$.
4) p is a *cH*-morphism which makes the right diagram commutative.

$$\begin{tikzcd} & O(\mathfrak{m}L') \arrow[d, "p"] \\ L' \arrow[ur, "e_{L'}"] \arrow[r, hook] & A' \end{tikzcd}$$

THEOREM 4.3. Under the condition C, there exists a unique continuous function $f:\mathfrak{m}L\to\mathfrak{m}A$ such that the left diagram commutes. In addition if $\phi(x)=\mathbb{0}$ implies $x=\mathbb{0}$, then f is surjective.

$$\begin{tikzcd} O(\mathfrak{m}L') \arrow[d, "p"'] & O(\mathfrak{m}A) \arrow[l, "f^{-1}"'] \\ A' & A^* \arrow[l, "\phi\restriction A^*"] \arrow[u, "e\restriction A^*"'] \end{tikzcd}$$

PROOF. Let f be the function defined in Lemma 4.10. Then, $f^{-1}e(x)=\bigcup_{u\in W_x} e_{L'}(\phi(u))$, where $u\in W_x \leftrightarrow u\in L$ and $u\wedge v=\mathbb{0}$ and $x\wedge v=\mathbb{1}$ for some v. On the otherhand, $x=\bigvee W_x$ for $x\in A^*$ by the compactness. So, $p\cdot f^{-1}e(x)=\bigvee_{u\in W_x}\phi(u)=\phi(x)$.

Suppose that a continuous function $g: \mathfrak{m}L'\to\mathfrak{m}A$ satisfies the diagram in the theorem and $g\neq f$. Then, $f(I)\neq g(I)$ for some $I\in\mathfrak{m}L'$. So, there exist u and u' in L such that $\bigvee f(I)\vee u=\bigvee g(I)\vee u'=\mathbb{1}$ and $u\wedge u'=\mathbb{0}$. Since $I\in f^{-1}e(u)\cap g^{-1}e(u')$, there is v in L' such that $I\in e_{L'}(v)\subseteq f^{-1}e(u)\cap g^{-1}e(u')$. Hence, $\mathbb{0}\neq v=p\cdot e_{L'}(v)\leq p\cdot f^{-1}e(u)\wedge p\cdot g^{-1}e(u')=\phi(u\wedge u')=\mathbb{0}$, which is a contradiction.

Since $\mathfrak{m}A$ is a Hausdorff space and $O(\mathfrak{m}A)$ is isomorphic to A^*, the uniqueness of the continuous function f has the same meaning of the uniqueness of the cH-morphism from A^* to $O(\mathfrak{m}L')$.

§5. Completions and co-products

In Def. 4.2, we have defined properties of BDL's. In this section we shall study about the preservation of such properties under the operations defined already.

THEOREM 5.1. Let A and B be BDL's. $A\otimes B$ is compact if and only if A and B are compact.

PROOF. Let I be a maximal ideal in $A\otimes B$ and $I_A=\{x; i_A(x)\in I\}$ and $I_B=\{y; i_B(y)\in I\}$. Then, I_A and I_B are maximal ideals. So, there are co-atoms a and b such that $a=\bigvee I_A$ and $b=\bigvee I_B$. Then, $i_A(a)\vee i_B(b)$ is a co-atom and belongs to I.

The compactness is not preserved under an infinite co-product. Let A_n be a compact BDL, c_n be a co-atom in A_n and i_n be the embedding: $A_n\to\bigotimes_{n<\omega} A_n$, for each $n<\omega$. Let I be the subset of $\bigotimes_{n<\omega} A_n$ such that $x\in I$ if and only if $x\leq\bigvee_{k<n} i_k(c_k)$ for some n. Then, I is an ideal and does not contain $\mathbb{1}$, but $\bigvee I=\mathbb{1}$ holds.

Differing from the compactness, the T-property and the normality are preserved under infinite co-products.

THEOREM 5.2. Let L_α be a BDL for each $\alpha\in A$. $\bigotimes_{\alpha\in A} L_\alpha$ has the T-property, if and only if L_α has the T-property for each $\alpha\in A$. $\bigotimes_{\alpha\in A} L_\alpha$ is normal, if and only if L_α is normal for each $\alpha\in A$.

PROOF. Let a and b be elements of $\bigotimes_{\alpha\in A} L_\alpha$ and $a\not\leq b$. By Lemma 3.1, $a=\bigvee_{j<m}\bigwedge_{\alpha\in F_j} i_\alpha(a_\alpha^j)$ and $b=\bigwedge_{k<n}\bigvee_{\alpha\in G_k} i_\alpha(b_\alpha^k)$ for some a_α^j, b_α^k, F_j and G_j. Then, $\bigwedge_{\alpha\in F_j} i_\alpha(a_\alpha^j)\not\leq\bigvee_{\alpha\in G_k} i_\alpha(b_\alpha^k)$

for some j and k. We now define c_α for $\alpha \in F_j$. If $\alpha \in G_k$, then $a_\alpha^j \nleq b_\alpha^k$. Let c_α be the element such that $a_\alpha^j \vee c_\alpha = \mathbb{1}$ and $b_\alpha^k \vee c_\alpha \neq \mathbb{1}$ for $\alpha \in G_k$. And if $\alpha \in G_k$, $a_\alpha^j \neq \mathbb{1}$. Let c_α be the element such that $a_\alpha^j \vee c_\alpha = \mathbb{1}$ and $c_\alpha \neq \mathbb{1}$ for $\alpha \in G_k$. Let $c = \bigvee_{\alpha \in F_j} i_\alpha(c_\alpha)$. Then, $a \vee c = \mathbb{1}$ and $b \vee c \neq \mathbb{1}$.

Let $c = \bigvee_{\alpha \in F_j} i_\alpha(c_\alpha)$. Then, $a \vee c = \mathbb{1}$ and $b \vee c \neq \mathbb{1}$.

Let a and b be elements of $\bigotimes_{\alpha \in A} L_\alpha$ and $a \vee b = \mathbb{1}$. By Lemma 3.1, $a = \bigwedge_{j<m} \bigvee_{\alpha \in F_j} i_\alpha(a_\alpha^j)$ and $b = \bigvee_{k<n} \bigvee_{\alpha \in G_k} i_\alpha(b_\alpha^k)$ for some a_α^j, b_α^k, F_j and G_k. Without any loss of generality, we can assume $F_j = G_k = F$ for $j<m$ and $k<n$. Then, $a_\alpha^j \vee b_\alpha^k = \mathbb{1}$ for some $\alpha \in F$, for each $j<m$ and $k<n$. By the normality of L_α for each $\alpha \in A$, there exist u_{jk} and v_{jk} such that $\bigvee_{\alpha \in F} i_\alpha(a_\alpha^j) \vee u_{jk} = \mathbb{1}$ and $\bigvee_{\alpha \in F} i_\alpha(b_\alpha^k) \vee v_{jk} = \mathbb{1}$ and $u_{jk} \vee v_{jk} = \mathbb{0}$ for $j<m$ and $k<n$. Let $u^* = \bigvee_{j<m} \bigwedge_{k<n} u_{jk}$ and $v^* = \bigvee_{k<n} \bigwedge_{j<m} v_{jk}$. Then, $a \vee u^* = b \vee v^* = \mathbb{1}$ and $u^* \wedge v^* = \mathbb{0}$.

