TSUKUBA J. MATH.
Vol. 4 (1980). 21–66

THE PRINCIPAL 2-BLOCKS OF FINITE GROUPS WITH ABELIAN SYLOW 2-SUBGROUPS

By

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Introduction

Let G be a finite group, p a prime number and B a p-block of G with defect group D. There is an important problem in representation theory of finite groups that is to give a description of B when the structure of D is given. Concerning with this problem there are some successful results. E.C. Dade [9] proved his results when D is cyclic. R. Brauer [6] proved his results for the case where $p=2$ and D is dihedral by making use of his powerful methods ([3], [4], [5]). Using Brauer's methods J.B. Olsson [18] obtained his results when $p=2$ and D is generalized quaternion or quasidihedral. In [3, IV] R. Brauer investigated B when $p=2$ and D is elementary abelian of order 4.

In the present paper we study B when $p=2$ and B is the principal 2-block of G with an abelian Sylow 2-subgroup P. Let $e(G)=|N_G(P):C_G(P)|$. Let $B_0(G)$ be the principal 2-block of G, and let $O(G)$ and $O'(G)$ be the maximal normal subgroup of G of odd order and the minimal normal subgroup of G of odd index, respectively. By the results on finite groups with abelian Sylow 2-subgroups ([2], [16], [17], [20], [21]), the structure of $O'(G/O(G))$ is almost determined. In general, however, $B_0(G)$ is different from $B_0(S)$ where $S=O'(G/O(G))$. The main purpose of this paper is to investigate the relation between $B_0(G)$ and $B_0(S)$. In particular we shall prove that $B_0(G)$ is isomorphic to $B_0(S)$ for the cases where $e(G)=e(S)=$prime, 9 and 21.

In section 1 we shall state several lemmas and propositions which will be useful for our aim. One of them is Alperin's theorem on isomorphic principal blocks [1]. Let $S=O'(G/O(G))$. In section 2 we shall consider $B_0(G)$ for the case where $e(G)=2^m-1$. In particular, we shall prove that if G is nonsolvable and if $e(G)$ is prime then $e(G)=2^m-1$ for some $m\geqq 2$ and $B_0(G)$ is isomorphic to $B_0(S)$. In sections 3 and 4 we shall investigate $B_0(G)$ for the cases when $e(G)=9$ and 21, respectively. Indeed, we shall prove that if $e(G)=e(S)=9$ or 21 then $B_0(G)$ is isomorphic to $B_0(S)$. It is noted that when $e(G)\neq e(S)$, $B_0(G)$ is not necessarily isomorphic to $B_0(S)$. In sections 5 and 6 we shall determine

Received October 17, 1979

$B_0(G)$ when P is elementary abelian of order 8 and 16, respectively.

Throughout this paper we shall use the following notation. When S is a subset of G, $N_G(S)$ and $C_G(S)$ denote the normalizer and the centralizer of S in G, respectively. Specially, for each $x \in G$ we write $C_G(x)$ for $C_G(\{x\})$. If $x, y \in G$, we write x^y for $y^{-1}xy$. When S is a subset of G, $\langle S \rangle$ denotes the subgroup of G generated by S. When $x_1, \cdots, x_n$ are elements of G and S is a subset of G, we also write $\langle x_1, \cdots, x_n, S \rangle$ for the subgroup of G generated by $\{x_1, \cdots, x_n\} \cup S$. The cyclic group of order n is denoted Z_n for a positive integer n. We write G' and $Z(G)$ for the commutator subgroup of G and the center of G, respectively. We denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G. Let us denote by $O_{p'}(G)$ the maximal normal subgroup of G of order prime to p, and by $O^{p'}(G)$ the minimal normal subgroup of G of index prime to p. In particular, for $p=2$ we write $O(G)$ and $O'(G)$ for $O_{2'}(G)$ and $O^{2'}(G)$, respectively. When P is an abelian Sylow 2-subgroup of G, we write $e(G)$ (or shortly e) for $|N_G(P) : C_G(P)|$. When B is a p-block of G, let us denote by $\mathrm{Irr}(B)$ the set of all irreducible complex characters in B, by $\mathrm{IBr}(B)$ the set of all irreducible Brauer characters in B, by $k(B)$ the number of elements of $\mathrm{Irr}(B)$, by $k'(B)$ the number of elements of $\mathrm{Irr}(B)$ with degree one, and by $l(B)$ the number of elements of $\mathrm{IBr}(B)$. We write $B_0(G)$ (or shortly B_0) for the principal p-block of G, and for each $x \in G$ we write b_x for $B_0(C_G(x))$. When ψ_1 and ψ_2 are complex characters of G, let $(\psi_1, \psi_2)=(1/|G|)\sum_{g\in G}\psi_1(g)\psi_2(g^{-1})$, that is to say, (ψ_1, ψ_2) is the inner product of ψ_1 and ψ_2. We write 1_G for the trivial complex (or Brauer) character of G. When H is a normal subgroup of G, $\psi|_H$ denotes the restriction of ψ to H for a character ψ of G, $W|_H$ denotes the restriction of W to H for a representation W of G, and $I_G(\tilde{\psi})$ denotes the inertial group of $\tilde{\psi}$ in G for a character $\tilde{\psi}$ of H, that is to say, $I_G(\tilde{\psi})=\{g \in G \mid \tilde{\psi}^g=\tilde{\psi}\}$, where $\tilde{\psi}^g$ is the conjugate of $\tilde{\psi}$.

1. Preliminaries

In this section we state some lemmas and propositions which will be needed for our aim. We fix a prime number p and we consider p-modular representations of a finite group G.

LEMMA 1.1. *Let G be a finite group with a Sylow p-subgroup P, and let $K=O_{p'}(G)$, $\bar{G}=G/K$ and $\bar{P}=(PK)/K$. Then we have the following.*

(i) $B_0(G)=B_0(\bar{G})$.

(ii) $N_G(P)/C_G(P) \cong N_{\bar{G}}(\bar{P})/C_{\bar{G}}(\bar{P})$.

PROOF. We get (i) by [10, Theorem 65.2] and [11, V (4.3)]. Since $N_{\bar{G}}(\bar{P})=(N_G(P)\cdot K)/K$ from [15, I 7.7 Hilfssatz (c)] and since $C_{\bar{G}}(\bar{P})=(C_G(P)\cdot K)/K$ from [19, Lemma 2.2], we easily get (ii).

We shall frequently use the next four propositions in order to prove our main theorems.

PROPOSITION 1.2. (Brauer). *Let $G=QC_G(Q)$ where Q is a p-group, and let $\bar{G}=G/Q$. Then $l(B_0(G))=l(B_0(\bar{G}))$.*

PROOF. See [10, Lemma 64.5 and Theorem 65.2(2)].

PROPOSITION 1.3 (Brauer). *Let H be a normal subgroup of G. If W is an ordinary or modular irreducible representation in $B_0(G)$, then any irreducible constituent of $W|_H$ lies in $B_0(H)$.*

PROOF. This is the special case of [3, I Lemma 1].

PROPOSITION 1.4 (Brauer). *Let H be a normal subgroup of G. Then for any $\tilde{\chi}\in \mathrm{Irr}(B_0(H))$, there is some $\chi\in \mathrm{Irr}(B_0(G))$ such that $(\chi|_H, \tilde{\chi})\neq 0$.*

PROOF. This is the special case of [3, II Lemma 1].

PROPOSITION 1.5 (Brauer). *Let P be a Sylow p-subgroup of G, and let $P\cdot C_G(P)=P\times V$. Then $k'(B_0(G))=|G:VG'|$.*

PROOF. See [3, IV Proposition (4G)].

Next, we state Alperin's theorems on isomorphic principal p-blocks which are very important for our aim.

Let F be an algebraically closed field of characteristic p and FG the group algebra of G over F. Let H be a normal subgroup of G with $p\nmid |G:H|$. We write $B_0(G)\cong B_0(H)$, if the category of all finitely generated FG-modules in $B_0(G)$ is isomorphic to the category of all finitely generated FH-modules in $B_0(H)$ and if the isomorphism is given by the restriction from G to H (cf. [1]).

PROPOSITION 1.6 (Alperin). *Let F be as above, and let P be a Sylow p-subgroup of G. If H is a normal subgroup of G which satisfies the conditions that $p\nmid |G:H|$, G/H is solvable and $G=H\cdot C_G(P)$, then we get the following.*

(i) $B_0(G)\cong B_0(H)$.

(ii) *$A_0(G)\cong A_0(H)$ as F-algebras, where $A_0(G)$ and $A_0(H)$ are the block ideals* *of FG and FH corresponding to* $B_0(G)$ *and* $B_0(H)$, *respectively.*

PROOF. See [1, Theorems 1 and 2].

COROLLARY 1.7 (Alperin). *Let H be a normal subgroup of G of prime index q with* $q\neq p$. *Let* $B_0=B_0(G)$ *and* $b_0=B_0(H)$. *Assume that* $k(B_0)=k(b_0)$ *and* $l(B_0)=l(b_0)$, *and that* $I_G(\tilde{\chi})=G$ *for every* $\tilde{\chi}\in \mathrm{Irr}(b_0)$. *Then we have the following.*

(i) *The correspondence* $\mathrm{Irr}(B_0)\to\mathrm{Irr}(b_0)$ *given by* $\chi\mapsto\chi|_H$ *is a bijection.*

(ii) *The correspondence* $\mathrm{IBr}(B_0)\to\mathrm{IBr}(b_0)$ *given by* $\phi\mapsto\phi|_H$ *is a bijection.*

(iii) $B_0\cong b_0$.

PROOF. (i) Since $I_G(\tilde{\chi})=G$ for every $\tilde{\chi}\in\mathrm{Irr}(b_0)$, the correspondence is surjective by Clifford's theorem, [8, (53.17) Theorem] and Propositions 1.3 and 1.4. Since $k(B_0)=k(b_0)$, we obtain (i).

(ii) By (i), [1, Lemma 1] holds. Thus, by the proof of [1, Lemma 3], the correspondence is surjective. Hence (ii) holds since $l(B_0)=l(b_0)$.

(iii) Since [1, Lemmas 1 and 3] hold, we get (iii) by the proofs of Alperin's theorems [1, Theorems 1 and 2].

In the remainder of this paper we assume $p=2$ and let G and P be a finite group and its abelian Sylow 2-subgroup of order 2^n, respectively. We use the notation B_0 and e for $B_0(G)$ and $e(G)$, respectively.

COROLLARY 1.8 (Alperin). *Let H be a normal subgroup of G of odd prime index. Let* $B_0=B_0(G)$ *and* $b_0=B_0(H)$. *Assume that* $k(B_0)=k(b_0)$ *and* $l(B_0)=l(b_0)$, *and that H has an involution x such that* $\chi(x)=\pm1$ *for every* $\chi\in\mathrm{Irr}(B_0)$ *and* $\tilde{\chi}(x)=\tilde{\chi}'(x)=\pm1$ *for all* $\tilde{\chi}, \tilde{\chi}'\in\mathrm{Irr}(b_0)$ *with* $\tilde{\chi}(1)=\tilde{\chi}'(1)$. *Then* $B_0\cong b_0$.

PROOF. By Clifford's theorem and Proposition 1.3, we have $\chi|_H\in\mathrm{Irr}(b_0)$ for all $\chi\in\mathrm{Irr}(B_0)$. Thus, by Proposition 1.4, $I_G(\tilde{\chi})=G$ for all $\tilde{\chi}\in\mathrm{Irr}(b_0)$. Thus the corollary is proved by Corollary 1.7 (iii).

LEMMA 1.9. *Let P be an abelian Sylow 2-subgroup of G. Suppose that* $k(B_0)=|P|$ *and that G has an involution x with* $l(b_x)=1$. *Then* $\chi(x)=\pm1$ *for all* $\chi\in\mathrm{Irr}(B_0)$.

PROOF. Since $l(b_x)=1$, b_x has the unique Cartan invariant $|P|$. Hence, by [10, Theorems 63.3(2), 63.2 and 65.4], we get $\sum\chi(x)^2=|P|$ where the sum runs through all $\chi\in\mathrm{Irr}(B_0)$. By [4, II (7A) and (4C)], $\chi(x)$ is a nonzero integer for every $\chi\in\mathrm{Irr}(B_0)$ since $|x|=2$. Therefore, the assumption $k(B_0)=|P|$ implies the lemma.

PROPOSITION 1.10 (Bender, Janko, Janko-Thompson, Walter, Ward). *If G has abelian Sylow 2-subgroups, then $O'(G/O(G))$ is a direct product of an abelian 2-group and simple groups of one of the following types;*

(1) *the special linear group $SL(2, 2^n)$ for $n\geqq 2$,*

(2) *the projective special linear group $L_2(q)$ for $q>3$ with $q\equiv 3$ or 5 (mod 8),*

(3) *the Janko's first simple group J_1,*

(4) *the simple group $R(q)$ of Ree type.*

PROOF. For groups of types (1) and (2), see [14, p. 40]. For J_1 see [16], and for $R(q)$ see [21]. The proposition is obtained from [2], [16], [17], [20] and [21].

In the rest of this paper we use the notation $SL(2, 2^n)$, $L_2(q)$, J_1 and $R(q)$ as in Proposition 1.10 (cf. [13, p. 415]). We also use the notation $GL(m, 2)$ for the general linear group (cf. [14, p. 40]).

The next lemma shows that Brauer's conjecture on heights of irreducible complex characters in p-blocks with abelian defect groups is affirmative for the principal 2-blocks of finite groups with abelian Sylow 2-subgroups.

LEMMA 1.11. *If G has abelian Sylow 2-subgroups, then all irreducible complex characters in $B_0(G)$ have height zero.*

PROOF. We may assume $O(G)=1$ by Lemma 1.1. Let H be a normal subgroup of G of odd index. If $\chi\in\mathrm{Irr}(B_0(G))$, then there is some $\tilde{\chi}\in\mathrm{Irr}(B_0(H))$ with $\chi(1)=m\tilde{\chi}(1)$ for a positive integer m from Clifford's theorem and Proposition 1.3. By [8, (53.17) Theorem], m divides $|G:H|$. This shows that if $\tilde{\chi}(1)$ is odd then $\chi(1)$ is also odd. Thus, we may assume $O'(G)=G$. Then, by Proposition 1.10, we can write $G=Q\times(\prod S_i)$ where Q is an abelian 2-group and each S_i is a simple group of one of the following types;

(i) $SL(2, 2^n)$ for $n\geqq 2$,

(ii) $L_2(q)$ for $q>3$ with $q\equiv 3$ or 5 (mod 8),

(iii) J_1,

(iv) $R(q)$.

