TSUKUBA J. MATH.
Vol. 4 (1980). 83—87

REFINABLE MAPS ONTO LOCALLY n-CONNECTED COMPACTA

By

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In [4], J. Ford and J. W. Rogers introduced the notion of refinable maps and they proved that each refinable map from a continuum to a locally connected continuum is monotone [4, Corollary 1.2]. In [5, Theorem 2.2], we proved that each refinable map from a compactum to an $\boldsymbol{FANR}$ induces a shape equivalence. In this paper we shall prove that if a map $r: X\rightarrow Y$ between compacta is refinable and $Y\in \boldsymbol{LC}^{n}$ $(n\geqq 0)$, then $r^{-1}(y)\in \boldsymbol{AC}^{n}$ for each $y\in Y$. Moreover if Y is an $\boldsymbol{ANR}$, then r is a $\boldsymbol{CE}$-map.

It is assumed that all spaces are metrizable and maps are continuous. A connected compactum is a continuum. A map $f: X\rightarrow Y$ between compacta is an *ε-mapping*, $\varepsilon>0$, if f is surjective and $\operatorname{diam} f^{-1}(y)<\varepsilon$ for each $y\in Y$. If x and y are points of a metric space, $d(x, y)$ denotes the distance from x to y. A map $r: X\rightarrow Y$ between compacta is *refinable* [4] if for any $\varepsilon>0$ there is an ε-mapping $f: X\rightarrow Y$ such that $d(r, f)=\sup\{d(r(x), f(x))\,|\,x\in X\}<\varepsilon$. Such a map f is called an *ε-refinement* of r. Note that every refinable map is surjective, every near homeomorphism is refinable and if there is a refinable map from a compactum X to a compactum Y, then X is Y-like. But simple examples show that any converse assertions of them are not true. A space X is *locally n-connected* ($X\in \boldsymbol{LC}^{n}$) if for each $x\in X$ and an open neighborhood U of x in X, there is an open set V with $x\in V\subset U$ such that each map $h: S^{k}\rightarrow V$ is null-homotopic in U for $0\leqq k\leqq n$, where S^{k} denotes the k-sphere. A compactum X in the Hilbert cube Q is *approximatively n-connected* ($X\in \boldsymbol{AC}^{n}$) if for each open neighborhood U of X in Q there is an open neighborhood $V\subset U$ of X in Q such that each map $h: S^{k}\rightarrow V$ is null-homotopic in U for $0\leqq k\leqq n$ (see [2]). A map $f: X\rightarrow Y$ between compacta is a *$\boldsymbol{CE}$-map* if f is surjective and $f^{-1}(y)$ is an $\boldsymbol{FAR}$ (see [2]) for each $y\in Y$.

The following lemma is well-known.

LEMMA 1 ([7, Lemma 1]). *Let f be a map from a compactum X to an $\boldsymbol{ANR}$ Y and $\varepsilon>0$. Then there is a positive number $\delta>0$ such that if g_1 is any δ-map-*

Received November 22, 1979.

ping from X to any compactum Z, then there is a map $g_2: Z \to Y$ *such that* $d(f, g_2 g_1) < \varepsilon$.

LEMMA 2. *Let X and Y be closed subsets of* $\boldsymbol{AR}$*-spaces M and N respectively, and let* $\hat{f}: M \to N$ *be an extension of a map* $f: X \to Y$. *If X and Y are locally n-connected and* $f:(X, x) \to (Y, f(x))$ *induces a zero-homomorphism* $\pi_k(f): \pi_k(X, x) \to \pi_k(Y, f(x))$ *for* $0 \leqq k \leqq n$, *then for each open neighborhood V of Y in N there is an open neighborhood U of X in M such that* $\pi_k(\hat{f}|U): \pi_k(U, x) \to \pi_k(V, f(x))$ *is a zero-homomorphism.*

PROOF. By [3, Theorem 8.7] the natural morphisms $i_k: \pi_k(X, x) \to \text{pro-}\pi_k(X, x)$ and $j_k: \pi_k(Y, y) \to \text{pro-}\pi_k(Y, y)$ are isomorphisms for $0 \leqq k \leqq n$. Since $j_k \pi_k(f) = \text{pro-}\pi_k(\hat{f}) i_k$, $\text{pro-}\pi_k(\hat{f}): \text{pro-}\pi_k(X, x) \to \text{pro-}\pi_k(Y, y)$ is a zero-homomorphism, which implies the existence of U in the statement of Lemma.