On the other hand, the T-property and the normality of L_α can be deduced from those of $\bigotimes_{\alpha \in A} L_\alpha$ respectively.

THEOREM 5.3. *Let A be an Ha. $\bar{A}$ has the T-property, if and only if A has the T-property.*

PROOF. Let i be the embedding: $A \to \bar{A}$. For any $a, b \in \bar{A}$ such that $a \nleq b$, there exists $a_0 \in A$ such that $i(a_0) \leq a$ and $i(a_0) \nleq b$. Let X be the subset of A defined by: $x \in X \leftrightarrow x \leq a_0$ and $i(x) \leq b$. Then, $\bigvee i''X = i(a_0) \wedge b < i(a_0)$. So, $\bigvee X = a_0$ does not hold. Since $x \leq a_0$ for each $x \in X$, there exists a_1 such that $x \leq a_1$ for each $x \in X$ and $a_0 \nleq a_1$. By the T-property, there exists c such that $a_0 \vee c = \mathbb{1}$ and $a_1 \vee c \neq \mathbb{1}$. Then, $a \vee i(c) = \mathbb{1}$. Suppose that $b \vee i(c) = \mathbb{1}$. Then, $(i(a_0) \wedge b) \vee i(c) = \mathbb{1}$ and so $\bigvee i''X \vee i(c) = \mathbb{1}$. Hence, $i(a_1 \vee c) = i(a_1) \vee i(c) = \mathbb{1}$, which is a contradiction. So, $b \vee i(c) \neq \mathbb{1}$.

Let a and b be elements of A such that $a \nleq b$. Then, there exists c in $\bar{A}$ such that $i(a) \vee c = \mathbb{1}$ and $i(b) \vee c \neq \mathbb{1}$. Since $c = \bigvee \{i(x); x \in A \text{ and } i(x) \leq c\}$ and $c \neq \mathbb{1}$, there exists c_0 in A such that $c \leq i(c_0)$ and $c_0 \neq \mathbb{1}$. Then, $a \vee c_0 = \mathbb{1}$ and $b \vee c_0 \neq \mathbb{1}$.

THEOREM 5.4. *Let L and L' be a normal BDL and a compact complete BDL respectively. If there exists an $\mathbb{0}, \mathbb{1}$-morphism $j: L \to L'$ such that $j''L$ completely generates L' and $j(x) = \mathbb{1}$ implies $x = \mathbb{1}$, then L' is normal.*

PROOF. Suppose that $x \vee y = \mathbb{1}$ for $x, y \in L'$. Then, by the compactness and the property of j, there exist a and b such that $j(a) \leq x$ and $j(b) \leq y$ and $a \vee b = \mathbb{1}$. By the normality, there exist u and v such that $a \vee u = b \vee v = \mathbb{1}$ and $u \wedge v = \mathbb{0}$. Then, $x \vee j(u) = y \vee j(v) = \mathbb{1}$ and $j(u) \wedge j(v) = \mathbb{0}$.

COROLLARY 5.1. *If L is a normal BDL, then $O(\mathfrak{m}L)$ is normal.*

PROOF. The morphism $e: L \to O(\mathfrak{m}L)$ satisfies the condition of the theorem.

COROLLARY 5.2. If A is a normal Ha then $\bar{A}$ is normal and compact.

PROOF. Since $\bar{A}$ is compact and the inclusion map: $A \to \bar{A}$ satisfies the condition of Th. 5.4, the corollary holds.

The normality of A does not always imply that of $\bar{A}$. We shall see such an example later.[i)]

THEOREM 5.5. $\overline{\otimes A_\alpha}$ is isomorphic to $\overline{\otimes \bar{A}_\alpha}$, where A_α is an Ha for each $\alpha \in \Lambda$.

PROOF. Let i_α and j_α be the embeddings defined in Def. 3.1 as indicated in the following diagram. Then, there exists a $\mathbb{0}, \mathbb{1}$-morphism ϕ that makes the following diagram commutative, by Th. 3.1.

$$\begin{tikzcd} & A_\alpha \arrow[r, hook] \arrow[dl, "i_\alpha"'] & \bar{A}_\alpha \arrow[dr, "j_\alpha"] & & \\ \bigotimes_{\alpha\in\Lambda} A_\alpha \arrow[rrr, dashed, "\phi"'] & & & \bigotimes_{\alpha\in\Lambda} \bar{A}_\alpha \arrow[r, hook] & \overline{\bigotimes_{\alpha\in\Lambda} \bar{A}_\alpha} \end{tikzcd}$$

By Th. 3.2, $\bigotimes_{\alpha\in\Lambda} A_\alpha$ is an Ha and so in the scope of Cor. 2.1 what we must prove is that ϕ is injective and complete and the range of ϕ completely generates $\overline{\bigotimes_{\alpha\in\Lambda} \bar{A}_\alpha}$. In this case, ϕ is clearly injective and the range of it completely generates $\overline{\bigotimes_{\alpha\in\Lambda} \bar{A}_\alpha}$. Suppose that ϕ is not complete. Then, there exist $a^\alpha, a^\alpha_\lambda$, finite F and finite F_λ such that $\bigwedge_{\alpha\in F} i_\alpha(a^\alpha) = \bigvee_{\lambda\in\Lambda} \bigwedge_{\alpha\in F_\lambda} i_\alpha(a^\alpha_\lambda)$ but $\bigwedge_{\alpha\in F} j_\alpha(a^\alpha) \nleq \bigvee_{\lambda\in\Lambda} \bigwedge_{\alpha\in F_\lambda} j_\alpha(a^\alpha_\lambda)$. So, there exist $\bar{b}^\alpha$ in $\bar{A}_\alpha$ and finite G such that $\bigwedge_{\alpha\in F_\lambda} j_\alpha(a^\alpha) \leq \bigvee_{\alpha\in G} j_\alpha(\bar{b}^\alpha)$ for each $\lambda\in\Lambda$ but $\bigwedge_{\alpha\in F} j_\alpha(a^\alpha) \nleq \bigvee_{\alpha\in G} j_\alpha(\bar{b}^\alpha)$. Here, $a^\alpha \neq \mathbb{0}$ for $\alpha\in F$ and $\alpha\notin G$, $a^\alpha \nleq \bar{b}^\alpha$ for $\alpha\in F$ and $\alpha\in G$ and $\bar{b}^\alpha \neq \mathbb{1}$ for $\alpha\notin F$ and $\alpha\in G$. Let c^α be the element of A_α for each $\alpha\in G$ such that $a^\alpha \wedge \bar{b}^\alpha \leq c^\alpha \leq a^\alpha$ for $\alpha\in F$ and $\bar{b}^\alpha \leq c^\alpha < \mathbb{1}$ for $\alpha\notin F$. Then, $\bigwedge_{\alpha\in F_\lambda} i_\alpha(a^\alpha_\lambda) \leq \bigvee_{\alpha\in G} i_\alpha(c^\alpha)$ for each $\lambda\in\Lambda$ but $\bigwedge_{\alpha\in F} i_\alpha(a^\alpha) \nleq \bigvee_{\alpha\in G} i_\alpha(c^\alpha)$, which is absurd. So, ϕ is complete.