When S_i is of type (i) or (ii), every $\chi\in\mathrm{Irr}(B_0(S_i))$ has odd degree from [10, Theorems 38.2 and 38.1]. When S_i is of type (iii) or (iv), every $\chi\in\mathrm{Irr}(B_0(S_i))$ has odd degree from [16, Lemma 5.1] and [21, Chap. I], respectively. These show that every $\chi\in\mathrm{Irr}(B_0(G))$ has odd degree. This completes the proof.

The next three lemmas are useful in order to obtain $e=e(G)$.

LEMMA 1.12. *Let P be a Sylow 2-subgroup of G.*

(i) *If $G=SL(2, 2^n)$ for $n\geqq 2$, then P is elementary abelian of order 2^n and $N_G(P)/C_G(P)$ is cyclic of order 2^n-1.*

(ii) *If $G=L_2(q)$ for $q>3$ with $q\equiv 3$ or 5 (mod 8), then P is noncyclic of order 4 and $N_G(P)/C_G(P)$ is cyclic of order 3.*

(iii) *If $G=J_1$ or $R(q)$, then P is elementary abelian of order 8 and $N_G(P)/C_G(P)$ is noncyclic of order 21.*

PROOF. (i) By [14, Theorems 2.8.1 and 2.8.3], P is elementary abelian of order 2^n. Let $q=2^n$, and let F_q be the finite field of q elements. We may assume that $P=\left\{\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} | f\in F_q\right\}$ (cf. the proof of [14, Theorem 2.8.3]). Clearly, $C_G(P)=P$. Let u be a generator of the multiplicative group $F_q-\{0\}$, and let $s=\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ in G. Then, $N_G(P)=\langle P, s\rangle$ and s has order $q-1$. Hence we get that $N_G(P)/P$ is cyclic of order $q-1$.

(ii) P is noncyclic of order 4 from [14, Lemma 15.1.1]. Hence Aut(P) is isomorphic to the symmetric group of degree 3. Since G is not 2-nilpotent, we get (ii).

(iii) If $G=J_1$, we obtain (iii) from [16, VI p. 160]. Assume $G=R(q)$. By [21, p. 63], P is elementary abelian of order 8 and $|N_G(P): C_G(P)|=21$. Then we know that $N_G(P)/C_G(P)$ is noncyclic since $\mathrm{Aut}(P)\cong GL(3, 2)\hookrightarrow GL(4, 2)\cong A_8$ from [15, II 2.5 Satz] where A_8 is the alternating group of degree 8.

LEMMA 1.13. (i) *$GL(4, 2)\cong A_8$, the alternating group of degree 8.*

(ii) *If H is a subgroup of A_8 of odd order, then $|H|=1, 3, 5, 7, 9, 15$ or 21.*

(iii) *A_8 has subgroups of orders 1, 3, 5, 7, 9, 15 and 21, and the subgroups of order 9 and the subgroups of order 21 are noncyclic.*

PROOF. (i) We have already showed (i) in the proof of Lemma 1.12(iii).

(ii) Since $|A_8|=2^6\cdot 3^2\cdot 5\cdot 7$, $|H|=1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105$ or 315. Since the groups of order 35 are cyclic, $|H|\neq 35$. By elementary calculations, A_8 has no subgroups of order 45, so that $|H|\neq 45$. Similarly, $|H|\neq 63$. If $|H|=105$, then H has an element of order 35. Evidently, this is a contradiction. Hence $|H|\neq 105$. If $|H|=315$, then H has an element of order 35, and this is a contradiction. So that $|H|\neq 315$.

(iii) By Sylow's theorem, A_8 has subgroups of orders 3, 5, 7 and 9. Since A_8 has no elements of order 9, Sylow 3-subgroups of A_8 are noncyclic of order

9. If $G=SL(2, 2^4)$, then P is elementary abelian of order 16 and $N_G(P)/C_G(P)$ is cyclic of order 15 from Lemma 1.12(i). Thus, by (i), A_8 has subgroups of order 15. Let $H=\langle(124)(536), (1234567)\rangle$. Then H is a noncyclic subgroup of A_8 of order 21. Since A_8 has no elements of order 21, all subgroups of A_8 of order 21 are noncyclic.

LEMMA 1.14. (i) *If H is a subgroup of $GL(3, 2)$ of odd order, then* $|H|=1, 3, 7$ *or* 21.

(ii) $GL(3, 2)$ *has subgroups of orders* 1, 3, 7 *and* 21, *and the subgroups of order* 21 *are noncyclic.*

PROOF. (i) By [10, Lemma 35.2(1)], $|GL(3, 2)|=2^3\cdot 3\cdot 7$. So that we easily get (i).

(ii) By the proof of (i) and Sylow's theorem, $GL(3, 2)$ has subgroups of orders 3 and 7. By Lemma 1.12(iii), $GL(3, 2)$ has noncyclic subgroups of order 21. Since $GL(3, 2)\hookrightarrow GL(4, 2)$, all subgroups of $GL(3, 2)$ of order 21 are noncyclic from Lemma 1.13(i) and (iii).

The next two lemmas are useful in order to determine B_0 when Sylow 2-subgroups of G are elementary abelian of order 8 or 16.

LEMMA 1.15. *Let P be an abelian Sylow 2-subgroup of G, and let $B_0=B_0(G)$. Assume that G has an involution x with $l(b_x)=1$.*

(1) *If* $|P|=8$, *then* $k(B_0)=8$.

(2) *If* $|P|=16$, *then* $k(B_0)=8$ *or* 16.

PROOF. Let $\{\chi_1, \cdots, \chi_{k(B_0)}\}=\mathrm{Irr}(B_0)$. Since $l(b_x)=1$, by [10, Theorems 63.2 and 65.4], for each χ_i let d_{i1}^x be the generalized decomposition number of B_0 relative to x. By Lemma 1.11 and [4, II (7A) and (4C)], every d_{i1}^x is an odd integer. Since b_x has the unique Cartan invariant $|P|$, by [10, Theorem 63.3], $\sum_{i=1}^{k(B_0)}(d_{i1}^x)^2=|P|$. These imply (1) and (2).

LEMMA 1.16. *Let $G=L_2(q)$ for $q>3$ with $q\equiv 3$ or 5 (mod 8), and let $B_0=B_0(G)$. Then we have the following.*

(i) $l(B_0)=3$ *and the degrees of all irreducible Brauer characters in B_0 are* 1, $(q-1)/2$ *and* $(q-1)/2$.

(ii) *The decomposition matrix of B_0 is as follows:*

1	0	0		1	0	0
0	1	0		1	1	0
0	0	1		1	0	1
1	1	1		1	1	1.
$3<q\equiv 3 \pmod 8$				$3<q\equiv 5 \pmod 8$		

PROOF. Since G is not 2-nilpotent, $l(B_0)>1$ from [10, Corollary 65.3]. Thus $k(B_0)=4$ and $l(B_0)=3$ by [3, IV Proposition (7D)].

Case 1. $3<q\equiv 3 \pmod 8$: Let $\mathrm{Irr}(B_0)=\{\chi_1, \cdots, \chi_4\}$. By [10, Theorem 38.1], we may assume $\chi_1=1_G$, $\chi_2(1)=\chi_3(1)=(q-1)/2$ and $\chi_4(1)=q$. By [14, Theorem 2.8.2], G has a Frobenius subgroup E of order $q(q-1)/2$. We know the character tables of E and $L_2(q)$ from [10, Theorems 13.8 and 38.1]. Thus, by [8, §84 Exercise 2], $\chi_2|_{G_0}$ and $\chi_3|_{G_0}$ are both irreducible Brauer characters of G, where $\chi_i|_{G_0}$ is the restriction of χ_i to the set G_0 of all $2'$-elements of G. Since $\chi_2\neq\chi_3$ on G_0, and since $\chi_4=\chi_1+\chi_2+\chi_3$ on G_0, we know (i) and the decomposition matrix of B_0. *Case* 2. $3<q\equiv 5 \pmod 8$: As in Case 1 we can prove the lemma.

REMARK 1. If G has an abelian Sylow 2-subgroup P and if $e(G)=1$, then $B_0(G)\cong B_0(P)$ since G is 2-nilpotent by [10, Theorem 18.7].

2. The case $e=2^m-1$

In this section we consider the case when $e=2^m-1$ for $m\geqq 2$. We use the notation G, P, n, e and B_0 as before, that is to say, P is an abelian Sylow 2-subgroup of G with order 2^n ($n\geqq 2$), $e=e(G)$ and $B_0=B_0(G)$. To begin with we state the next three lemmas which will be needed for the main result of this section.

LEMMA 2.1. *Let S be a normal subgroup of G of odd index such that $S\cong SL(2, 2^n)$ for some $n\geqq 3$. Assume $e=2^n-1$. Then $B_0\cong B_0(S)$.*

PROOF. We may assume $S=SL(2, 2^n)$. There are an element $t\in N_S(P)$ and an involution $x\in P$ such that $N_S(P)=\langle t, C_S(P)\rangle$ and $P=\{1, x, x^t, \cdots, x^{t^{2^n-2}}\}$ (cf. the proof of Lemma 1.12(i)). Since $e=2^n-1$, $N_G(P)=\langle t, C_G(P)\rangle$. Clearly $y^t\neq y$ for all $y\in P-\{1\}$, so that $N_M(P)=C_M(P)$ where $M=C_G(x)$. Hence M is 2-nilpotent from [10, Theorem 18.7]. Thus, by [10, Corollary 65.3], $l(b_x)=l(B_0(M))=1$. Now, we prove the lemma by induction on $|G|$. Suppose $G\neq S$. Since $|G/S|$ is odd, by [12, Theorem], G has a normal subgroup H of odd prime index l with $S\subseteqq H$. Let $b_0=B_0(H)$. By induction, $b_0\cong B_0(S)$. Hence, by the character table of $SL(2, 2^n)$ [10, Theorem 38.2], we get

	1	x	
1_H	1	1	
$\tilde{\theta}_i$	2^n-1	-1	for $i=1, \cdots, 2^{n-1}$
$\tilde{\chi}_j$	2^n+1	1	for $j=1, \cdots, 2^{n-1}-1$

where $\{1_H, \tilde{\theta}_i, \tilde{\chi}_j | i=1, \cdots, 2^{n-1}; j=1, \cdots, 2^{n-1}-1\}=\mathrm{Irr}(b_0)$. Let $C_G(P)=P\times V$. If $G=VH$, then $G=C_G(P)\cdot H$, so that $B_0\cong b_0$ from Proposition 1.6. Hence we may assume $G\neq VH$. Then $H=VH$, so that $C_H(P)=P\times V$. Thus, by Proposition 1.5, $k'(b_0)=|H: VH'|$. Since $b_0\cong B_0(S)$, $k'(b_0)=1$. Thus, $H=VH'$. This implies $H=VG'$ since G/H is cyclic. Hence $k'(B_0)=l$ from Proposition 1.5. By Clifford's theorem and Proposition 1.3, for each $\chi\in\mathrm{Irr}(B_0)$ one of the following five cases occurs:

(a) $\chi|_H=1_H$,

(b) $\chi|_H=\tilde{\theta}_i$ for some i,

(c) $\chi|_H=\tilde{\theta}_{i_1}+\cdots+\tilde{\theta}_{i_l}$ for $i_1<\cdots<i_l$, and all $\tilde{\theta}_{i_k}$ are G-conjugate,

(d) $\chi|_H=\tilde{\chi}_j$ for some j,

(e) $\chi|_H=\tilde{\chi}_{j_1}+\cdots+\tilde{\chi}_{j_l}$ for $j_1<\cdots j_l$, and all $\tilde{\chi}_{j_k}$ are G-conjugate.

Since $k'(b_0)=1$, for each $\chi\in\mathrm{Irr}(B_0)$ $\chi(1)=1$ if and only if $\chi|_H=1_H$. Let r, s, u and v be the numbers of $\chi\in\mathrm{Irr}(B_0)$ of types (b), (c), (d) and (e), respectively. Since $l(b_x)=1$, as in the proof of Lemma 1.9, $\sum\chi(x)^2=2^n$ where the sum runs through all $\chi\in\mathrm{Irr}(B_0)$. This shows $l+r+sl^2+u+vl^2=2^n$. On the other hand, by Proposition 1.4, for every $\tilde{\chi}\in\mathrm{Irr}(b_0)$ there is some $\chi\in\mathrm{Irr}(B_0)$ with $(\chi|_H, \tilde{\chi})\neq 0$. So that $k(b_0)\leqq 1+r+sl+u+vl$. Since $k(b_0)=2^n$, we have a contradiction. This completes the proof.

REMARK 1. We can not remove the assumption $e=2^n-1$ in Lemma 2.1. Indeed, let $S=SL(2, 8)$ and $P=\left\{\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix} | f\in F_8\right\}$ where F_8 is the finite field of 8 elements. Let u be a generator of the multiplicative group $F_8-\{0\}$. There is an automorphism h of F_8 with $h(u)=u^2$. For each $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in S$ let $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^h=\begin{pmatrix}h(a) & h(b)\\ h(c) & h(d)\end{pmatrix}$. Then we can consider $h\in\mathrm{Aut}(S)$ and $h|_P\in\mathrm{Aut}(P)$ where $h|_P$ is the restriction of h to P. Hence there is a semi-direct product G of its normal subgroup S by $\langle h\rangle$. Then $O'(G)=S=SL(2, 8)$ and $e(G)=21\neq 2^3-1$. By [10, Theorem 38.2], $l(B_0(S))=7$. But we shall afterwards show that $l(B_0(G))=5$, and this shows $B_0(G)\not\cong B_0(S)$.