THEOREM. *Let X and Y be compacta and* $r: X \to Y$ *be a refinable map. If* $Y \in \boldsymbol{LC}^n$ $(n \geqq 0)$, *then* $r^{-1}(y) \in \boldsymbol{AC}^n$ *for each* $y \in Y$. *Moreover if Y is an* $\boldsymbol{ANR}$, *then r is a* $\boldsymbol{CE}$*-map.*

PROOF. Since X is a compactum, X can be embedded into the Hilbert cube Q. Let $y \in Y$ and let G be any open neighborhood of $r^{-1}(y)$ in Q. Choose a compact $\boldsymbol{ANR}$ U such that $r^{-1}(y) \subset \text{Int}_Q U \subset U \subset G$. Since U is a compact $\boldsymbol{ANR}$, there is a positive number $\varepsilon_1 > 0$ such that any ε_1-near maps to U are homotopic. Let $\varepsilon_2 = d(r^{-1}(y), Q-U) = \inf\{d(x_1, x_2) \mid x_1 \in r^{-1}(y),\ x_2 \in Q-U\} > 0$. Since $Y \in \boldsymbol{LC}^n$, there is a sequence $V_1, V_2, V_3, \cdots$ of open sets in Y such that

(1) $V_1 \supset \bar{V}_2 \supset V_2 \supset \bar{V}_3 \supset \cdots$,

(2) $\bigcap_{i=1}^{\infty} \bar{V}_i = \{y\}$,

(3) each map $h: S^k \to V_{i+1}$ $(0 \leqq k \leqq n)$ is null-homotopic in V_i.

Since r is refinable, there are maps r_i: $X \to Y$ such that each r_i is an $(1/i)$-refinement of r and

(4) $r_i(r^{-1}(y)) \subset V_{i+2}$ for each i.

Then we shall show that $\lim [r_i^{-1}(\bar{V}_i)] = r^{-1}(y)$. In fact, suppose, on the contrary, that there is a sequence $x_{n_i} \in r_{n_i}^{-1}(\bar{V}_{n_i})$ such that $\lim x_{n_i} = x_0$ and $r(x_0) \neq y$. Choose an open neighborhood W of x_0 in X such that $r(W) \subset S_\delta(r(x_0))$, where $\delta = (1/4)d(r(x_0), y) > 0$ and for a set A $S_\delta(A)$ denotes the δ-neighborhood of A. By (2), choose a sufficiently large integer n_i such that $x_{n_i} \in W$, $d(r, r_{n_i}) < \delta$ and $V_{n_i} \subset S_\delta(y)$.

Then $r(x_{n_i}) \in S_\delta(r(x_0))$ and $r_{n_i}(x_{n_i}) \in \bar{V}_{n_i} \subset S_\delta(y)$, hence

$$d(r(x_0), y) \leqq d(r(x_0), r(x_{n_i})) + d(r(x_{n_i}), r_{n_i}(x_{n_i})) + d(r_{n_i}(x_{n_i}), y)$$
$$< \delta + \delta + \delta = 3\delta, \quad \text{which implies the contradiction.}$$

Let $0 < \varepsilon < \operatorname{Min}\{\varepsilon_1, \varepsilon_2\}$. Since $\lim [r_i^{-1}(\bar{V}_i)] = r^{-1}(y)$, there is a natural number i_0 such that

$$(5) \quad r_i^{-1}(\bar{V}_i) \subset S_{\varepsilon/3}(r^{-1}(y)) \qquad \text{for each } i \geqq i_0 .$$

By Lemma 1, there is a natural number $m \geqq i_0$ such that there is a map $g_m: Y \to Q$ such that

$$(6) \quad d(i_X, g_m r_m) < \varepsilon/3, \qquad \text{where } i_X: X \to Q \text{ is the inclusion.}$$

Then we shall show

$$(7) \quad g_m(V_m) \subset g_m(\bar{V}_m) \subset U .$$

In fact, for each $x \in r_m^{-1}(\bar{V}_m)$, by (5) and (6) we have

$$d(g_m r_m(x), r^{-1}(y)) \leqq d(g_m r_m(x), x) + d(x, r^{-1}(y)) < \varepsilon/3 + \varepsilon/3 < \varepsilon ,$$

hence $g_m r_m(x) \in S_\varepsilon(r^{-1}(y)) \subset U$.

Now, take two $\boldsymbol{AR}$-spaces M and N containing V_{m+1} and V_m respectively as closed subsets, and let $\hat{i}: M \to N$ be an extension of the inclusion $i: V_{m+1} \to V_m$. Since U is an $\boldsymbol{ANR}$, by (7) there is an open neighborhood V'_m of V_m in N and an extension $\hat{g}_m: V'_m \to U$ of $g_m | V_m: V_m \to U$. Since $V_{m+1}, V_m \in \boldsymbol{LC}^n$, by Lemma 2 and (3) there is an open neighborhood V'_{m+1} of V_{m+1} in M such that

$$(8) \quad \pi_k(\hat{i} | V'_{m+1}): \pi_k(V'_{m+1}) \longrightarrow \pi_k(V'_m) \text{ is a zero-homomorphism}$$
$$\text{for each } 0 \leqq k \leqq n .$$