In the rest of this section, we shall investigate about open algebras.

THEOREM 5.6. Let X and Y be topological spaces. Then, $O(X\times Y)$ is isomorphic to $\overline{O(X)\otimes O(Y)}$.

i) After the completion of this paper, the author has found some results of C. H. Dowker, D. Strauss and H. Simmons in [1], [2] and [11]. He has noticed that many separation axioms there are preserved under the canonical completion and the co-products.

PROOF. Let $j_X : O(X) \to O(X\times Y)$ and $j_Y : O(Y) \to O(X\times Y)$ be the cH-morphisms induced by the projections. Then, by Th. 3.1, there exists a unique $\mathbb{0}, \mathbb{1}$-morphism ϕ that makes the left diagram commutative, where i_X and i_Y are the embeddings defined in Def. 3.1.

$$\begin{tikzcd} & O(X) \arrow[dl, "i_X"'] \arrow[dr, "j_X"] & \\ O(X)\otimes O(Y) \arrow[rr, dashed, "\phi"] & & O(X\times Y) \\ & O(Y) \arrow[ul, "i_Y"] \arrow[ur, "j_Y"'] & \end{tikzcd}$$

By Th. 3.2, $O(X)\otimes O(Y)$ is an Ha. So, what we must prove is that ϕ is injective and ϕ preserves infinite sums and the range of ϕ completely generates $O(X\times Y)$.

Suppose that $j_X(u)\wedge j_Y(v)\leq j_X(u')\vee j_Y(v')$. That is $u\times v\subseteq u'\times Y\cup X\times v'$. So, $u\subseteq u'$ or $v\subseteq v'$. Hence, ϕ is injective.

To show the preservation of infinite sums, it is enough to treat the case $i_X(u)\wedge i_Y(v)=\bigvee_{\alpha\in A} i_X(u_\alpha)\wedge i_Y(v_\alpha)$. Suppose that $j_X(u)\cap j_Y(v)\supsetneqq\bigcup_{\alpha\in A} j_X(u_\alpha)\cap j_Y(v_\alpha)$. Then, there exist x and y such that $(x,y)\in u\times v$ and $(x,y)\notin u_\alpha\times v_\alpha$ for any $\alpha\in A$. $i_X(u_\alpha)\wedge i_Y(v_\alpha)\leq i_X(X-\overline{\{x\}})\vee i_Y(Y-\overline{\{y\}})$ for each $\alpha\in A$, but $i_X(u)\wedge i_Y(v)\nleq i_X(X-\overline{\{x\}})\vee i_Y(Y-\overline{\{y\}})$. These contradict to the fact $i_X(u)\wedge i_Y(v)=\bigvee_{\alpha\in A} i_X(u_\alpha)\wedge i_Y(v_\alpha)$.

The range of ϕ forms a base of $X\times Y$ and so completely generates $O(X\times Y)$.

LEMMA 5.1. $\mathfrak{m}\bigotimes_{\alpha\in A} L_\alpha$ is homeomorphic to $\prod_{\alpha\in A}\mathfrak{m}L_\alpha$.

PROOF. We use the same notation as in the proof of Lemma 3.2. By Lemma 3.2, it is sufficient to prove that $\bigotimes_{\alpha\in A} I_\alpha$ belongs to $\mathfrak{m}\bigotimes_{\alpha\in A} L_\alpha$ for maximal ideals I_α's ($\alpha\in A$) and $(I)_\alpha$ is maximal for a maximal ideal I for each $\alpha\in A$. Suppose that $\bigvee_{j<m}\bigwedge_{\alpha\in F_j} i_\alpha(a_j^\alpha)\notin\bigotimes_{\alpha\in A} I_\alpha$. Then, $\bigwedge_{\alpha\in F_j} i_\alpha(a_j^\alpha)\notin\bigotimes_{\alpha\in A} I_\alpha$ for some $j<m$. So, $a_j^\alpha\in I_\alpha$ for each $\alpha\in F_j$. These imply that $\bigotimes_{\alpha\in A} I_\alpha$ is maximal. The other implication is obvious.

COROLLARY 5.3. $O(\mathfrak{p}(L\otimes L'))$ is isomorphic to $\overline{O(\mathfrak{p}L)\otimes O(\mathfrak{p}L')}$. And $O(\mathfrak{m}(L\otimes L'))$ is isomorphic to $\overline{O(\mathfrak{m}L)\otimes O(\mathfrak{m}L')}$.

PROOF. It is clear by Lemma 3.2, Lemma 5.1 and Th. 5.6.

We have proved that the canonical completion of a finite co-product of open algebras is an open algebra. However, in the case of an infinite co-product, that does not hold.

Let X_n be a discrete space of two elements for each n. Then, $O(X_n)$ is a Boolean algebra and consequently $\bigotimes_{n<\omega} O(X_n)$ is a Boolean algebra and is co-atomless. So, $\overline{\bigotimes_{n<\omega} O(X_n)}$ is a complete Boolean algebra and atomless. This is the regular open algebra of the Cantor space. It is well-known that an atomless complete Boolean algebra cannot be an open algebra. (Remind that there exist no cH-morphism from an atomless complete Boolean algebra to $\{\mathbb{0}, \mathbb{1}\}$.) So, in most cases, $\overline{\bigotimes_{\alpha\in A} O(X_\alpha)}$ is not an open algebra for an infinite A. It is contrasted with the case of regular open algebras. For, $\overline{\bigotimes_{\alpha\in A} RO(X_\alpha)}$ is isomorphic to $RO(\prod_{\alpha\in A} X_\alpha)$. (cf. [3])

As stated before, the normality of an Ha A does not imply that of $\bar{A}$. Let X be a normal space such that $X\times X$ is not normal. Then, $O(X)\otimes O(X)$ is normal by Th. 5.2. On the other hand, $\overline{O(X)\otimes O(X)}$ is isomorphic to $O(X\times X)$ and so is not normal.