LEMMA 2.2. *Let S be a normal subgroup of G of odd index such that $S\cong L_2(q)\times(P/(Z_2\times Z_2))$ for some $q>3$ with $q\equiv 3$ or* 5 (mod 8), *or* $S\cong SL(2, 2^m)$

$\times(P/(\underbrace{Z_2\times\cdots\times Z_2}_{m}))$ *for some* $m\geqq 3$. *Assume* $e=e(S)$. *Then* $k(B_0)=2^n$ *and* $l(B_0)=e$.

PROOF. Let $L=L_2(q)$ for $m=2$, and let $L=SL(2, 2^m)$ for $m\geqq 3$. Let R be a Sylow 2-subgroup of L. We can write $S=L\times Q$ and $P=R\times Q$. We use induction on n. If $n=m=2$, then the lemma is proved by [3, IV Proposition (7D)]. If $n=m\geqq 3$, by Lemma 2.1, $B_0\cong B_0(S)$, so that $k(B_0)=2^n$ and $l(B_0)=2^n-1$ $=2^m-1$ (cf. [10, Theorem 38.2]). Next, suppose $n>m$. There are an element $t\in N_L(R)$ and an involution $x\in R$ such that $N_L(R)=\langle t, C_L(R)\rangle$ and $R=\{1, x, x^t, \cdots x^{t^{2^m-2}}\}$. Since $e=e(S)$, $N_G(P)=\langle t, C_G(P)\rangle$. Let $Q=\{1=y_1, y_2, \cdots, y_{2^{n-m}}\}$. Then, by [10, Lemma 18.5], the G-conjugate classes of P are as follows:

$$\{1\}$$
$$\{y_i\} \text{ for } i=2, \cdots, 2^{n-m}$$
$$\{xy_i, x^t y_i, \cdots, x^{t^{2^m-2}}y_i\} \text{ for } i=1, \cdots, 2^{n-m}.$$

Then, by [10, Theorems 68.4 and 65.4],

$$k(B_0)=l(B_0)+\sum_{i=2}^{2^{n-m}} l(b_{y_i})+\sum_{i=1}^{2^{n-m}} l(b_{xy_i}).$$

Fix any i with $2\leqq i\leqq 2^{n-m}$, and let $M=C_G(y_i)$. Since $y_i\in Z(S)$, let $\bar{S}=S/\langle y_i\rangle$. Similarly, let $\bar{M}=M/\langle y_i\rangle$, $\bar{P}=P/\langle y_i\rangle$ and $\bar{Q}=Q/\langle y_i\rangle$. Since $\bar{S}\cong L\times\bar{Q}$, we get $e(\bar{S})=e(L)=2^m-1$. Since $\bar{S}\subseteqq\bar{M}$, the canonical homomorphism $N_{\bar{S}}(\bar{P})/C_{\bar{S}}(\bar{P})\to N_{\bar{M}}(\bar{P})/C_{\bar{M}}(\bar{P})$ is monomorphic. This shows $(2^m-1)|e(\bar{M})$. On the other hand, by [15, I 7.7 Hilfssatz (c)], we get $N_{\bar{M}}(\bar{P})=(N_M(P)\cdot\langle y_i\rangle)/\langle y_i\rangle$. This implies that the canonical homomorphism $N_M(P)/C_M(P)\to N_{\bar{M}}(\bar{P})/C_{\bar{M}}(\bar{P})$ is epimorphic. Hence $e(\bar{M})|e(M)$. Since $S\subseteqq M\subseteqq G$ and $e=e(S)=2^m-1$, we have $e(M)=e(S)=2^m-1$ by considering the canonical monomorphisms as above. Thus $e(\bar{M})=2^m-1$. Hence we get $l(B_0(\bar{M}))=2^m-1$ by induction. Thus $l(b_{y_i})=l(B_0(M))=2^m-1$ from Proposition 1.2. We may assume $O(G)=1$ by Lemma 1.1. Since $Q\neq 1$, there is an involution $y_j\in Q$. By Z^*-theorem [10, Theorem 67.1], $y_j\in Z(G)$. Hence $l(B_0)$ $=l(b_{y_j})=2^m-1$. Next, we consider $l(b_{xy_i})$ for each $i=1, \cdots, 2^{n-m}$. For an integer k it is seen that $(xy_i)^{t^k}=xy_i$ if and only if $(2^m-1)|k$. Hence $N_U(P)=C_U(P)$ where $U=C_G(xy_i)$. Then U is 2-nilpotent from [10, Theorem 18.7], so that $l(b_{xy_i})=l(B_0(U))=1$ by [10, Corollary 65.3]. These imply $k(B_0)=2^n$.

LEMMA 2.3. *Assume as in Lemma* 2.2. *Then* $B_0\cong B_0(S)$.

PROOF. We use the same notation as in the proof of Lemma 2.2. We prove the lemma by induction on $|G|$. Suppose $G\neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index with $S\subseteqq H$. Let $b_0=B_0(H)$. By induction, $b_0\cong B_0(S)$. It follows from Lemma 2.2 that $k(B_0)=k(b_0)=2^n$ and that

$l(B_0)=l(b_0)=2^m-1$. By the proof of Lemma 2.2, there is an involution $x\in G$ with $l(b_x)=1$. Hence $\chi(x)=\pm 1$ for all $\chi\in \mathrm{Irr}(B_0)$ from Lemma 1.9. Thus, by Corollary 1.8, it is sufficient to show that

$$(*)\qquad \begin{array}{l}\text{if } \tilde{\chi},\ \tilde{\chi}'\in \mathrm{Irr}(b_0) \text{ with } \tilde{\chi}(1)=\tilde{\chi}'(1),\\ \text{then } \tilde{\chi}(x)=\tilde{\chi}'(x)=\pm 1.\end{array}$$

Let $\{\theta_1, \cdots, \theta_{2n-m}\}$ be the set of all irreducible complex characters of Q.

Case 1. $m=2$: By the character table of $L_2(q)$ (cf. [10, Theorem 38.1]), we can write

$$\begin{array}{ccc} & 1 & x \\ \zeta_1 & 1 & 1 \\ \zeta_2 & (q+\varepsilon)/2 & -\varepsilon \\ \zeta_3 & (q+\varepsilon)/2 & -\varepsilon \\ \zeta_4 & q & \varepsilon\,, \end{array} \qquad \varepsilon=\begin{cases}-1 & \text{if } q\equiv 3 \pmod 8\\ \ \ 1 & \text{if } q\equiv 5 \pmod 8\end{cases}$$

where $\{\zeta_1, \cdots, \zeta_4\}=\mathrm{Irr}(B_0(L_2(q)))$. Since $b_0\cong B_0(S)$ and since $S=L_2(q)\times Q$, we may write $\mathrm{Irr}(b_0)=\{\tilde{\chi}_{ij}\mid i=1, \cdots, 4;\ j=1, \cdots, 2^{n-2}\}$ such that $\tilde{\chi}_{ij}|_S=\zeta_i\theta_j$ for all i, j. Then

$$\tilde{\chi}_{ij}(1)=\begin{cases}1 & \text{for } i=1\\ (q+\varepsilon)/2 & \text{for } i=2, 3\\ q & \text{for } i=4\end{cases}$$

and

$$\tilde{\chi}_{ij}(x)=\begin{cases}1 & \text{for } i=1\\ -\varepsilon & \text{for } i=2, 3\\ \varepsilon & \text{for } i=4\,.\end{cases}$$

These imply (*).

Case 2. $m\geqq 3$: By the character table of $SL(2, 2^m)$ (cf. [10, Theorem 38.2]), we know

$$\begin{array}{cccl} & 1 & x & \\ 1 & 1 & 1 & \\ \tilde{\theta}_i & 2^m-1 & -1 & \text{for } i=1, \cdots, 2^{m-1}\\ \tilde{\chi}_j & 2^m+1 & 1 & \text{for } j=1, \cdots, 2^{m-1}-1\end{array}$$

where $\{1, \tilde{\theta}_i, \tilde{\chi}_j\mid i=1, \cdots, 2^{m-1};\ j=1, \cdots, 2^{m-1}-1\}=\mathrm{Irr}(B_0(SL(2, 2^m)))$. Using this we can show (*) as in Case 1. This completes the proof.

Now, the above lemmas imply the next main result of this section.

THEOREM 2.4. *Let P be an abelian Sylow 2-subgroup of G. Assume that e is prime. Then we have the following.*

(1) $l(B_0)=e$. *And if G is nonsolvable. then* $k(B_0)=|P|$.

(2) *When G is nonsolvable, one of the following holds*:

(i) $e=3$, *and* $B_0 \cong B_0(L_2(q)\times(P/(Z_2\times Z_2)))$ *for some* $q>3$ *with* $q\equiv 3$ *or* 5 (mod 8),

(ii) $e=2^m-1$ *for some* $m\geqq 3$, *and* $B_0\cong B_0(SL(2, 2^m)\times(P/(\underbrace{Z_2\times\cdots\times Z_2}_{m})))$.

PROOF. We can assume $O(G)=1$ by Lemma 1.1. Let $S=O'(G)$. Firstly assume that S is solvable. Then $S=P$, so that $C_G(P)=P$. Hence G is a semi-direct product of its normal subgroup P by Z_e. This shows $l(B_0)=e$. So it is enough to consider the case where G is nonsolvable. Since e is prime, $e=e(S)$. By Proposition 1.10 and Lemma 1.12, one of the following two cases occurs:

(i) $e(S)=3$, and $S\cong L_2(q)\times(P/(Z_2\times Z_2))$ for some $q>3$ with $q\equiv 3$ or 5 (mod 8),

(ii) $e(S)=2^m-1$ for some $m\geqq 3$, and $S\cong SL(2, 2^m)\times(P/(\underbrace{Z_2\times\cdots\times Z_2}_{m}))$.

Hence we obtain (1) and (2) from Lemmas 2.2 and 2.3, respectively.

REMARK 2. For the case where G is solvable, the latter half of Theorem 2.4(1) does not hold in general. Indeed, let P be an elementary abelian group of order 16 with $P=\langle x, y, z, w\rangle$. Let $t\in \mathrm{Aut}(P)$ such that $x^t=y$, $y^t=xy$, $z^t=w$ and $w^t=zw$. There is a semi-direct product G of its normal subgroup P by $\langle t\rangle$. Then G is solvable and $e=|G:P|=3$. Since $u^t\neq u$ for all $u\in P-\{1\}$, we shall show that $k(B_0)=8\neq 16$ (cf. Proposition 6.1). As another example, let P be the same as above, and let $t\in\mathrm{Aut}(P)$ with $|t|=5$. If G is a semi-direct product of P by $\langle t\rangle$ and G is not the direct product $P\times Z_5$, then we shall show that $k(B_0)=8\neq 16$ (cf. Proposition 6.3).

3. The case $e=9$

In this section we consider the case when $e=e(S)=9$, where $S=O'(G/O(G))$. We use the notation G, P, n, e and B_0 as in § 2.

LEMMA 3.1. *Let P be an elementary abelian Sylow 2-subgroup of G of order 16. If $e=9$, then $k(B_0)=16$ and $l(B_0)=9$.*

PROOF. By Lemma 1.13, Aut(P) has noncyclic Sylow 3-subgroups of order 9. Hence we may assume that $N_G(P)=\langle s, t, C_G(P)\rangle$ for some $s, t\in N_G(P)$, $P=\langle x, y, z, w\rangle$, $x^s=x$, $y^s=y$, $z^s=w$, $w^s=zw$, $x^t=y$, $y^t=xy$, $z^t=z$ and $w^t=w$. By [10, Lemma 18.5 and Theorems 68.4 and 65.4],

$$k(B_0)=l(B_0)+l(b_x)+l(b_z)+l(b_{xz}).$$

Since $e(C_G(xz))=1$, $l(b_{xz})=1$ from [10, Theorem 18.7 and Corollary 65.3]. Since $e(C_G(x))=e(C_G(z))=3$, it follows from Theorem 2.4 that $l(b_x)=l(b_z)=3$. By [10, Corollary 65.3], $l(B_0)\geqq 2$ since $e=9$. Hence, by Lemma 1.15(2), $k(B_0)=16$, so that $l(B_0)=9$.

LEMMA 3.2. *Let S be a normal subgroup of G of odd index such that $S\cong L_2(q)\times L_2(q')\times(P/(Z_2\times Z_2\times Z_2\times Z_2))$ for some $q, q'>3$ with $q\equiv 3$ or 5 (mod 8) and $q'\equiv 3$ or 5 (mod 8). If $e=9$, then $k(B_0)=2^n$ and $l(B_0)=9$.*

PROOF. We may assume $S=L_2(q)\times L_2(q')\times Q$ where $Q\cong P/(Z_2\times Z_2\times Z_2\times Z_2)$. We use induction on n. If $n=4$, Sylow 2-subgroups of G are elementary abelian of order 16, so that the lemma is proved by Lemma 3.1. Suppose $n>4$. Let R_1 and R_2 be Sylow 2-subgroups of $L_2(q)$ and $L_2(q')$, respectively. We may assume $P=R_1\times R_2\times Q$. We can write $R_1=\{1, x, x^s, x^{s^2}\}$ for some $s\in L_2(q)$ and for an involution $x\in R_1$. Similarly, $R_2=\{1, y, y^t, y^{t^2}\}$ for some $t\in L_2(q')$ and for an involution $y\in R_2$. Since $e=e(S)=9$, we know that $N_G(P)=\langle s, t, C_G(P)\rangle$ and that $N_G(P)/C_G(P)$ is elementary abelian of order 9. Let $Q=\{1=z_1, z_2, \cdots, z_{2^{n-4}}\}$. By [10, Lemma 18.5], $\{z_i, xz_i, yz_i, xyz_i \mid i=1, \cdots, 2^{n-4}\}$ is the set of all representatives of G-conjugate classes of P. Thus, by [10, Theorems 68.4 and 65.4],

$$k(B_0)=l(B_0)+\sum_{i=2}^{2^{n-4}} l(b_{z_i})$$
$$+\sum_{i=1}^{2^{n-4}}\{l(b_{xz_i})+l(b_{yz_i})+l(b_{xyz_i})\}.$$

As in the proof of Lemma 2.2, by induction, we get $l(b_{z_i})=9$ for all $i=2, \cdots, 2^{n-4}$. By Lemma 1.1, we may assume $O(G)=1$. Since $Q\neq 1$, as in the proof of Lemma 2.2, by making use of Z^*-theorem [10, Theorem 67.1], we have $l(B_0)=9$. Since $s\notin C_G(xz_i)$ and since $t\in C_G(xz_i)$, we obtain $e(C_G(xz_i))=3$. Hence $l(b_{xz_i})=3$ for all $i=1, \cdots, 2^{n-4}$ from Theorem 2.4(1). Similarly, by Theorem 2.4(1), $l(b_{yz_i})=3$ for all $i=1, \cdots, 2^{n-4}$. Fix any i with $1\leqq i\leqq 2^{n-4}$. For integers j and k, it is seen that $(xyz_i)^{s^j t^k}=xyz_i$ if and only if $3|j$ and $3|k$. Hence as in the proof of Lemma 2.2, $l(b_{xyz_i})=1$ for all $i=1, \cdots, 2^{n-4}$. Thus $k(B_0)=2^n$. This finishes the proof.