Let U' be an open neighborhood of $r_m^{-1}(\bar{V}_{m+2})$ in Q such that $U' \subset U$ and there is an extension $\hat{r}_m: U' \to V'_{m+1}$ of $r_m | r_m^{-1}(\bar{V}_{m+2}): r_m^{-1}(\bar{V}_{m+2}) \to V_{m+1}$. Since $\hat{g}_m \hat{i} \hat{r}_m | r_m^{-1}(\bar{V}_{m+2}) = g_m i r_m | r_m^{-1}(\bar{V}_{m+2})$, by (6) there is an open neighborhood $U'' \subset U'$ of $r_m^{-1}(\bar{V}_{m+2})$ in Q such that

$$(9) \quad d(\hat{g}_m \hat{i} \hat{r}_m | U'', i_{U''}) < \varepsilon, \text{ where } i_{U''}: U'' \to U \text{ is the inclusion.}$$

By (9), we have

$$(10) \quad \hat{g}_m \hat{i} \hat{r}_m | U'' \simeq i_{U''} \text{ in } U .$$

By (8) and (10), $\pi_k(i_{U''}): \pi_k(U'') \to \pi_k(U)$ is a zero-homomorphism. Note that

$r^{-1}(y)\subset r_m^{-1}(\bar{V}_{m+2})\subset U''$. Hence $r^{-1}(y)\in \boldsymbol{AC}^n$.

If Y is a compact $\boldsymbol{ANR}$, the proof is similar. This completes the proof.

REMARK 1. Note that if $n=0$, Theorem implies the result of J. Ford and J. W. Rogers.

By Theorem and [3, Theorem 8.5], we have the following.

COROLLARY 1. *If a map $r: X\to Y$ between compacta is refinable and $Y\in \boldsymbol{LC}^n$ $(n\geqq 1)$, for any compactum $B\subset Y$ and $x\in r^{-1}(B)$, pro-$\pi_k(r|r^{-1}(B))$: pro-$\pi_k(r^{-1}(B), x)$ $\to$pro-$\pi_k(B, r(x))$ is an isomorphism of pro-groups for $1\leqq k\leqq n$ and an epimorphism of pro-groups for $k=n+1$.*

COROLLARY 2. *Let X and Y be compacta and $r: X\to Y$ be a refinable map. If $Y\in \boldsymbol{LC}^n$ and* $\mathrm{Fd}(Y)\leqq n$ (see [2]), *then r induces a shape equivalence.*

PROOF. By [5, Theorem 1.8], $\mathrm{Fd}(X)=\mathrm{Fd}(Y)\leqq n$. By Theorem and the result of [3, Theorem 8.14], [6] or [8], r induces a shape equivalence.

COROLLARY 3. *If a map $r: X\to Y$ between compacta is refinable and Y is a finite-dimensional $\boldsymbol{ANR}$, then r induces a hereditary shape equivalence,* i.e., *for any compactum B, $r|r^{-1}(B)$: $r^{-1}(B)\to B$ induces a shape equivalence.*

COROLLARY 4. *Let r be a map from a $(S_1\vee S_2\vee \cdots \vee S_n)$-like continuum onto $S_1\vee S_2\vee \cdots \vee S_n$, where $S_1\vee S_2\vee \cdots \vee S_n$ denotes the one point union of n circles. Then the followings are equivalent.*

(1) *r is refinable.*

(2) *r is a* ***CE**-map.*

(3) *r is monotone.*

PROOF. By [5, Theorem 3.2], (1) and (3) are equivalent. By Theorem, (1) implies (2). Obviously (2) implies (3).

REMARK 2. In the statement of Theorem, we cannot replace $\boldsymbol{AC}^n$ by $\boldsymbol{C}^n$ (n-connected).

REMARK 3. By [4, p. 264], there is a refinable map $r: X\to Y$ such that X, Y are 1-dimensional continua and $r^{-1}(y_0)\notin \boldsymbol{AC}^0$ for some $y_0\in Y$ (cf. [5, Example 2.7]). In [5, Example 2.6], for each $n=1, 2, 3, \cdots$, we constructed a refinable map $r: X\to Y$ such that X and Y are n-dimensional continua, $Y\in \boldsymbol{LC}^{n-1}$ and $\mathrm{Sh}(X)\neq \mathrm{Sh}(Y)$. In fact, for some $y_0\in Y$, $r^{-1}(y_0)\notin \boldsymbol{AC}^n$. Thus those show that in the statement of Theorem we cannot replace $\boldsymbol{LC}^n$ by $\boldsymbol{LC}^{n-1}$. Moreover, in [5, Example 2.8], we constructed a near homeomorphism $h: X \to X$ such that X is a n-dimensional continuum, $X \in \boldsymbol{LC}^{n-1}$ and r does not induce a shape equivalence. In fact, for some $y_0 \in X$, $r^{-1}(y_0) \notin \boldsymbol{AC}^n$.

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