§ 6. Complete Heyting algebras in a Heyting extension

In this section, we shall study about a completion of a co-product and a completion in a Heyting extension. So, we assume that the readers are familiar with an extension of a universe of the set theory with a cHa. (cf. Grayson [7] and Takeuti [13])

We shall use the notation $[\![\Phi]\!]^{(H)}$ or $[\![\Phi]\!]$ for the value of Φ in a cHa H. As in [12], we assume that $V^{(H)}$ is separated, i.e. "$x=y$" is equivalent to "$[\![x=y]\!]=\mathbb{1}$". Just as in a Boolean extension, $\check{x}$ is the element of $V^{(H)}$ such that dom $\check{x}=\{\check{y}\,;\,y\in x\}$ and range $\check{x}\subseteq\{\mathbb{1}^H\}$. $\hat{x}$ is the set $\{y\,;\,[\![y\in x]\!]=\mathbb{1}\}$. We say "$\Phi$ is H-valid", if $[\![\Phi]\!]^{(H)}=\mathbb{1}$.

As indicated in [13], the maximal principle does not always hold in $V^{(H)}$, but a weak form of it holds. Next three lemmas can be proved as in the case of a Boolean extension and the proofs can be seen in [13]. So, we omit them.

LEMMA 6.1. If $[\![\exists!\,x\Phi(x)]\!]^{(H)}=\mathbb{1}$, then there exists u in $V^{(H)}$ such that $[\![\Phi(u)]\!]^{(H)}=\mathbb{1}$.

LEMMA 6.2. Let $\Phi(x_0,\cdots,x_n)$ be a Δ_0-formula. Then, $[\![\Phi(\check{x}_0,\cdots,\check{x}_n)]\!]=\mathbb{1}$ if and only if $\Phi(x_0,\cdots,x_n)$ holds and $[\![\Phi(x_0,\cdots,x_n)]\!]=\mathbb{0}$ if and only if $\Phi(x_0,\cdots,x_n)$ does not hold.

LEMMA 6.3. If "Ω is a cHa" is H-valid, $\hat{\Omega}$ is a cHa.

THEOREM 6.1. Let H and Ω be a cHa and an Ha respectively. And let $\bar{\check{\Omega}}^H$ be the canonical completion of $\check{\Omega}$ in $V^{(H)}$.

Then, $\hat{\bar{\check{\Omega}}}^H$ is isomorphic to $\overline{R(H)\otimes\Omega}$.

PROOF. Let $\Phi(I)$ be the formula that asserts "I is a closed ideal in $\check{\Omega}$", i.e.

$\forall x\in\check{\Omega}\forall y\in\check{\Omega}(x\in I$ and $y\leq x\to y\in I)$ and $\forall x\in\check{\Omega}(\forall u\in\Omega(\forall y\in\check{\Omega}(y\in I$ and $y\leq x\to y\leq u)$ $\to x\leq u)\to x\in I)$.

Suppose that $\Phi(I)$ is H-valid. Then, we can assume that dom $I=$dom $\check{\Omega}$ and $x\leq y$ implies $I(\check{y})\leq I(\check{x})$ and $I(\check{x})\wedge I(\check{y})=I(\check{x}\vee\check{y})$ for each $x,y\in\Omega$. For each $x\in\Omega$, $\bigwedge_{u\in\Omega}\bigwedge_{y\in\Omega}(I(\check{y})\wedge[\![\check{y}\leq\check{x}]\!]\Rightarrow[\![\check{y}\leq\check{u}]\!])\Rightarrow[\![\check{x}\leq\check{u}]\!])\leq I(\check{x})$. So, $\bigwedge_{u<x}(\bigwedge_{\substack{y\leq x\\ y\nleq u}}(I(\check{y})\Rightarrow\mathbb{0})\Rightarrow\mathbb{0})\leq I(\check{x})$. The left part of this inequality is equal to or greater than $(I(\check{x})\Rightarrow\mathbb{0})\Rightarrow\mathbb{0}$. Hence, $I(\check{x})$ is a regular element of H. So, we can assume that an element of $\overset{\wedge H}{\check{\Omega}}$ is a function which maps dom $\check{\Omega}$ into $R(H)$.[i]

We now define $j_{R(H)}(p)$ and $j_{\Omega}(q)$ for $p\in R(H)$ and $q\in\Omega$ as follows.

The domains of $j_{R(H)}(p)$ and $j_{\Omega}(q)$ are both dom $\check{\Omega}$.

$$
\begin{aligned}
j_{R(H)}(p)(\check{x}) &= \mathbb{1} \quad \text{for } x=\mathbb{0},\\
&= p \quad \text{otherwise.}\\
j_{\Omega}(q)(\check{x}) &= \mathbb{1} \quad \text{for } x\leq q,\\
&= \mathbb{0} \quad \text{otherwise.}
\end{aligned}
$$

Then, $\Phi(j_{R(H)}(p))$ and $\Phi(j_{\Omega}(q))$ are H-valid for each $p\in R(H)$ and each $q\in\Omega$.

A straight calculation shows the following:

$$
\begin{aligned}
j_{R(H)}(p)\wedge j_{\Omega}(q)(\check{x}) &= \mathbb{1} \quad \text{for } x=\mathbb{0},\\
&= p \quad \text{for } x\leq q \text{ and } x\neq\mathbb{0},\\
&= \mathbb{0} \quad \text{otherwise.}
\end{aligned}
$$

$$
\begin{tikzcd}
 & R(H) \arrow[dl, "i_{R(H)}"'] \arrow[dr, "j_{R(H)}"] & \\
R(H) \arrow[rr, dashed, "\phi"] & & \overset{\wedge H}{\check{\Omega}} \\
 & \Omega \arrow[ul, "i_{\Omega}"] \arrow[ur, "j_{\Omega}"'] &
\end{tikzcd}
$$

$j_{R(H)}$ and j_{Ω} are $\mathbb{0},\mathbb{1}$-morphisms. So, by Th. 3.1, there exists a unique $\mathbb{0},\mathbb{1}$-morphism ϕ such that the left diagram commutes.

Suppose that $j_{R(H)}(p)\wedge j_{\Omega}(q)\leq j_{R(H)}(p')\vee j_{\Omega}(q')$ and $p\nleq p'$. Then, there exists $r\in R(H)$ such that $\mathbb{0}\neq r\leq[\![\check{q}\in j_{\Omega}(q')]\!]$. So, $q\leq q'$ holds by the definition of j_{Ω} and hence ϕ is injective.