LEMMA 3.3. *Assume as in Lemma 3.2. Then $B_0\cong B_0(S)$.*

PROOF. We use the same notation as in the proof of Lemma 3.2. We prove the lemma by induction on $|G|$. Assume $G\neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index with $S\subseteqq H$. Let $b_0=B_0(H)$. By induction, $b_0 \cong B_0(S)$. By the proof of Lemma 3.2, there is an involution $xy \in G$ with $l(b_{xy})=1$. It follows from Lemmas 3.2 and 1.9 that $\chi(xy)=\pm 1$ for all $\chi \in \mathrm{Irr}(B_0)$. By Lemma 3.2, $k(B_0)=k(b_0)$ and $l(B_0)=l(b_0)$. Thus, by Corollary 1.8, it is enough to prove that

$$(*)\qquad \text{if } \tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_0) \text{ with } \tilde{\chi}(1)=\tilde{\chi}'(1), \text{ then } \tilde{\chi}(xy)=\tilde{\chi}'(xy)=\pm 1.$$

As in the proof of Lemma 2.3 we know the character tables of $L_2(q)$ and $L_2(q')$. Thus we can write

	1	x
η_1	1	1
η_2	$(q+\varepsilon)/2$	$-\varepsilon$
η_3	$(q+\varepsilon)/2$	$-\varepsilon$
η_4	q	ε

$$\varepsilon=\begin{cases} -1 & \text{if } q\equiv 3 \pmod 8 \\ 1 & \text{if } q\equiv 5 \pmod 8 \end{cases}$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\}=\mathrm{Irr}(B_0(L_2(q)))$, and

	1	y
ζ_1	1	1
ζ_2	$(q'+\varepsilon')/2$	$-\varepsilon'$
ζ_3	$(q'+\varepsilon')/2$	$-\varepsilon'$
ζ_4	q'	ε'

$$\varepsilon'=\begin{cases} -1 & \text{if } q'\equiv 3 \pmod 8 \\ 1 & \text{if } q'\equiv 5 \pmod 8 \end{cases}$$

where $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}=\mathrm{Irr}(B_0(L_2(q')))$. Let $\{\theta_1, \cdots \theta_{2n-4}\}$ be the set of all irreducible complex characters of Q. Since $b_0 \cong B_0(S)$, we may write $\mathrm{Irr}(b_0)=\{\tilde{\chi}_{ijk} \mid i=1, \cdots, 4; j=1, \cdots, 4; k=1, \cdots, 2^{n-4}\}$ such that $\tilde{\chi}_{ijk}|_S=\eta_i\zeta_j\theta_k$ for all i, j, k.

Case 1. $\varepsilon=-1$ and $\varepsilon'=1$: In order to show (*) it is enough to prove that $\{1, (q-1)/2, q', (q-1)q'/2, q(q'+1)/2\} \cap \{(q'+1)/2, q, (q-1)(q'+1)/4, qq'\}=\varnothing$ since $\tilde{\chi}_{ijk}(1)=\eta_i(1)\zeta_j(1)$ and $\tilde{\chi}_{ijk}(xy)=\eta_i(x)\zeta_j(y)$ for all i, j, k. We can prove it.

Case 2. $\varepsilon=\varepsilon'=-1$: We know that $\{1, (q-1)/2, (q'-1)/2, (q-1)(q'-1)/4, qq'\} \cap \{q, q', (q-1)q'/2, q(q'-1)/2\}=\varnothing$. This implies (*) as in Case 1.

Case 3. $\varepsilon=\varepsilon'=1$: Since $\{1, q, q', (q+1)(q'+1)/4, qq'\} \cap \{(q+1)/2, (q'+1)/2, (q+1)q'/2, q(q'+1)/2\}=\varnothing$, we can show (*). This completes the proof of the lemma.

The above lemmas imply the next main result of this section.

THEOREM 3.4. *Let P be an abelian Sylow 2-subgroup of G. Assume $e=e(S)$*

$=9$, *where* $S=O'(G/O(G))$. *Then we have the following.*

(1) $k(B_0)=|P|$ *and* $l(B_0)=9$.

(2) $B_0\cong B_0(L_2(q)\times L_2(q')\times(P/(Z_2\times Z_2\times Z_2\times Z_2)))$ *for some* $q, q'>3$ *with* $q\equiv 3$ *or* 5 (mod 8) *and* $q'\equiv 3$ *or* 5 (mod 8).

RROOF. We may assume $O(G)=1$ by Lemma 1.1. Since $e(S)=9$, by Proposition 1.10 and Lemma 1.12, we get that $S\cong L_2(q)\times L_2(q')\times(P/(Z_2\times Z_2\times Z_2\times Z_2))$ for some $q, q'>3$ with $q\equiv 3$ or 5 (mod 8) and $q'\equiv 3$ or 5 (mod 8). Hence we obtain (1) and (2) from Lemmas 3.2 and 3.3, respectively.

4. The case $e=21$

In this section we deal with the case when $e=e(S)=21$, where $S=O'(G/O(G))$. As in §1, let J_1 and $R(q)$ be the Janko's first simple group and the simple groups of Ree type, respectively (cf. [16], [21] and [13]). We use the notation G, P, n, e and B_0 as before.

LEMMA 4.1. *Let* P *be an elementary abelian Sylow 2-subgroup of* G *of order* 8. *If* $e=21$, *then* $k(B_0)=8$ *and* $l(B_0)=5$.

PROOF. By Lemma 1.14, $N_G(P)/C_G(P)$ is noncyclic of order 21. Hence we can write that $N_G(P)=\langle s, t, C_G(P)\rangle$, $P=\{1, x, x^s, x^{s^2}, z, xz, x^s z, x^{s^2}z\}=\{1, z, z^t, \cdots, z^{t^6}\}$ for some $s, t\in N_G(P)$ and involutions $x, z\in P$ with $z^s=z$. Then, by [10, Theorems 68.4 and 65.4], $k(B_0)=l(B_0)+l(b_z)$. Since $e(C_G(z))=3$, $l(b_z)=3$ from Theorem 2.4(1). The calculation of the generalized decomposition matrix of B_0 relative to z is due to J. B. Olsson [18, Theorems 3.15, 3.16 and 3.17]. Let $M=C_G(z)$, $\bar{M}=M/\langle z\rangle$ and $\bar{b}_z=B_0(\bar{M})$. By [10, Theorem 66.3], there is a basic set $\overline{W}$ of $\bar{b}_z$ such that $\overline{W}$ contains the trivial Brauer character and the Cartan matrix of $\bar{b}_z$ with respect to $\overline{W}$ has the form

$$\begin{matrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2. \end{matrix}$$

Then, by [10, Lemma 66.1], there is a basic set W of b_z such that W contains the trivial Brauer character and the Cartan matrix C_z of b_z with respect to W has the form

$$(*) \qquad \begin{matrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4. \end{matrix}$$

We use the following notation here. For an integer $r \geqq 0$ and a p-block B, let $E_B(p^r)$ denote the multiplicity of p^r as an elementary divisor of the Cartan matrix of B. If Q is a p-subgroup of a finite group A and if B is a p-block of A, let $n_B(Q)$ denote the multiplicity of Q as a lower defect group of B (cf. [5]. In [5], $n_B(Q)$ is denoted by $m_B^{(1)}(Q)$). By [8, (89.8) Theorem], $E_{B_0}(8)=1$. Since all involutions in G are conjugate, by [5, (7G)], [18, Proposition 1.2] and [10, Theorem 65.4], we get $E_{B_0}(2)=n_{b_z}(\langle z\rangle)$. Since every lower defect group of a 2-block of G contains all 2-subgroups U of G with $U \subseteqq Z(G)$, by [5, (7G)], $E_{b_z}(2)=n_{b_z}(\langle z\rangle)$. By (*), $E_{b_z}(2)=2$. Thus $E_{B_0}(2)=2$, so that $l(B_0)\geqq 3$. This shows $k(B_0)\geqq 6$. Let $\{\chi_i \mid i=1, \cdots, k(B_0)\}=\mathrm{Irr}(B_0)$. Since $l(b_z)=3$, let $N=(n_{i\alpha})_{\substack{1\leq i\leq k(B_0)\\ 1\leq\alpha\leq 3}}$ be the matrix of the generalized decomposition numbers of B_0 relative to z with respect to W. Since $|z|=2$, every $n_{i\alpha}$ is an integer. By [4, II (7A) and (4C)], $(n_{i1}, n_{i2}, n_{i3})\neq(0, 0, 0)$ for every χ_i. For χ_i, χ_j let $a_{ij}=\sum_{1\leq\alpha,\beta\leq 3} 8n_{i\alpha}u_{\alpha\beta}n_{j\beta}$, where $C_z^{-1}=(u_{\alpha\beta})_{1\leq\alpha,\beta\leq 3}$. By Lemma 1.11 and [4, II (7A) and (5G)], all a_{ii} are odd integers. Hence $n_{i1}+n_{i2}+n_{i3}$ is odd for every χ_i. Let N_α be the α-th column of N for each α, and let $N_\alpha N_\beta=\sum_{i=1}^{k(B_0)} n_{i\alpha}n_{i\beta}$ for all α, β. By [10, Theorem 63.3(2)], ${}^tNN=C_z$ where ${}^tN$ is the transposed matrix of N. So $N_\alpha N_\beta=4$ if $\alpha=\beta$, and $N_\alpha N_\beta=2$ if $\alpha\neq\beta$. Clearly, $12=\mathrm{tr}(C_z)=\sum_{i,\alpha} n_{i\alpha}{}^2$ where $\mathrm{tr}(C_z)$ is the trace of C_z. Then the next three possibilities arise for the nonzero entries of N:

(i) 2 entries are ± 2, and 4 entries are ± 1.

(ii) 1 entry is ± 2, and 8 entries are ± 1.

(iii) 12 entries are ± 1.

By elementary calculations as in [18, Theorems 3.15, 3.16 and 3.17] we can write

$$N=\begin{matrix} \delta_1 & 0 & 0 \\ \delta_2 & 0 & 0 \\ 0 & \delta_3 & 0 \\ 0 & \delta_4 & 0 \\ 0 & 0 & \delta_5 \\ 0 & 0 & \delta_6 \\ \delta_7 & \delta_7 & \delta_7 \\ \delta_8 & \delta_8 & \delta_8 \end{matrix}$$

where $\delta_i=\pm 1$. This shows $k(B_0)=8$, so that $l(B_0)=5$. This completes the proof.

LEMMA 4.2. *Let S be a normal subgroup of G of odd index such that $S\cong J_1\times(P/(Z_2\times Z_2\times Z_2))$ or $S\cong R(q)\times(P/(Z_2\times Z_2\times Z_2))$. If $e=21$, then $k(B_0)=2^n$ and $l(B_0)=5$.*

PROOF. We may assume $S=R\times Q$ where $R=J_1$ or $R(q)$ and $Q\cong P/(Z_2\times Z_2\times Z_2)$. Let T be a Sylow 2-subgroup of R with $T\times Q=P$. By Lemma 1.12(iii), $N_R(T)/C_R(T)$ is noncyclic of order 21. Hence we can write $N_R(T)=\langle s, t, C_R(T)\rangle$ and $T=\{1, x, x^s, x^{s^2}, z, xz, x^s z, x^{s^2}z\}=\{1, x, x^t, \cdots, x^{t^6}\}$ for some $s, t\in N_R(T)$ and for involutions $x, z\in T$ with $z^s=z$. Since $e=21$, $N_G(P)=\langle s, t, C_G(P)\rangle$. We prove the lemma by induction on n. If $n=3$, the lemma is proved from Lemma 4.1 because $P=T$ and P is elementary abelian of order 8 from Lemma 1.12(iii). Suppose $n>3$. Let $Q=\{1=y_1, y_2, \cdots, y_{2^{n-3}}\}$. By [10, Lemma 18.5], $\{y_i, zy_i \mid i=1, \cdots, 2^{n-3}\}$ is the set of all representatives of G-conjugate classes of P. Then, by [10, Theorems 68.4 and 65.4],

$$k(B_0)=l(B_0)+\sum_{i=2}^{2^{n-3}} l(b_{y_i})+\sum_{i=1}^{2^{n-3}} l(b_{zy_i}).$$

As in the proof of Lemma 2.2, by induction we get $l(b_{y_i})=5$ for all $i=2, \cdots, 2^{n-3}$. We can assume $O(G)=1$ by Lemma 1.1. Since $Q\neq 1$, it follows from Z^*-theorem that $l(B_0)=5$. Since $s\in C_G(zy_i)$ and $t\notin C_G(zy_i)$, we have $e(C_G(zy_i))=3$. Hence $l(b_{zy_i})=3$ for all $i=1, \cdots, 2^{n-3}$ from Theorem 2.4(1). Thus $k(B_0)=2^n$.

LEMMA 4.3. *Let S be a normal subgroup of G of odd index such that $S\cong J_1\times(P/(Z_2\times Z_2\times Z_2))$. If $e=21$, then $B_0\cong B_0(S)$.*

PROOF. We can assume $S=J_1\times Q$ where $Q\cong P/(Z_2\times Z_2\times Z_2)$. We use induction on $|G|$. Assume $G\neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index l with $S\subseteq H$. Let $b_0=B_0(H)$. By induction, $b_0\cong B_0(S)$. Let s, t, x, z and y_i be the same as in the proof of Lemma 4.2. Since z is an involution in J_1, by [16, Theorem], $C_{J_1}(z)=A_5\times\langle z\rangle$ where A_5 is the alternating group of degree 5. Hence $C_S(z)=A_5\times\langle z\rangle\times Q$. Let $M=C_G(z)$. Clearly $C_S(z)\cong A_5\times(P/(Z_2\times Z_2))$ and $C_S(z)$ is a normal subgroup of M of odd index. By the proof of Lemma 4.2, $e(M)=3$. Hence, by Lemma 2.3, we get that $b_z=B_0(M)\cong B_0(A_5\times(P/Z_2\times Z_2)))$ since $A_5\cong L_2(5)$. By Lemma 1.16(ii), the Cartan matrix of $B_0(A_5)$ has the form

$$\begin{array}{cccc} & 1 & & \\ 1 & 4 & 2 & 2 \\ & 2 & 2 & 1 \\ & 2 & 1 & 2 . \end{array}$$

Thus, by [10, Lemma 66.1], the Cartan matrix C_z of b_z has the form

$$\begin{array}{cccc} & 1 & & \\ 1 & 2^n & 2^{n-1} & 2^{n-1} \\ & 2^{n-1} & 2^{n-1} & 2^{n-2} \\ & 2^{n-1} & 2^{n-2} & 2^{n-1} . \end{array}$$

By Lemma 4.2, $k(B_0)=2^n$. Let $\{\chi_1, \cdots, \chi_{2^n}\}=\mathrm{Irr}(B_0)$. We can write $\mathrm{IBr}(b_z)=\{\phi_1^z=1_M, \phi_2^z, \phi_3^z\}$ with $\phi_2^z(1)=\phi_3^z(1)=2$ from Lemma 1.16(i). For each χ_i and ϕ_α^z, let $n_{i\alpha}=d_{i\alpha}^z$ be the generalized decomposition number of B_0 relative to z. Since $|z|=2$, every $n_{i\alpha}$ is an integer. Let $N=(n_{i\alpha})_{1\leq i\leq 2^n, 1\leq\alpha\leq 3}$, $N_\alpha=(n_{i\alpha})_{1\leq i\leq 2^n}$ for each α, and $N_\alpha N_\beta=\sum_{i=1}^{2^n} n_{i\alpha}n_{i\beta}$ for each α, β. It follows from [10, Theorems 63.3(2), 63.2 and 65.4] that $N_1N_1=2^n$, $N_2N_2=N_3N_3=2^{n-1}$, $N_1N_2=N_1N_3=2^{n-1}$ and $N_2N_3=2^{n-2}$. For each χ_i, χ_j, let $a_{ij}=\sum_{1\leq\alpha,\beta\leq 3} 2^n n_{i\alpha}u_{\alpha\beta}n_{j\beta}$, where $C_z^{-1}=(u_{\alpha\beta})_{1\leq\alpha,\beta\leq 3}$. Then

$$\begin{aligned} a_{ii}&=3n_{i1}^2+4(n_{i2}^2+n_{i3}^2)-4(n_{i1}n_{i2}+n_{i1}n_{i3}) \\ &\equiv n_{i1}^2\equiv n_{i1} \pmod 2 \end{aligned}$$

for all χ_i. By Lemma 1.11, every χ_i has height zero. Hence, by [4, II (7A) and (5G)], every a_{ii} is odd, so that n_{i1} is odd for all $i=1, \cdots, 2^n$. Since $N_1N_1=2^n$, $n_{i1}=\pm 1$ for all $i=1, \cdots, 2^n$. Let $\delta_i=n_{i1}$ and $u_i=n_{i2}\delta_i$ for each i. Since $N_1N_2=N_2N_2=2^{n-1}$, $\sum_{i=1}^{2^n}u_i=\sum_{i=1}^{2^n}u_i^2$. Thus, $u_i=1$ or 0 for all $i=1, \cdots, 2^n$. Hence exactly 2^{n-1} u_i's are 1 and the other u_i's are 0 since $N_1N_2=2^{n-1}$. Then we may assume

$$n_{i2}=\begin{cases} \delta_i & \text{for } i=1, \cdots, 2^{n-1} \\ 0 & \text{for } i=2^{n-1}+1, \cdots, 2^n. \end{cases}$$

Similarly, exactly 2^{n-1} $(n_{i3}\delta_i)$'s are 1 and the other $(n_{i3}\delta_i)$'s are 0. Since $N_2N_3=2^{n-2}$, we may assume

$$n_{i3}=\begin{cases} \delta_i & \text{for } i=1, \cdots, 2^{n-2} \text{ and for } i=2^{n-1}+1, \cdots, 3\cdot 2^{n-2} \\ 0 & \text{for } i=2^{n-2}+1, \cdots, 2^{n-1} \text{ and for } i=3\cdot 2^{n-2}+1, \cdots, 2^n. \end{cases}$$

Since $\chi_i(z)=n_{i1}+2(n_{i2}+n_{i3})$ for each i, we get

$$\chi_i(z)=\begin{cases} \pm 5 & \text{for } i=1, \cdots, 2^{n-2} \\ \pm 3 & \text{for } i=2^{n-2}+1, \cdots, 3\cdot 2^{n-2} \\ \pm 1 & \text{for } i=3\cdot 2^{n-2}+1, \cdots, 2^n. \end{cases}$$

Let $C_G(P)=P\times V$. When $G=VH$, $G=C_G(P)\cdot H$, so that $B_0\cong b_0$ from Proposition 1.6. Thus, we may assume $G\neq VH$. Hence $C_H(P)=P\times V$. Since $b_0\cong B_0(S)$, it follows from Proposition 1.5 that $|H: VH'|=k'(b_0)=2^{n-3}$. By [10, Theorem 18.4], $P\cap G'=\{1, x, x^t, \cdots, x^{t^6}\}$. Then the order of Sylow 2-subgroups of G' is 8. This implies $2^{n-3} \mid |G: VG'|$ and $2^{n-2} \nmid |G: VG'|$. Thus, by Proposition 1.5, $k'(B_0)=|G: VG'|=l\cdot 2^{n-3}$ where $l=|G: H|$. Since $b_0\cong B_0(S)$, by Clifford's theorem, Proposition 1.3 and the character table of J_1 [16, p. 148], we get that $\chi_i(z)=1$ for every $\chi_i\in\mathrm{Irr}(B_0)$ with degree one. These show that the number of

$\chi_i \in \mathrm{Irr}(B_0)$ with $\chi_i(z)=1$ is at least $l\cdot 2^{n-3}$. However, $\chi_i(z)=\pm 1$ only for $i=3\cdot 2^{n-2}+1, \cdots, 2^n$. This is a contradiction since $l\cdot 2^{n-3}>2^{n-2}$. This completes the proof.

LEMMA 4.4. *Let S be a normal subgroup of G of odd index such that $S\cong R(q)\times(P/(Z_2\times Z_2\times Z_2))$. If $e=21$, then $B_0\cong B_0(S)$.*

PROOF. Let $R=R(q)$. We may assume $S=R\times Q$ where $Q\cong P/(Z_2\times Z_2\times Z_2)$. We prove the lemma by induction on $|G|$. Assume $G\neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index l with $S\subseteq H$. Let $b_0=B_0(H)$. By induction, $b_0\cong B_0(S)$. Let s, t, x, z and y_i be the same as in the proof of Lemma 4.2. Since z is an involution in R, $C_R(z)=L_2(q)\times\langle z\rangle$ from [21, p. 62 III]. (It is noted that we use the notation $R(q)$ as in the sense of [13]). Hence $C_S(z)=L_2(q)\times\langle z\rangle\times Q$. Let $M=C_G(z)$. Then $C_S(z)$ is a normal subgroup of M of odd index and $C_S(z)\cong L_2(q)\times(P/(Z_2\times Z_2))$. By the proof of Lemma 4.2, $e(M)=3$. Then, by Lemma 2.3, $b_z=B_0(M)\cong B_0(L_2(q)\times(P/(Z_2\times Z_2)))$. By [21, Theorem (1)], $3<q\equiv 3 \pmod 8$, so that as in the proof of Lemma 4.3 the Cartan matrix C_z of b_z has the form

$$\begin{matrix} 2^{n-1} & 2^{n-2} & 2^{n-2} \\ 2^{n-2} & 2^{n-1} & 2^{n-2} \\ 2^{n-2} & 2^{n-2} & 2^{n-1}. \end{matrix}$$

By Lemma 4.2, $k(B_0)=2^n$. Let $\{\chi_1, \cdots, \chi_{2^n}\}=\mathrm{Irr}(B_0)$. We can write $\mathrm{IBr}(b_z)=\{\phi_1^z=1_M, \phi_2^z, \phi_3^z\}$ with $\phi_2^z(1)=\phi_3^z(1)=(q-1)/2$ from Lemma 1.16(i). Let $n_{i\alpha}$, N, N_α and $N_\alpha N_\beta$ be the same as in the proof of Lemma 4.3. Every $n_{i\alpha}$ is an integer. As in the proof of Lemma 4.3 we get $N_\alpha N_\alpha=2^{n-1}$ for all $\alpha=1, 2, 3$, and $N_\alpha N_\beta=2^{n-2}$ if $\alpha\neq\beta$. Let $C_G(P)=P\times V$. As in the proof of Lemma 4.3 we may assume $G\neq VH$. Since $b_0\cong B_0(S)$, $k'(b_0)=2^{n-3}$. So that $k'(B_0)=|G:VG'|=l\cdot 2^{n-3}$ as in the proof of Lemma 4.3, where $l=|G:H|$. Since $b_0\cong B_0(S)$, by [21, p. 74 and pp. 87-88], we can write $\{\tilde{\chi}_{ij}\,|\, i=1, \cdots, 8;\ j=1, \cdots, 2^{n-3}\}=\mathrm{Irr}(b_0)$ and

	1	z
$\tilde{\chi}_{1j}$	1	1
$\tilde{\chi}_{2j}$	q^2-q+1	-1
$\tilde{\chi}_{3j}$	q^3	q
$\tilde{\chi}_{4j}$	$q(q^2-q+1)$	$-q$
$\tilde{\chi}_{5j}$	$(q-1)m(q+1+3m)/2$	$-(q-1)/2$
$\tilde{\chi}_{6j}$	$(q-1)m(q+1+3m)/2$	$-(q-1)/2$
$\tilde{\chi}_{7j}$	$(q-1)m(q+1-3m)/2$	$(q-1)/2$
$\tilde{\chi}_{8j}$	$(q-1)m(q+1-3m)/2$	$(q-1)/2$

for $j=1, \cdots, 2^{n-3}$, where $q=3^{2k+1}$ and $m=3^k$ for some $k\geqq 1$ (cf. [21, Theorem]). By Clifford's theorem, Proposition 1.3 and the above table, we know that if $\chi_i(1)=1$ then $\chi_i(z)=1$. When $n_{i1}=0$, $\chi_i(z)=(n_{i2}+n_{i3})(q-1)/2$. Thus $n_{i1}\neq 0$ if $\chi_i(z)=\pm 1$. Hence the number of $\chi_i\in \mathrm{Irr}(B_0)$ with $n_{i1}\neq 0$ is at least $l\cdot 2^{n-3}$. Since $N_1N_1=2^{n-1}$, we get $l=3$. Fix any χ_i. If $\chi_i|_H=\tilde{\chi}_{2j}$ for some j with $1\leqq j\leqq 2^{n-3}$, then $n_{i1}{}^2\geqq 1$ since $\chi_i(z)=-1$. If $\chi_i|_H=\tilde{\chi}_{2j}+\tilde{\chi}_{2j'}+\tilde{\chi}_{2j''}$ for some j, j', j'' with $1\leqq j<j'<j''\leqq 2^{n-3}$, then $n_{i1}{}^2\geqq 9$ since $\chi_i(z)=-3$. Let u be the number of $\chi_i\in \mathrm{Irr}(B_0)$ with $\chi_i|_H=\tilde{\chi}_{2j}$, and let v be the number of $\chi_i\in \mathrm{Irr}(B_0)$ with $\chi_i|_H=\tilde{\chi}_{2j}+\tilde{\chi}_{2j'}+\tilde{\chi}_{2j''}$ for $j<j'<j''$. Since $N_1N_1=2^{n-1}$, and since $1<q^2-q+1<3(q^2-q+1)$, we have

$$2^{n-1}=\textstyle\sum_{i=1}^{2^n} n_{i1}{}^2\geqq k'(B_0)+u+9v=3\cdot 2^{n-3}+u+9v\,.$$

Then $2^{n-3}\geqq u+9v$. By Proposition 1.4, for every $\tilde{\chi}_{2j}$ there is some χ_i with $(\chi_i|_H, \tilde{\chi}_{2j})\neq 0$, so that, by Clifford's theorem and Proposition 1.3, $\chi_i|_H=\tilde{\chi}_{2j}$ or $\chi_i|_H=\tilde{\chi}_{2j}+\tilde{\chi}_{2j}{}^g+\tilde{\chi}_{2j}{}^{g^2}$ where g is an element of G with $G=\langle g, H\rangle$. By considering the degrees of $\tilde{\chi}_{ij}$, we get that $\tilde{\chi}_{2j}{}^g$ and $\tilde{\chi}_{2j}{}^{g^2}$ are both in $\{\tilde{\chi}_{2j'}\mid j'=1, \cdots, 2^{n-3}\}$. Thus $2^{n-3}\leqq u+3v$, so that $v=0$ and $u=2^{n-3}$. This implies that the number of $\chi_i\in \mathrm{Irr}(B_0)$ with $\chi_i(z)=-1$ is at least 2^{n-3}, so that the number of $\chi_i\in \mathrm{Irr}(B_0)$ with $\chi_i(z)=\pm 1$ is at least 2^{n-1}. Then the number of $\chi_i\in \mathrm{Irr}(B_0)$ with $n_{i1}\neq 0$ is at least 2^{n-1}. Since $N_1N_1=2^{n-1}$, we may assume

$$n_{i1}=\begin{cases}\delta_i & \text{for } i=1, \cdots, 2^{n-1}\\ 0 & \text{for } i=2^{n-1}+1, \cdots, 2^n\end{cases}$$

where $\delta_i=\pm 1$. Thus $\chi_i(z)=\pm 1$ for all $i=1, \cdots, 2^{n-1}$. For all $i=1, \cdots, 2^{n-1}$, $\chi_i(z)=\delta_i+(n_{i2}+n_{i3})(q-1)/2$, so that $n_{i2}+n_{i3}=0$ since $(q-1)/2\geqq 13$. Consequently, $N_1N_2+N_1N_3=\sum_{i=1}^{2^n} n_{i1}(n_{i2}+n_{i3})=\sum_{i=1}^{2^{n-1}}\delta_i(n_{i2}+n_{i3})=0$. This is a contradiction since $N_1N_2=N_1N_3=2^{n-2}$. This completes the proof.