Let $\bigvee_{(p,q)\in A} i_{R(H)}(p)\wedge i_{\Omega}(q)=i_{R(H)}(p_0)\wedge i_{\Omega}(q_0)$. And let A be the subset of $\check{\Omega}$ in $V^{(H)}$ defined by the following. $A(\check{x})=\bigvee^{R(H)}\{p\,;\,\exists q\,(x\leq q$ and $(p,q)\in A)\}$ for $x\neq\mathbb{0}$ and $A(\check{\mathbb{0}})=\mathbb{1}$. Let I_0 be $j_{R(H)}(p_0)\wedge j_{\Omega}(q_0)$. If $[\![\Phi(I)]\!]=\mathbb{1}$ and $[\![i_{R(H)}(p)\wedge i_{\Omega}(q)\subseteq I]\!]=\mathbb{1}$ for each $(p,q)\in A$, then $[\![A\subseteq I]\!]=\mathbb{1}$. By Lemma 2.2, the unique minimal closed ideal $\bar{I}$ that contains A exists. By Lemma 6.1, we can assume that $\bar{I}$ is an element of $\overset{\wedge H}{\check{\Omega}}$.

i) Dr. Hayashi has pointed out that after this point we may work in $V^{(R(H))}$, because $\overset{\wedge H}{\check{\Omega}}\cong\overset{\wedge R(H)}{\check{\Omega}}$. However, regular elements have an important role for the calculation in $V^{(H)}$ and here is an example. So, we present the original proof.

Clearly $[\![\bar{I} \subseteq I_0]\!] = \mathbb{1}$.

Since $[\![\forall x \in \check{\Omega} \forall y \in A(x \leq y \rightarrow x \in A)]\!] = \mathbb{1}$, $[\![x \in \bar{I}]\!] = [\![x \in \check{\Omega}$ and $x = \bigvee\{y\,;\, y \in \check{\Omega}$ and $y \leq x$ and $y \in A\}]\!]$ by Lemma 2.2. We claim that I_0 is equal to $\bar{I}$. Suppose that $p_0 \leq [\![\check{q}_0 \in \bar{I}]\!]$ does not hold. Then, since $[\![\check{q}_0 \in \bar{I}]\!]$ is a regular element, there exists $p' \in R(H)$ such that $\mathbb{0} \neq p' \leq p_0$ and $p' \wedge [\![\check{q}_0 \in \bar{I}]\!] = \mathbb{0}$. $[\![\check{q}_0 \in \bar{I}]\!] = \bigwedge_{u<q_0} \bigwedge_{\substack{y \leq q_0 \\ y \nleq u}} (A(\check{y}) \Rightarrow \mathbb{0}) \Rightarrow \mathbb{0}) = (\bigvee_{u<q_0} \bigwedge_{\substack{y \leq q_0 \\ y \nleq u}} (A(\check{y}) \Rightarrow \mathbb{0})) \Rightarrow \mathbb{0}$. So, there exists $u < q_0$ such that $p'' = p' \wedge \bigwedge_{\substack{y \leq q_0 \\ y \nleq u}} (A(\check{y}) \Rightarrow \mathbb{0}) \neq \mathbb{0}$. Then, $p'' \in R(H)$ and $p'' \wedge A(\check{y}) = \mathbb{0}$ for any $y\,;\, y \leq q_0$ and $y \nleq u$. Since $i_{R(H)}$ is a cH-morphism,

$$\bigvee_{(p,q) \in A} i_{R(H)}(p) \wedge i_\Omega(q) = \bigvee_{q \in \Omega} i_{R(H)}(A(\check{q})) \wedge i_\Omega(q).$$

And so, $i_{R(H)}(p_0) \wedge i_\Omega(q_0) = \bigvee_{q \leq q_0} i_{R(H)}(A(\check{q}) \wedge i_\Omega(q))$.

Hence,
$$\begin{aligned} I_{R(H)}(p'') \wedge i_\Omega(q_0) &= \bigvee_{q \leq q_0} i_{R(H)}(A(\check{q}) \wedge p'') \wedge i_\Omega(q) \\ &= \bigvee_{q \leq u} i_{R(H)}(A(\check{q}) \wedge p'') \wedge i_\Omega(q) \\ &\leq i_{R(H)}(p'') \wedge i_\Omega(u), \end{aligned}$$

which contradicts to the fact that $u < q_0$. So, ϕ preserves infinite sums.

Let $I' = \bigvee\{j_{R(H)}(p) \wedge j_\Omega(q)\,;\, [\![j_{R(H)}(p) \wedge j_\Omega(q) \subseteq I]\!] = \mathbb{1}\}$ for $I \in \hat{\check{\Omega}}^{H}$. Since $j_{R(H)}(I(\check{q})) \wedge j_\Omega(q) \leq I$, $I(\check{q}) \leq [\![\check{q} \in I']\!]$ for $q \in \Omega$. So, $I = I'$.

COROLLARY 6.1. Let B be a complete Boolean algebra and Ω be an Ha. Then, $\hat{\check{\Omega}}^{B}$ is isomorphic to $\overline{B \otimes \Omega}$.

PROOF. It is clear from the theorem and the fact: $B = R(B)$.

For Boolean algebras B and C, $B \otimes C$ is isomorphic to the co-product in the category of Boolean algebras and the canonical completion of B as an Ha is isomorphic to the canonical completion as a Boolean algebra. So, Th. 6.1 is a generalization of the next result of Kunen and Scott.

COROLLARY 6.2. (Kunen and Scott [12]) Let B and C be complete Boolean algebras. Then, $\hat{\check{C}}^{B}$ is isomorphic to $\overline{B \otimes C}$.

THEOREM 6.2. Let L and L' be distributive lattices. Then, $\widehat{\mathfrak{I}\check{L}'}^{\mathfrak{I}L}$ is isomorphic to $\mathfrak{I}(L \otimes L')$, where $\mathfrak{I}\check{L}'^{\mathfrak{I}L}$ is in $V^{(\mathfrak{I}L)}$.

PROOF. By Lemma 1.1, "$\mathfrak{I}\check{L}'^{\mathfrak{I}L}$ is a cHa" is $\mathfrak{I}L$-valid. Let j_L and $j_{L'}$ be the following $\mathbb{0}, \mathbb{1}$-morphism:

$$j_L : L \to \widehat{\mathfrak{J}L'}^{\mathfrak{J}L} \text{ and } j_L(a)(\check{p}) = \mathbb{1} \quad \text{for } p = \mathbb{0},$$
$$= I_a \quad \text{otherwise,}$$

$$j_{L'} : L' \to \widehat{\mathfrak{J}L'}^{\mathfrak{J}L} \text{ and } j_{L'}(q)(\check{p}) = \mathbb{1} \quad \text{for } p \leq q$$
$$= \mathbb{0} \quad \text{otherwise.}$$

Then, $j_L(a) \wedge j_{L'}(q)\ (\check{p}) = \mathbb{1}$ for $p = \mathbb{0}$,

$$= I_a \quad \text{for } \mathbb{0} \neq p \leq q$$
$$= \mathbb{0} \quad \text{otherwise.}$$

for each $a \in L$ and $p, q \in L'$.

By Th. 3.1, there exists a unique $\mathbb{0}-\mathbb{1}$ morphism ϕ such that the following diagram (1) commutes. Then, there exists a unique cH-morphism ψ such that the following diagram (2) commutes, by Th. 1.2 and Cor. 1.2.