LEMMA 4.5. *Let S be a normal subgroup of G of odd index such that $S\cong L_2(q)\times SL(2, 8)$ for some $q>3$ with $q\equiv 3$ or 5 (mod 8). If $e=21$, then $B_0\cong B_0(S)$.*

PROOF. Let R_1 and R_2 be Sylow 2-subgroups of $L_2(q)$ and $SL(2, 8)$, respectively. We may assume $S=L_2(q)\times SL(2, 8)$ and $P=R_1\times R_2$. There are an element $s\in L_2(q)$ and an involution $x\in R_1$ with $R_1=\{1, x, x^s, x^{s^2}\}$. Similarly, we can write $R_2=\{1, y, y^t, \cdots, y^{t^6}\}$ for some $t\in SL(2, 8)$ and for an involution $y\in R_2$. Since $e=21$, $N_G(P)=\langle s, t, C_G(P)\rangle$ and $N_G(P)/C_G(P)$ is cyclic of order 21. By [10, Lemma 18.5], $\{1, x, y, xy\}$ is the set of all representatives of G-conjugate classes of P. Hence, by [10, Theorems 68.4 and 65.4], $k(B_0)=l(B_0)+l(b_x)$

$+l(b_y)+l(b_{xy})$. Since $s\notin C_G(x)$ and $t\in C_G(x)$, we have $e(C_G(x))=7$. Thus $l(b_x)=7$ from Theorem 2.4(1). Similarly, $l(b_y)=3$ from Theorem 2.4(1). For integers i and j, $(xy)^{s^i t^j}=xy$ if and only if $3|i$ and $7|j$. This implies $N_M(P)=C_M(P)$ where $M=C_G(xy)$. Thus, by [10, Theorem 18.7 and Corollary 65.3], $l(b_{xy})=l(B_0(M))=1$. Since G is nonsolvable, $l(B_0)\geqq 2$ from [10, Corollary 65.3], so that $k(B_0)\geqq 13$.

Now, we prove the lemma by induction on $|G|$. Assume $G\neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index l with $S\subseteqq H$. Let $b_0=B_0(H)$. We know $b_0\cong B_0(S)$ by induction. From the character tables of $L_2(q)$ and $SL(2, 8)$ (cf. [10, Theorems 38.1 and 38.2]), we can write

	1	x
θ_1	1	1
θ_2	$(q+\varepsilon)/2$	$-\varepsilon$
θ_3	$(q+\varepsilon)/2$	$-\varepsilon$
θ_4	q	ε

$$\varepsilon=\begin{cases}-1 & \text{if } q\equiv 3 \pmod 8\\ \ \ 1 & \text{if } q\equiv 5 \pmod 8\end{cases}$$

where $\{\theta_1, \theta_2, \theta_3, \theta_4\}=\mathrm{Irr}(B_0(L_2(q)))$ and

	1	y	
ζ_1	1	1	
ζ_j	7	-1	for $j=2, 3, 4, 5$
ζ_j	9	1	for $j=6, 7, 8$

where $\{\zeta_1, \cdots, \zeta_8\}=\mathrm{Irr}(B_0(SL(2, 8)))$. Since $b_0\cong B_0(S)$, we may write $\mathrm{Irr}(b_0)=\{\tilde{\chi}_{ij}|i=1, \cdots, 4; j=1, \cdots, 8\}$ with $\tilde{\chi}_{ij}|_S=\theta_i\zeta_j$ for all i, j. Hence the degrees of all $\tilde{\chi}_{ij}$ are 1, 7, 9, $(q+\varepsilon)/2$, $7(q+\varepsilon)/2$, $9(q+\varepsilon)/2$, q, $7q$ and $9q$. Next, we want to show that

$$(*)\qquad \begin{array}{l}\text{if } \tilde{\chi}, \tilde{\chi}'\in\mathrm{Irr}(b_0) \text{ with } \tilde{\chi}(1)=\tilde{\chi}'(1)\\ \text{then } \tilde{\chi}(xy)=\tilde{\chi}'(xy)=\pm 1.\end{array}$$

Case 1. $\varepsilon=1$: Clearly $\{1, 9, 7(q+1)/2, 9, 9q\}\cap\{7, (q+1)/2, 9(q+1)/2, 7q\}=\varnothing$. Hence, by considering the values $\tilde{\chi}_{ij}(1)$ and $\tilde{\chi}_{ij}(xy)$, we get (*).

Case 2. $\varepsilon=-1$: Since $\{1, 9, (q-1)/2, 9(q-1)/2, 7q\}\cap\{7, 7(q-1)/2, q, 9q\}=\varnothing$, we obtain (*) as in Case 1.

We get from Clifford's theorem, Proposition 1.3, (*) and the above character tables of $L_2(q)$ and $SL(2, 8)$ that $\chi(xy)=\pm 1$ or $\pm l$ for every $\chi\in\mathrm{Irr}(B_0)$. Let $k=k(B_0)$, and let m be the number of $\chi\in\mathrm{Irr}(B_0)$ with $\chi(xy)=\pm 1$. Hence we can write $\mathrm{Irr}(B_0)=\{\chi_1=1_G, \chi_2, \cdots, \chi_m, \chi_{m+1}, \cdots, \chi_k\}$ such that

$$\chi_i(xy)=\begin{cases}\pm 1 & \text{for } i=1, \cdots, m \\ \pm l & \text{for } i=m+1, \cdots, k.\end{cases}$$

Since $l(b_{xy})=1$, as in the proof of Lemma 1.9,

(**) $$32=\sum_{i=1}^{k}\chi_i(xy)^2=m+(k-m)l^2.$$

Firstly, suppose $k=m$. Then $\chi_i(xy)=\pm 1$ for all $\chi_i\in \mathrm{Irr}(B_0)$. Since $k=m=32$ and since $b_0\cong B_0(S)$, we have $k(B_0)=k(b_0)=32$. Hence $l(B_0)=21$, so that $l(B_0)=l(b_0)$ since $b_0\cong B_0(S)$. Thus, by (*) and Corollary 1.8, $B_0\cong b_0$. Thus, we may assume $k>m$. Since $k\geqq 13$, by (**), $l=3$. So that $k-m=1$ or 2. Let $C_G(P)=P\times V$. Since $k>m$ and $b_0\cong B_0(S)$, we know $B_0\not\cong b_0$. Hence $G\neq VH$ from Proposition 1.6. This shows $C_H(P)=P\times V$. Thus, by Proposition 1.5, $|H:VH'|=k'(b_0)=1$ since $b_0\cong B_0(S)$. Then $H=VH'$. Since G/H is cyclic, $VG'=VH'=H$. Hence $k'(B_0)=|G:VG'|=l=3$ by Proposition 1.5. So that we may assume that $\chi_1(1)=\chi_2(1)=\chi_3(1)=1$ and $\chi_i(1)>1$ for all $i=4, \cdots, k$.

Case A. $k-m=1$: By (**), we get $m=23$ and $k=24$. Then $\chi_i(xy)=\pm 1$ for $i=1, \cdots, 23$ and $\chi_{24}(xy)=\pm 3$. Since $b_0\cong B_0(S)$, by Clifford's theorem and Proposition 1.3, $\chi_i|_H=1_H$ for $i=1, 2, 3$, $\chi_i|_H\neq 1_H$ and $\chi_i|_H\in \mathrm{Irr}(b_0)$ for $i=4, \cdots, 23$, $\chi_{24}|_H=\tilde{\chi}_{i_1}+\tilde{\chi}_{i_2}+\tilde{\chi}_{i_3}$ where $\tilde{\chi}_{i_1}$, $\tilde{\chi}_{i_2}$ and $\tilde{\chi}_{i_3}$ are distinct G-conjugate elements in $\mathrm{Irr}(b_0)$. On the other hand, it follows from Proposition 1.4 that for every $\tilde{\chi}\in \mathrm{Irr}(b_0)$ there is some $\chi_i\in \mathrm{Irr}(B_0)$ with $(\chi_i|_H, \tilde{\chi})\neq 0$. These show $k(b_0)\leqq 1+20+3=24$. But $k(b_0)=32$ since $b_0\cong B_0(S)$. Then we have a contradiction.

Case B. $k-m=2$: We have from (**) that $\chi_i(xy)=\pm 1$ for $i=1, \cdots, 14$, $\chi_{15}(xy)=\pm 3$ and $\chi_{16}(xy)=\pm 3$. Hence as in Case A we get $k(b_0)\leqq 1+11+6=18$. This is a contradiction as in Case A. This completes the proof.

LEMMA 4.6. *Let S be a normal subgroup of G of odd index such that $S\cong L_2(q)\times SL(2, 8)\times (P/(Z_2\times Z_2\times Z_2\times Z_2\times Z_2))$ for some $q>3$ with $q\equiv 3$ or* 5 (mod 8). *If $e=21$, then $k(B_0)=2^n$ and $l(B_0)=21$.*

PROOF. If $n=5$, we can prove the lemma by Lemma 4.5 (cf. Lemma 1.12 and Theorem 2.4). If $n>5$, we can verify the lemma by induction on n as in the proof of Lemma 4.2.

LEMMA 4.7. *Assume as in Lemma* 4.6. *Then $B_0\cong B_0(S)$.*

PROOF. We may assume $S=L_2(q)\times SL(2, 8)\times Q$ with $Q\cong P/(Z_2\times Z_2\times Z_2\times Z_2\times Z_2)$. We use induction on $|G|$ as before. Assume $G\neq S$. Hence G has a normal subgroup H of odd prime index with $S\subseteqq H$ from [12, Theorem]. Let $b_0=B_0(H)$. By induction, $b_0\cong B_0(S)$. Let x and y be involutions in $L_2(q)$ and

$SL(2, 8)$, respectively. As in the proof of Lemma 4.2, $l(b_{xy})=1$. By Lemma 4.6, $k(B_0)=2^n$. Thus, $\chi(xy)=\pm 1$ for all $\chi \in \mathrm{Irr}(B_0)$ from Lemma 1.9. By Lemma 4.6, $k(B_0)=k(b_0)$ and $l(B_0)=l(b_0)$. Since $b_0 \cong B_0(S)$, as in the proof of Lemma 4.5, we get that if $\tilde{\chi}, \tilde{\chi}' \in \mathrm{Irr}(b_0)$ with $\tilde{\chi}(1)=\tilde{\chi}'(1)$ then $\tilde{\chi}(xy)=\tilde{\chi}'(xy)=\pm 1$. These imply $B_0 \cong b_0$ from Corollary 1.8. This completes the proof.

Next, we state the following main result of this section. That is proved by making use of Lemmas 4.2–4.7.

THEOREM 4.8. *Let P be an abelian Sylow 2-subgroup of G, and let $S=O'(G/O(G))$. If $e=e(S)=21$, then we have the following.*

(1) $k(B_0)=|P|$ *and*

$$l(B_0)=\begin{cases} 5 & \text{if } N_G(P)/C_G(P) \text{ is noncyclic} \\ 21 & \text{if } N_G(P)/C_G(P) \text{ is cyclic.} \end{cases}$$

(2) *One of the following holds:*

(i) $B_0 \cong B_0(J_1 \times (P/(Z_2 \times Z_2 \times Z_2)))$,

(ii) $B_0 \cong B_0(R(q) \times (P/(Z_2 \times Z_2 \times Z_2)))$,

(iii) $B_0 \cong B_0(L_2(q) \times SL(2, 8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2)))$ *for some* $q>3$ *with* $q \equiv 3$ *or* $5 \pmod 8$.

PROOF. By Lemma 1.1, we may assume $O(G)=1$. By Proposition 1.10 and Lemma 1.12, one of the following holds:

(i) $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$,

(ii) $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$,

(iii) $S \cong L_2(q) \times SL(2, 8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q>3$ with $q \equiv 3$ or $5 \pmod 8$. Then we can prove the theorem by Lemmas 4.2–4.7.

5. The case when P is elementary abelian of order 8

In this section we consider the case when G has elementary abelian Sylow 2-subgroups of order 8. In particular, we shall determine B_0 in the case when G is nonsolvable, $e=21$ and $e(S) \neq 21$ where $S=O'(G/O(G))$. Throughout this section we assume that G has an elementary abelian Sylow 2-subgroup P of order 8 and we use the notation e and B_0 as before.

By Lemma 1.14 and Remark 1 of § 1, it is sufficient to consider the cases when $e=3$, 7 and 21.

PROPOSITION 5.1. (i) *If $e=3$, then $k(B_0)=8$ and $l(B_0)=3$.*

(ii) *If $e=7$, then $k(B_0)=8$ and $l(B_0)=7$.*

(iii) *If* $e=21$, *then* $k(B_0)=8$ *and* $l(B_0)=5$.

PROOF. (i) We can write $N_G(P)=\langle s, C_G(P)\rangle$ for some $s\in N_G(P)$. There is an involution $x\in P$ with $x^s\neq x$. Hence $l(b_x)=1$ as in the proof of Lemma 2.1. Then $k(B_0)=8$ from Lemma 1.15(1). On the other hand, $l(B_0)=3$ by Theorem 2.4(1).