(1)
$$\begin{tikzcd} & L \arrow[dl, "i_L"'] \arrow[dr, "j_L"] & \\ L \otimes L' \arrow[rr, dashed, "\phi"] & & \widehat{\mathcal{J}L'}^{\mathcal{J}L} \\ & L' \arrow[ul, "i_{L'}"] \arrow[ur, "j_{L'}"'] & \end{tikzcd}$$

(2)
$$\begin{tikzcd} \mathcal{J}(L \otimes L') \arrow[dr, dashed, "\psi"] & \\ L \otimes L' \arrow[u, "i"] \arrow[r, "\phi"'] & \widehat{\mathcal{J}L'}^{\mathcal{J}L} \end{tikzcd}$$

Let $I \in \widehat{\mathfrak{J}L'}^{\mathfrak{J}L}$, then $a \in I(\check{p})$ implies $j_L(a) \wedge j_{L'}(p) \leq I$ for each $a \in L$ and $p \in L'$. So, $\bigvee\{j_L(a) \wedge j_{L'}(p) ; j_L(a) \wedge j_{L'}(p) \leq I\} = I$. This means that the range of ϕ completely generates $\widehat{\mathfrak{J}L'}^{\mathfrak{J}L}$. Hence, ψ is surjective by Th. 1.2.

Now, let X_p be the set; $\{a ; \exists F\ (F$ is a finite subset of Λ and $a \leq \bigwedge(F)_0$ and $p \leq \bigvee(F)_1)\}$, where $(F)_k$ is the set of the k-th co-ordinates of elements of F.

Claim) Let $J = \bigvee\{i(i_L(a) \wedge i_{L'}(p)) ; (a, p) \in \Lambda\}$ for $J \in \mathfrak{J}(L \otimes L')$. Then, $\psi(J)(\check{p}) = I(X_p)$.

Let $K(\check{p}) = I(X_p)$ for each $p \in L'$. Then, $K(\check{\mathbb{0}}) = L = \mathbb{1}$ and $p \leq q$ implies $K(\check{q}) \leq K(\check{p})$. Let $x \in K(\check{p}) \wedge K(\check{p}')$, then there exist $h_1, \cdots, h_m, h_1', \cdots$ and h_n' such that $x \leq \bigvee_{1 \leq i \leq m} h_i \wedge \bigvee_{1 \leq j \leq n} h_j' = \bigvee_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} h_i \wedge h_j'$ and $h_i \in X_p$ for $1 \leq i \leq m$ and $h_j' \in X_{p'}$, for $1 \leq j \leq n$. Since $h_i \wedge h_j' \in X_{p \vee p'}$, for each $1 \leq i \leq m$ and $1 \leq j \leq n$, $x \in K(\widecheck{p \vee p'})$. So, $[\![K \in \mathfrak{J}\check{L}']\!] = \mathbb{1}$. $[\![j_L(a) \wedge j_{L'}(p) \subseteq K]\!] = \mathbb{1}$ for each $(a, p) \in \Lambda$. So, "$\psi(J) \subseteq K$" is H-valid. On the other hand, $[\![K \subseteq \psi(J)]\!] = \mathbb{1}$. Hence, $\psi(J) = K$. Now, we have proved the claim.

Suppose that $\psi(I) = \psi(J)$ and $I \not\subseteq J$. Then, there exist a and p such that $i_L(a) \wedge i_{L'}(p) \in I$ but $i_L(a) \wedge i_{L'}(p) \notin J$. By the claim, a $\psi(I)(\check{p}) = \psi(J)(\check{p})$. Let $\Delta = \{(a, p) ; i_L(a) \wedge i_{L'}(p) \in J\}$, then there exist $a_1, \cdots, a_n$ and finite subsets $F_1, \cdots, F_n$

of Δ such that $a\le a_1\vee\cdots\vee a_n$, $a_i\le\bigwedge(F_i)_0$ and $p\le\bigvee(F_i)_1$ for $1\le i\le n$. Since J is an ideal, $(\bigwedge(F_i)_0, \bigvee(F_i)_1)\in\Delta$ for each $1\le i\le n$. So, $(a_i,p)\in\Delta$ for each $1\le i\le n$ and so $(\bigvee_{1\le i\le n} a_i, p)\in\Delta$. Hence, $(a,p)\in\Delta$, which is a contradiction. So, ϕ is injective.

COROLLARY 6.3. Let L and L' be distributive lattices. Then, $\widehat{\mathfrak{J}L'}^{\,\mathfrak{J}L}$ is isomorphic to $\overline{\mathfrak{J}L\otimes\mathfrak{J}L'}$.

PROOF. $\mathfrak{J}(L\otimes L')$ is isomorphic to $\overline{\mathfrak{J}L\otimes\mathfrak{J}L'}$, by Cor. 1.2, Lemma 3.2 and Th. 5.6. Now, the corollary is clear from the theorem.

The next lemma is an easy consequence of Lemma 6.2.

LEMMA 6.4. Let X be a topological space. Then, " $\check{X}$ is a topological space with a base $\check{O(X)}$ " is H-valid.

THEOREM 6.3. Let H be a cHa and X be a topological space. Then, $\widehat{O(\check{X})}$ is isomorphic to $\overline{H\otimes O(X)}$, where $O(\check{X})$ is an open algebra of the topological space $\check{X}$ with its base $\check{O(X)}$ in $V^{(H)}$.

PROOF. Let j_H and j_o be functions defined by the following: $j_H: H\to\widehat{O(\check{X})}$ and $j_H(h)(\check{x})=h$ for each $x\in X$, $j_o: O(X)\to\widehat{O(\check{X})}$ and $j_o(P)(\check{x})=\mathbb{1}$ for $x\in P$ $j_o(p)(\check{x})=\mathbb{0}$ otherwise. Then, j_H and j_o are $\mathbb{0},\mathbb{1}$-morphisms. So, there exists a $\mathbb{0},\mathbb{1}$-morphism ϕ such that the following diagram commutes, by Th. 3.1.

$$\begin{tikzcd} & H \arrow[dl, "i_H"'] \arrow[dr, "j_H"] & \\ H\otimes O(X) \arrow[rr, dashed, "\phi"'] & & \widehat{O(\check{X})} \\ & O(X) \arrow[ul, "i_0"] \arrow[ur, "j_0"'] & \end{tikzcd}$$

Let $U\in\widehat{O(\check{X})}$, then $[\![\check{x}\in U]\!]=[\![\exists P\in\check{O(X)}; \check{x}\in P \text{ and } P\subseteq U]\!]=\bigvee_{\substack{x\in P\\ P\in O(X)}}[\![\check{P}\subseteq U]\!]$ for $x\in X$. So, $U=\bigvee\{j_H(h)\wedge j_o(P); j_H(h)\wedge j_o(P)\le U\}$, i.e. the range of ϕ completely generates $\widehat{O(\check{X})}$.

Suppose that $j_H(h)\wedge j_o(P)\le j_H(h')\vee j_o(P')$ and $P\nsubseteq P'$ for some h, P, h' and P'. Then, there is x_0 in P that is not in P'. $j_H(h)\wedge j_o(P)(\check{x}_0)=h$ and $j_H(h')\vee j_o(P')(\check{x}_0)=h'$ and so $h\le h'$. Hence, ϕ is injective.

Let $i_H(h_0)\wedge i_o(P_0)=\bigvee\{i_H(h)\wedge i_o(P); (h,P)\in\Lambda\}$ for some Λ. Suppose that $j_H(h_0)\wedge j_o(P_0)\nleq\bigvee\{j_H(h)\wedge j_o(P); (h,P)\in\Lambda\}$. Then, there exists $x_0\in P_0$ such that $h_0\nleq h_0'$, where

$h_0'=\bigvee\{h\,;\,x_0\in P$ and $(h,P)\in A$ for some $P\}$. $x_0\notin P$ implies $\overline{\{x_0\}}\cap P=\phi$ and so implies $P\subseteq X-\overline{\{x_0\}}$. So, $i_H(h)\wedge i_0(P)\leq i_H(h_0')\vee i_0(X-\overline{\{x_0\}})$ for each $(h,P)\in A$. On the other hand, $i_H(h_0)\wedge i_0(P_0)\wedge(i_H(h_0')\vee i_0(X-\overline{\{x_0\}}))=(i_H(h_0')\wedge i(P_0))\vee(i_H(h_0)\wedge i_0(P_0-\overline{\{x_0\}}))<i_H(h_0)\wedge i_0(P_0)$, which is a contradiction.

COROLLARY 6.4. Let X and Y be topological spaces and $O(\check{X})$ be an open algebra of the topological space X with its base $\check{O(X)}$ in $V^{(O(Y))}$.

Then, $\widehat{O(\check{X})}$ is isomorphic to $O(X\times Y)$.

PROOF. By Th. 6.3, $\widehat{O(\check{X})}$ is isomorphic to $\overline{O(X)\otimes O(Y)}$ and so is isomorphic to $O(X\times Y)$.

In the preceding three theorems, we have investigated the structure of $\hat{\Omega}$ for some *cHa* Ω in $V^{(H)}$. By the theorem of Fourman and Scott [4], our result can be internalized into $V^{(H)}$ in some sense. For that, we shall introduce their results by a different presentation. In many cases we shall omit the proofs, since they are in [4] and essentially as same as in the case of Boolean extensions. [12]

LEMMA 6.5.* Let Ω be an *Ha* and F be a filter of it. Then, Ω/F is an *Ha*, where Ω/F is the quotient by the equivalence relation $\{(a,b)\,;\,a\Rightarrow b\in F$ and $b\Rightarrow a\in F\}$.

PROOF. Let $\pi:\Omega\to\Omega/F$ be the canonical quotient map. Then, π is a strong H-morphism and Ω/F is an *Ha*.

Let H and Ω be *cHa*'s. And let F_ε be the element of $V^{(H)}$ such that $\operatorname{dom}F_\varepsilon=\operatorname{dom}\check{\Omega}$ and $F_\varepsilon(\check{p})=\bigvee\{h\,;\,\varepsilon(h)\leq p\}$, where $\varepsilon:H\to\Omega$ is a cH-morphism. Then, "F_ε is a filter of $\check{\Omega}$" is H-valid. By Lemma 6.5, "$\check{\Omega}/F_\varepsilon$ is an *Ha*" is H-valid. We denote the canonical quotient map by π.

LEMMA 6.6. $[\![\pi(\check{p})\leq\pi(\check{q})]\!]=\bigvee\{h\,;\,\varepsilon(h)\leq p\Rightarrow q\}$ for $p,q\in\Omega$. Consequently,

a) $h\leq[\![\pi(\check{p})\leq\pi(\check{q})]\!]$ if and only if $\varepsilon(h)\leq p\Rightarrow q$,

b) $h\leq[\![\pi(\check{p})=\pi(\check{q})]\!]$ if and only if $\varepsilon(h)\wedge p=\varepsilon(h)\wedge q$.

PROOF. $[\![\pi(\check{p})\leq\pi(\check{q})]\!]=[\![\pi(\check{1})\leq\pi((p\Rightarrow q)\check{})]\!]=[\![(p\Rightarrow q)\check{}\in F_\varepsilon]\!]=\bigvee\{h\,;\,\varepsilon(h)\leq p\Rightarrow q\}$. By the completeness of ε, a) and b) are clear.

By Lemma 6.6, we understand that $\check{\Omega}/F_\varepsilon$ is the same thing defined in Th. 8.13 of [4].

THEOREM 6.4. "$\check{\Omega}/F_\varepsilon$ is a *cHa*" is H-valid and $\widehat{\check{\Omega}/F_\varepsilon}$ is isomorphic to Ω.

LEMMA 6.7. (Fourman and Scott [4]) Let "Ω is a cHa" is H-valid. Then, for any $x\in V^{(H)}$, there exists $x'\in\hat{\Omega}$ such that $[\![x=x']\!]=[\![x\in\Omega]\!]$.

This lemma implies that as far as concerning cHa's many things will go well like a Boolean extension.

Let "Ω is a cHa" be H-valid and e be the function such that $e: H\to\hat{\Omega}$ and "$e(h)=\bigvee^{\Omega}\{\mathbb{1}\,; h\}$" is H-valid. And let Ω' and e' be defined similarly.

LEMMA 6.8. (Fourman and Scott [4]) e is a cH-morphism.

THEOREM 6.5. (Fourman and Scott [4]) Let "$\phi: \Omega\to\Omega'$ is a cH-morphism" be H-valid. And let $\hat{\phi}: \hat{\Omega}\to\hat{\Omega}'$ be the function; $[\![\hat{\phi}(x)=\phi(x)]\!]=\mathbb{1}$ for $x\in\hat{\Omega}$. Then, $\hat{\phi}$ is a cH-morphism and $e\cdot\hat{\phi}=e'$. And conversely, let $\hat{\phi}: \hat{\Omega}\to\hat{\Omega}'$ be the cH-morphism that satisfies $e\cdot\hat{\phi}=e'$. Then, there exists ϕ in $V^{(H)}$ such that "$\phi: \Omega\to\Omega'$ is a cH-morphism" is H-valid and $[\![\phi(x)=\hat{\phi}(x)]\!]=\mathbb{1}$ for $x\in\hat{\Omega}$.

In the above, "ϕ is injective" is H-valid, if and only if $\hat{\phi}$ is injective. And "ϕ is surjective" is H-valid, if and only if $\hat{\phi}$ is surjective.

LEMMA 6.9. $h\leq[\![p\leq q]\!]\leftrightarrow e(h)\leq p\Rightarrow q$, for $p, q\in\hat{\Omega}$. So, if Ω is $\check{\Omega}^*/F_\varepsilon$, $e(h)=\pi(\check{\varepsilon(h)})$.