(ii) We can verify (ii) as in (i).

(iii) We have already proved (iii) in Lemma 4.1.

PROPOSITION 5.2. *There is a basic set* W *of* B_0 *such that* W *contains the trivial Brauer character and the decomposition matrix of* B_0 *with respect to* W *has the form*

$$
\begin{array}{cccc}
1_G & 1 & 0 & 0 \\
 & \delta_2 & 0 & 0 \\
 & 0 & \delta_3 & 0 \\
 & 0 & \delta_4 & 0 \\
 & 0 & 0 & \delta_5 \\
 & 0 & 0 & \delta_6 \\
 & \delta_7 & \delta_7 & \delta_7 \\
 & \delta_8 & \delta_8 & \delta_8 ,
\end{array}
\qquad
\begin{array}{ccccccc}
1 & & & & & & \\
 & \delta_2 & & & & & \\
 & & \delta_3 & & \mathbf{0} & & \\
 & & & \ddots & & & \\
 & & \mathbf{0} & & \ddots & & \\
 & & & & & \ddots & \\
 & & & & & & \delta_7 \\
\delta_8 & \delta_8 & \delta_8 & \cdots & \cdots & \cdots & \delta_8 ,
\end{array}
\qquad
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 & 0 \\
0 & 0 & \delta_3 & 0 & 0 \\
0 & 0 & 0 & \delta_4 & 0 \\
0 & 0 & 0 & 0 & \delta_5 \\
\delta_6 & 0 & 0 & \delta_6 & \delta_6 \\
0 & \delta_7 & 0 & \delta_7 & \delta_7 \\
0 & 0 & \delta_8 & \delta_8 & \delta_8
\end{array}
$$

$$e=3 \qquad\qquad e=7 \qquad\qquad e=21$$

where $\delta_i=\pm 1$.

PROOF. *Case* 1. $e=3$: Clear from Proposition 5.1(i) and the proof of Lemma 4.1.

Case 2. $e=7$: By Proposition 5.1(ii), $k(B_0)=8$. Let $\{\chi_1, \cdots, \chi_8\}=\mathrm{Irr}(B_0)$. By the proof of Proposition 5.1(ii), G has an involution x with $l(b_x)=1$. By Lemma 1.9, we get $\chi_i(x)=\pm 1$ for all i. On the other hand, $\sum_{i=1}^{8}\chi_i(x)\chi_i=0$ on $2'$-elements of G from [10, Theorem 63.3(1)]. Thus, the assertion is proved.

Case. 3 $e=21$: Let z be an involution in G. By the proof of Lemma 4.1, the generalized decomposition matrix of B_0 relative to z with respect to some basic set of b_z has the same form as in Case 1. Hence, by [10, Theorem 63.3(1)], we can verify the proposition.

LEMMA 5.3. *Assume* $e=21$, $O(G)=1$, $O'(G)=SL(2, 8)$ *and* G *has a normal subgroup* H *of odd prime index with* $e(H)=7$. *Then for any involution* z *in* G *we get*

$$\chi_i(1)=\begin{cases} 1 & \textit{for } i=1,2,3 \\ 7 & \textit{for } i=4,5,6 \\ 21 & \textit{for } i=7 \\ 27 & \textit{for } i=8, \end{cases} \qquad \chi_i(z)=\begin{cases} 1 & \textit{for } i=1,2,3 \\ -1 & \textit{for } i=4,5,6 \\ -3 & \textit{for } i=7 \\ 3 & \textit{for } i=8 \end{cases}$$

where $\{\chi_1=1_G, \chi_2, \cdots, \chi_8\}=\mathrm{Irr}(B_0)$.

PROOF. Let $S=O'(G)=SL(2, 8)$. By Lemmas 1.12 and 1.14, we can write $N_S(P)=\langle s, C_S(P)\rangle$, $N_H(P)=\langle s, C_H(P)\rangle$ and $N_G(P)=\langle s, t, C_G(P)\rangle$ for some $s\in N_S(P)$ and $t\in N_G(P)$ such that s and t have orders 7 and 3 modulo $C_G(P)$, respectively. Clearly, $G/H=\langle tH\rangle$. Let $b_0=B_0(H)$, and let $C_G(P)=P\times V$. By Proposition 5.2, $B_0\not\equiv b_0$. Hence $VH=H$ from Proposition 1.6. Then $C_G(P)=C_H(P)$ and $|G:H|=3$. We may assume $z\in P$. Let $M=C_G(z)$. By the proof of Lemma 2.1, $C_S(z)$ is a 2-nilpotent normal subgroup of M, so that M is solvable. By the proof of Lemma 4.1, $e(M)=3$. Thus, by Lemma 1.1, $B_0(M)\cong B_0(P\cdot Z_3)$ where $P\cdot Z_3$ is the semi-direct product of its normal subgroup P by Z_3 and it is not the direct product $P\times Z_3$. Thus, as in the proof of Lemma 4.1 we know the generalized decomposition numbers of B_0 relative to z. So we can write

$$(*) \qquad \chi_i(z)=\begin{cases} \pm 1 & \text{for } i=1, \cdots, 6 \\ \pm 3 & \text{for } i=7, 8 \end{cases}$$

for suitable indexing of $\chi_2, \cdots, \chi_8$. By Lemma 2.1, $b_0\cong B_0(S)$. Hence, by [10, Theorem 38.2],

$$(**) \qquad \tilde{\chi}_i(1)=\begin{cases} 1 & \text{for } i=1 \\ 7 & \text{for } i=2, \cdots, 5 \\ 9 & \text{for } i=6, 7, 8 \end{cases} \qquad \tilde{\chi}_i(z)=\begin{cases} 1 & \text{for } i=1 \\ -1 & \text{for } i=2, \cdots, 5 \\ 1 & \text{for } i=6, 7, 8 \end{cases}$$

where $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_8\}=\mathrm{Irr}(b_0)$. Since $|G:VH|=|G:H|=3$, we get $3 \mid |G:VG'|$. By Proposition 1.5, $k'(B_0)=|G:VG'|$. By (**), $k'(b_0)=1$, so that $|G:VG'|=3$ from Frobenius reciprocity. So we may assume that $\chi_1|_H=\chi_2|_H=\chi_3|_H=\tilde{\chi}_1$ from (*), (**) and Proposition 1.3. Similarly, we may also assume that $\chi_7|_H=\tilde{\chi}_3+\tilde{\chi}_4+\tilde{\chi}_5$ and $\chi_8|_H=\tilde{\chi}_6+\tilde{\chi}_7+\tilde{\chi}_8$. Then we get $\chi_4|_H=\chi_5|_H=\chi_6|_H=\tilde{\chi}_2$. This completes the proof.

The next theorem is the main result of this section.

THEOREM 5.4. *Let* $\bar{G}=G/O(G)$ *and* $S=O'(\bar{G})$. *If* G *is nonsolvable,* $e=21$ *and* $e(S)\neq 21$, *then we have the following.*

(i) $S\cong SL(2, 8)$.

(ii) *For any subnormal subgroup* $\bar{L}$ *of* $\bar{G}$ *of odd index with* $e(\bar{L})=21$,

$B_0 \cong B_0(\bar{L})$.

PROOF. We may assume $O(G)=1$ by Lemma 1.1, so that $S=O'(G)$.

(i) Noncyclic groups of order 21 have no normal subgroups of order 3. Thus, by Lemma 1.14, $e(S)=7$. Then $S \cong SL(2, 8)$ from Proposition 1.10 and Lemma 1.12.

(ii) Firstly, we want to show that

$$(*) \quad \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \text{ is an odd prime and} \\ e(L)=21 \text{ and if } H \text{ is a normal subgroup of } L \text{ such that } |L:H| \text{ is an} \\ \text{odd prime and } e(H)=7, \text{ then } B_0 \cong B_0(L). \end{cases}$$

Let $b_0=B_0(L)$, and let z be an involution in G. By Lemma 5.3, we get

$$(**) \qquad \tilde{\chi}_i(1)=\begin{cases} 1 & \text{for } i=1, 2, 3 \\ 7 & \text{for } i=4, 5, 6 \\ 21 & \text{for } i=7 \\ 27 & \text{for } i=8, \end{cases} \qquad \tilde{\chi}_i(z)=\begin{cases} 1 & \text{for } i=1, 2, 3 \\ -1 & \text{for } i=4, 5, 6 \\ -3 & \text{for } i=7 \\ 3 & \text{for } i=8 \end{cases}$$

where $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_8\}=\mathrm{Irr}(b_0)$. As in the proof of Lemma 5.3, using the generalized decomposition numbers of B_0 relative to z,

$$(***) \qquad \chi_i(z)=\begin{cases} \pm 1 & \text{for } i=1, \cdots, 6 \\ \pm 3 & \text{for } i=7, 8 \end{cases}$$

where $\{\chi_1, \cdots, \chi_8\}=\mathrm{Irr}(B_0)$. Since $|G:L|$ is an odd prime, $I_G(\tilde{\chi}_7)=I_G(\tilde{\chi}_8)=G$ from (**). Thus, by Proposition 1.4, Clifford's theorem, (**) and (***), we may assume that $\chi_7|_L=\tilde{\chi}_7$ and $\chi_8|_L=\tilde{\chi}_8$. By Clifford's theorem, Proposition 1.3, (***) and (**), we have $\chi_i|_L \in \mathrm{Irr}(b_0)$ for $i=1, \cdots, 6$. Thus, by Proposition 1.4, we may assume that $\chi_i|_L=\tilde{\chi}_i$ for $i=1, \cdots, 6$. These show $I_G(\tilde{\chi}_j)=G$ for all $\tilde{\chi}_j \in \mathrm{Irr}(b_0)$. By Proposition 5.1(3), $k(B_0)=k(b_0)$ and $l(B_0)=l(b_0)$. Thus, $B_0 \cong b_0$ from Corollary 1.7. Then, (*) is proved. Since G/S is solvable by [12, Theorem], by repeating the above way, we can prove (ii).

REMARK 1. If G is solvable, we easily know B_0 since we may assume $O(G)=1$ from Lemma 1.1. Assume G is nonsolvable. If $e=3$ or 7, we know B_0 from Theorem 2.4. If $e=21$, we know B_0 from Theorems 4.8 and 5.4.

REMARK 2. By Remark 1 of §2, there is a finite group G with elementary abelian Sylow 2-subgroups of order 8 such that $e(G)=21$ and $e(S)=7$ where $S=O'(G/O(G))$.

6. The case when P is elementary abelian of order 16

In this section we deal with the case when G has elementary abelian Sylow 2-subgroups of order 16. Specially, we are interested in the case where e is not prime. When e is 9 or 21, the similar phenomenon to Theorem 5.4 occurs. Throughout this section we assume that G has an elementary abelian Sylow 2-subgroup P of order 16 and we use the notation e and B_0 as usual.

By Lemma 1.13 and Remark 1 of § 1, it is enough to consider the cases when $e=3$, 5, 7, 9, 15 and 21.

PROPOSITION 6.1. *If G is solvable and $e=3$, then one of the following holds.*

(i) *$B_0 \cong B_0(M)$ where M is a semi-direct product of its normal subgroup P by $\langle t\rangle$ such that $P=\langle x, y, z, w\rangle$ is elementary abelian of order* 16, *$\langle t\rangle$ is cyclic of order* 3, *$x^t=y$, $y^t=xy$, $z^t=w$ and $w^t=zw$. In this case $k(B_0)=8$.*

(ii) *$B_0 \cong B_0(L)$ where L is a semi-direct product of its normal subgroup P by $\langle t\rangle$ such that $P=\langle x, y, z, w\rangle$ is elementary abelian of order* 16, *$\langle t\rangle$ is cyclic of order* 3, *$x^t=x$, $y^t=y$, $z^t=w$ and $w^t=zw$. In this case $k(B_0)=16$.*

PROOF. By Lemma 1.1, we may assume $O(G)=1$. Hence G is a semi-direct product of its normal subgroup P by Z_3 and G is not the direct product $P\times Z_3$. Let $G=P\langle t\rangle$ where $\langle t\rangle$ is cyclic of order 3, and let $P=\langle x, y, z, w\rangle$. We may assume that

(i) $x^t=y,\ y^t=xy,\ z^t=w,\ w^t=zw$

or

(ii) $x^t=x,\ y^t=y,\ z^t=w,\ w^t=zw$.

Then we can easily prove the assertion.

PROPOSITION 6.2. *Let D be the decomposition matrix of B_0. If $e=3$, then we have the following.*

(i) *When G is solvable, D has the form*

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1, \end{matrix} \quad \textit{or} \quad \begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$k(B_0)=8$

$$\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1. \end{matrix}$$

$k(B_0)=16$

(ii) *When G is nonsolvable, we obtain D from Theorem* 2.4(2) *and Lemma* 1.16(ii).

PROOF. The assertion is proved by Proposition 6.1.

PROPOSITION 6.3. *If $e=5$, then G is solvable, $B_0 \cong B_0(P\cdot Z_5)$ where $P\cdot Z_5$ is the semi-direct product of its normal subgroup P by Z_5 and it is not the direct product $P\times Z_5$, and the decomposition matrix of B_0 has the form*

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1. \end{matrix}$$

PROOF. By Proposition 1.10 and Lemma 1.12, G is solvable since we may assume $O(G)=1$ by Lemma 1.1. Hence G is the semi-direct product of P by Z_5, and it is not the direct product $P\times Z_5$. The decomposition matrix of B_0 is easily obtained.

PROPOSITION 6.4. *If $e=7$, then there is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$
\begin{array}{c|cccccc}
 & 1_G & & & & & \\
1_G & 1 & & & & & \\
 & & \delta_2 & & & \mathbf{0} & \\
 & & & \delta_3 & & & \\
 & & & & \delta_4 & & \\
 & & & & & \ddots & \\
 & & \mathbf{0} & & & & \delta_{13} \\
 & & & & & & \delta_{14} \\
 & \delta_{15} & \delta_{15} & \cdots & & & \delta_{15} \\
 & \delta_{16} & \delta_{16} & \cdots & & & \delta_{16}
\end{array}
$$

where $\delta_i = \pm 1$.