THEOREM 6.6. Let ε be $i_{R(H)}\cdot R: H\to\overline{R(H)\otimes\Omega}$ and Ω^* be $\overline{R(H)\otimes\Omega}$. Then, "$\check{\Omega}$ is isomorphic to $\bar{\check{\Omega}}^*/F_\varepsilon$" is H-valid.

PROOF. We use the notations in the proof of Th. 6.1. The left diagram commutes and $\bar{\phi}$ and $\pi(\check{\ })$ are isomorphisms. And $\bigvee^{\hat{\check{\Omega}}}\{\mathbb{1}\,; h\}=j_{R(H)}\cdot R(h)$. So, the theorem holds by Lemma 6.9, Th. 6.1 and Th. 6.5.

$$\begin{tikzcd} \hat{\check{\Omega}} & \Omega^* \arrow[l, "\bar{\phi}"'] \arrow[r, "\pi(\check{\ })"] & \widehat{\check{\Omega}^*/F_\varepsilon} \\ & H \arrow[ul, "j_{R(H)}\cdot R"] \arrow[u, "\varepsilon"] \arrow[ur, "e"'] & \end{tikzcd}$$

THEOREM 6.7. Let ε be the unique cH-morphism that makes the diagram (1) commutative and Ω^* be $\mathfrak{J}(L\otimes L')$. Then, "$\mathfrak{J}\check{L}'$ is isomorphic to $\check{\Omega}^*/F_\varepsilon$" is $\mathfrak{J}L$-valid.

$$(1)\quad \begin{tikzcd} \mathcal{J}(L\otimes L') & L\otimes L' \arrow[l] \\ \mathcal{J}L \arrow[u, "\varepsilon"] & L \arrow[l] \arrow[u, "i_L"'] \end{tikzcd} \qquad (2)\quad \begin{tikzcd} \mathcal{J}L \arrow[dr, dashed, "\bar{j}_L"] & \\ L \arrow[u] \arrow[r, "j_L"'] & \mathcal{J}L' \end{tikzcd}$$

PROOF. We use the notation in the proof of Th. 6.2. Let $\bar{j}_L$ be the unique cH-morphism that makes the diagram 2) commutative. Then, $\bar{j}_L=\phi\cdot\varepsilon$ and $\bigvee^{\mathfrak{J}\check{L}'}\{\mathbb{1}\,; I\}=\bar{j}_L(I)$. So, the theorem holds.

THEOREM 6.8. Let ε be i_H: $H \to \overline{H \otimes O(X)}$ and Ω^* be $\overline{H \otimes O(X)}$. Then, "$O(\check{X})$ is isomorphic to $\check{\Omega}^*/F_\varepsilon$" is H-valid.

PROOF. Similarly as the proofs of Th. 6.6 and 6.7, $\bigvee^{O(\check{X})}\{\mathbb{1}; h\} = j_H(h)$ and so the theorem holds.

COROLLARY 6.5. Let L and L' be distributive lattices. Then, "$O(\check{\mathfrak{p}L'})$ is isomorphic to $\mathfrak{J}L'$" is $\mathfrak{J}L$-valid.

PROOF. It is clear from Th. 6.7, Th. 6.8, Cor. 6.3 and Cor. 1.2.

Next we shall roughly state the relationship between $\check{X}$ and X_T in $V^{O(T)}$, where X_T is a sheaf representation of X in $V^{O(T)}$. It is known that $\widehat{O(X_T)}$ is isomorphic to $O(X \times T)$. By Cor. 6.4, it is isomorphic to $\widehat{O(\check{X})}$. We now internalize this fact.

f belongs to $({}^T X)^p$ if and only if f is a continuous function from an open subset of T to X. For $f \in ({}^T X)^p$, $\tilde{f}$ is the element of $V^{(H)}$ such that $\operatorname{dom} \tilde{f} = \operatorname{dom} \check{O(X)}$ and $\tilde{f}(\check{P}) = f^{-1}P$. $\operatorname{dom} X_T = \{\tilde{f}; f \in ({}^T X)^p\}$ and $X_T(\tilde{f}) = \operatorname{dom} f$. For $P \in O(X)$, $\operatorname{dom} \tilde{P} = \operatorname{dom} X_T$ and $\tilde{P}(\tilde{f}) = f^{-1}P$. And $\operatorname{dom} B = \{\tilde{P}; P \in O(X)\}$ and $B(\tilde{P}) = \mathbb{1}$. Then, "$X_T$ is a topological space with a base B" is $O(T)$-valid. And "$\check{X}$ is a dense subset of X_T" is $O(T)$-valid, if we embed $\check{X}$ into X_T naturally.

THEOREM 6.9. "$O(\check{X})$ is isomorphic to $O(X_T)$" is $O(T)$-valid.

PROOF. Similarly as Th. 6.3, we can prove that $O(X_T)$ is isomorphic to $\overline{O(T) \otimes O(X)}$. Next, we internalize this just like Th. 6.8. $[\![\tilde{f} \in V^{O(X_T)}\{\mathbb{1}; P\}]\!] = P \cap \operatorname{dom} f$, for $f \in ({}^T X)^p$. And so, the theorem holds by Th. 6.6.

Let R be the set of real numbers and $R^{(H)}$ be the set of Dedekind real numbers in $V^{(H)}$. Then, "$R^{(O(T))} = R_T$" is $O(T)$-valid. And so "$O(\check{R})$ is isomorphic to $O(R^{(O(T))})$" is $O(T)$-valid. However, this of course does not hold for many Boolean extensions. Let B be the complete Boolean algebra that satisfies $[\![\check{R} = R^{(B)}]\!]^{(B)} = \mathbb{0}$. Then, "$R^{(B)}$ is connected, but $\check{R}$ is not connected" is B-valid. So, "$O(\check{R})$ is not isomorphic to $O(R^{(B)})$" is B-valid.

A similar situation occurs concerning Th. 6.7 and Cor. 6.5. Let B be the complete Boolean algebra such that "The generic filter does not belong to $\check{V}$" is B-valid. Then, "$\mathfrak{J}\check{B}$ is isomorphic to $O(\mathfrak{p}\check{B})$ and $O(\mathfrak{m}\check{B})$" is B-valid, by Cor. 1.2. Since" $O(\check{\mathfrak{p}B})$ is not compact" is B-valid, "$O(\check{\mathfrak{p}B})$ is isomorphic to $\mathfrak{J}\check{B}$" is not B-valid. Let ε be i_B: $B \to \overline{B \otimes \mathfrak{J}B}$ and $\Omega^* = \overline{B \otimes \mathfrak{J}B}$. Then, "$\check{\Omega}^*/F_\varepsilon$ is isomorphic to $O(\check{\mathfrak{p}B})$" is B-valid, by Th. 6.8. So, "$\check{\Omega}^*/F_\varepsilon$ is not isomorphic to $\mathfrak{J}\check{B}$" is B-valid.

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