PROOF. As in the proof of Lemma 4.1 we can prove the assertion by Proposition 5.2.

PROPOSITION 6.5. *Suppose* $k(B_0)=16$.

(1) *If G has an involution x with* $b_x \cong B_0(P \cdot Z_3)$ *where* $P \cdot Z_3$ *is a semi-direct product of P by* Z_3 *and it is not the direct product* $P \times Z_3$, *then the generalized decomposition matrix* D^x *of* B_0 *relative to x has the form* (*).

(2) *If G has an involution x with* $b_x \cong B_0(Z_2 \times Z_2 \times L_2(q))$ *for some* $q>3$ *with* $q \equiv 3$ *or* 5 (mod 8), *then the generalized decomposition matrix* D^x *of* B_0 *relative to x is as follows*:

(i) *When* $3<q\equiv 3$ (mod 8), D^x *has the form* (*).

(ii) *When* $3<q\equiv 5$ (mod 8), D^x *has the form* (**).

	1_M		
1_G	1	0	0
	δ_2	0	0
	δ_3	0	0
	δ_4	0	0
	0	δ_5	0
	0	δ_6	0
	0	δ_7	0
	0	δ_8	0
	0	0	δ_9
	0	0	δ_{10}
	0	0	δ_{11}
	0	0	δ_{12}
	δ_{13}	δ_{13}	δ_{13}
	δ_{14}	δ_{14}	δ_{14}

	1_M		
1_G	1	0	0
	δ_2	0	0
	δ_3	0	0
	δ_4	0	0
	δ_5	δ_5	0
	δ_6	δ_6	0
	δ_7	δ_7	0
	δ_8	δ_8	0
	δ_9	0	δ_9
	δ_{10}	0	δ_{10}
	δ_{11}	0	δ_{11}
	δ_{12}	0	δ_{12}
	δ_{13}	δ_{13}	δ_{13}
	δ_{14}	δ_{14}	δ_{14}

$$\begin{matrix}\delta_{15} & \delta_{15} & \delta_{15}\\ \delta_{16} & \delta_{16} & \delta_{16}\end{matrix} \qquad\qquad \begin{matrix}\delta_{15} & \delta_{15} & \delta_{15}\\ \delta_{16} & \delta_{16} & \delta_{16}\end{matrix}$$

(*) (**)

where $\delta_i=\pm 1$ *and* $M=C_G(x)$.

PROOF. (1) By Proposition 6.2(i), we know the Cartan matrix of b_x. Hence the assertion is proved as in the proof of Lemma 4.1.

(2) We obtain the Cartan matrix of b_x from Lemma 1.16(ii). Thus we can verify (2) as in the proof of (1).

LEMMA 6.6. *Assume* $e=9$, $O(G)=1$, $O'(G)=Z_2\times Z_2\times L_2(q)$ *for some* $q>3$ *with* $q\equiv 3$ *or* 5 (mod 8), *and G has a normal subgroup H of odd prime index with* $e(H)=3$. *Let* $b_0=B_0(H)$, *and let x and z be involutions in* $Z(O'(G))$ *and* $L_2(q)$, *respectively. Then we have the following.*

(i) $\chi_i|_H=\chi_{i+1}|_H=\chi_{i+2}|_H=\tilde{\chi}_i$ *for* $i=1, 5, 9, 13$

$\chi_j|_H=\tilde{\chi}_{j-2}+\tilde{\chi}_{j-1}+\tilde{\chi}_j$ *for* $j=4, 8, 12, 16$

and the values $\tilde{\chi}_i(1)$, $\tilde{\chi}_i(x)$, $\tilde{\chi}_i(z)$ *and* $\tilde{\chi}_i(xz)$ *are as follows*:

	1	x	z	xz
$\tilde{\chi}_1, \tilde{\chi}_2$	1	1	1	1
$\tilde{\chi}_3, \tilde{\chi}_4$	1	-1	1	-1
$\tilde{\chi}_5, \tilde{\chi}_6$	$(q+\varepsilon)/2$	$(q+\varepsilon)/2$	$-\varepsilon$	$-\varepsilon$
$\tilde{\chi}_7, \tilde{\chi}_8$	$(q+\varepsilon)/2$	$-(q+\varepsilon)/2$	$-\varepsilon$	ε
$\tilde{\chi}_9, \tilde{\chi}_{10}$	$(q+\varepsilon)/2$	$(q+\varepsilon)/2$	$-\varepsilon$	$-\varepsilon$
$\tilde{\chi}_{11}, \tilde{\chi}_{12}$	$(q+\varepsilon)/2$	$-(q+\varepsilon)/2$	$-\varepsilon$	ε
$\tilde{\chi}_{13}, \tilde{\chi}_{14}$	q	q	ε	ε
$\tilde{\chi}_{15}, \tilde{\chi}_{16}$	q	$-q$	ε	$-\varepsilon$

where $\{\chi_1=1_G, \chi_2, \cdots, \chi_{16}\}=\mathrm{Irr}(B_0)$, $\{\tilde{\chi}_1=1_H, \tilde{\chi}_2, \cdots, \tilde{\chi}_{16}\}=\mathrm{Irr}(b_0)$ *and* $\varepsilon=-1$ *if* $q\equiv 3$ (mod 8); $\varepsilon=1$ *if* $q\equiv 5$ (mod 8).

(ii) $\phi_i|_H=\phi_{i+1}|_H=\phi_{i+2}|_H=\tilde{\phi}_{(i+2)/3}$ *for* $i=1, 4, 7$

$\tilde{\phi}^G_{j/3}=\phi_{j-2}+\phi_{j-1}+\phi_j$ *for* $j=3, 6, 9$

where $\{\phi_1=1_G, \phi_2, \cdots, \phi_9\}=\mathrm{IBr}(B_0)$ *and* $\{\tilde{\phi}_1=1_H, \tilde{\phi}_2, \tilde{\phi}_3\}=\mathrm{IBr}(b_0)$.

PROOF. Let $S=O'(G)=\langle x, y\rangle\times L_2(q)$ and $P=\langle x, y, z, w\rangle$ where $\langle z, w\rangle$ is a 2-Sylow subgroup of $L_2(q)$. We can write $N_S(P)=\langle s, C_S(P)\rangle$ for some $s\in N_S(P)$. We may assume $z^s=w$ and $w^s=zw$. We can also write $N_G(P)=\langle s, t, C_G(P)\rangle$ for some $t\in N_G(P)$ where s and t have order 3 modulo $C_G(P)$ since $e=9$ (cf. Lemma 1.13). We may assume $x^t=y$, $y^t=xy$, $z^t=z$ and $w^t=w$. As in the proof of Lemma 5.3, we get $G/H=\langle tH\rangle$, $C_G(P)=C_H(P)$ and $|G:H|=3$. By [10, Lemma

18.5], $\{1, x, z, xz\}$ is the set of all representatives of G-conjugate classes of P. As before, $l(b_x)=l(b_z)=3$ and $l(b_{xz})=1$. By Lemma 3.1, $k(B_0)=16$. Since S is normal in $C_G(x)$ and $e(C_G(x))=3$, it follows from Lemma 2.3 that $b_x\cong B_0(Z_2\times Z_2\times L_2(q))$. Thus, by Lemmas 1.16(i) and 6.5(2), we may assume

$$(*)\qquad \chi_i(x)=\begin{cases}\pm 1 & \text{for } i=1,\cdots,4\\ \pm(q+\varepsilon)/2 & \text{for } i=5,\cdots,12\\ \pm q & \text{for } i=13,\cdots,16.\end{cases}$$

Since $e(H)=3$, by Lemma 2.3, $b_0\cong B_0(S)$. Let $C_G(P)=P\times V$. By [10, Theorem 18.4], $P\cap G'=P$, so that $|G:VG'|$ is odd. Since $b_0\cong B_0(S)$ and since $C_H(P)=P\times V$, by Proposition 1.5, $|H:VH'|=4$. Thus, $|G:VG'|=3$, so that $k'(B_0)=3$ from Proposition 1.5. Since $b_0\cong B_0(S)$, by [10, Theorem 38.1], we know the values of $\tilde{\chi}_i|_S$ for all i. Then we get the table in (i). Using this we may assume that

$$I_G(\tilde{\chi}_i)=G \quad \text{for } i=1, 5, 9, 13$$

$$(**)\qquad \left.\begin{array}{l} I_G(\tilde{\chi}_{j-2})=I_G(\tilde{\chi}_{j-1})=I_G(\tilde{\chi}_j)=H \\ \tilde{\chi}_{j-2}^t=\tilde{\chi}_{j-1},\ \tilde{\chi}_{j-1}^t=\tilde{\chi}_j \end{array}\right\} \text{ for } j=4, 8, 12, 16.$$

By (*) and (**), we may assume that $\chi_1|_H=\chi_2|_H=\chi_3|_H=\tilde{\chi}_1$. Since $\tilde{\chi}_2(x)+\tilde{\chi}_3(x)+\tilde{\chi}_4(x)=-1$, by Proposition 1.4, (*) and (**), we get $\chi_4|_H=\tilde{\chi}_2+\tilde{\chi}_3+\tilde{\chi}_4$. Similarly, we may assume that $\chi_j|_H=\tilde{\chi}_{j-2}+\tilde{\chi}_{j-1}+\tilde{\chi}_j$ for $j=8, 12, 16$. We may also assume that $\chi_i|_H=\chi_{i+1}|_H=\chi_{i+2}|_H=\tilde{\chi}_i$ for $i=5, 9, 13$ using Frobenius reciprocity (*) and (**). This completes the proof of (i). Since $b_0\cong B_0(S)$, by Lemma 1.16(i), $\tilde{\phi}_2(1)=\tilde{\phi}_3(1)=(q-1)/2$. Thus $I_G(\tilde{\phi}_j)=G$ for $j=1, 2, 3$ since $|G:H|=3$. For all $\phi_i\in \mathrm{IBr}(B_0)$ we have $\phi_i|_H\in \mathrm{IBr}(b_0)$ by Clifford's theorem since $|G:H|=3$. Thus, by [15, V 16.6 Satz], we get (ii) for suitable indexing of $\phi_2, \cdots, \phi_9$. This completes the proof of the lemma.

PROPOSITION 6.7. *Assume as in Lemma* 6.6. *Then the decomposition matrix D of B_0 is as follows.*

(i) $3<q\equiv 3 \pmod 8$:

$$D=\begin{array}{ccc}
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} \\
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix}
\end{array}.$$

(ii) $3<q\equiv 5 \pmod 8$:

$$D=\begin{array}{ccc}
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \mathbf{0} & \mathbf{0} \\
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \mathbf{0} \\
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \mathbf{0} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} \\
\begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix} & \begin{matrix}(*)\\1\quad 1\quad 1\end{matrix} & \begin{matrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{matrix}
\end{array},$$

where (*) *is one of the following types*

$$\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix},\quad \begin{matrix}1&0&0\\0&0&1\\0&1&0\end{matrix},\quad \begin{matrix}0&1&0\\1&0&0\\0&0&1\end{matrix},\quad \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix},\quad \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix},\quad \begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}.$$

PROOF. We use the same notation as in Lemma 6.6. Let $\tilde{D}$ be the decomposition matrix of b_0, and let $D=(d_{j\lambda})_{j,\lambda}$. Let $\chi_j|_H=\sum_i l_{ij}\tilde{\chi}_i$ for each j, and $L=(l_{ij})_{i,j}$. Similarly, let $\tilde{\phi}_\kappa^G=\sum_\lambda \beta_{\kappa\lambda}\phi_\lambda$ for each κ, and $B=(\beta_{\kappa\lambda})_{\kappa,\lambda}$. By [7, § 26],

$$\tilde{D}B=LD. \tag{1}$$

(i) Since $b_0 \cong B_0(S)$, by (1) and Lemmas 1.16(ii) and 6.6,

$$D=\begin{array}{ccccccccc}
1&0&0&&&&&&\\
0&1&0&&\mathbf{0}&&&\mathbf{0}&\\
0&0&1&&&&&&\\
1&1&1&&&&&&\\
&&&1&0&0&&&\\
&\mathbf{0}&&0&1&0&&\mathbf{0}&\\
&&&0&0&1&&&\\
&&&1&1&1&&&\\
&&&&&&1&0&0\\
&\mathbf{0}&&&\mathbf{0}&&0&1&0\\
&&&&&&0&0&1\\
&&&&&&1&1&1\\
&?&&&?&&&?&\\
1&1&1&1&1&1&1&1&1
\end{array} \tag{2}$$

and

$$d_{13,\lambda}+d_{14,\lambda}+d_{15,\lambda}=1 \quad \text{for all } \lambda=1,\cdots,9. \tag{3}$$

By Lemma 6.6,

$$D^{xz}=(d_{i1}^{xz})_i=\begin{array}{r}
1\\ 1\\ 1\\ -1\\ 1\\ 1\\ 1\\ -1\\ 1\\ 1\\ 1\\ -1\\ -1\\ -1\\ -1\\ 1
\end{array} \tag{4}$$

where D^{xz} is the generalized decomposition matrix of B_0 relative to xz. Clearly, S is normal in $C_G(x)$. By the proof of Lemma 3.1, $e(C_G(x))=3$. Since $q\equiv 3 \pmod 8$, by Lemmas 2.3 and 6.5(2),

$$(5)\qquad (d^x_{\nu_i,\alpha})_{i,\alpha}=\begin{array}{c|ccc} & \phi^x_1 & \phi^x_2 & \phi^x_3 \\ 1_G=\chi_{\nu_1} & 1 & 0 & 0 \\ \chi_{\nu_2} & \delta_2 & 0 & 0 \\ \chi_{\nu_3} & \delta_3 & 0 & 0 \\ \chi_{\nu_4} & \delta_4 & 0 & 0 \\ \chi_{\nu_5} & 0 & \delta_5 & 0 \\ \chi_{\nu_6} & 0 & \delta_6 & 0 \\ \chi_{\nu_7} & 0 & \delta_7 & 0 \\ \chi_{\nu_8} & 0 & \delta_8 & 0 \\ \chi_{\nu_9} & 0 & 0 & \delta_9 \\ \chi_{\nu_{10}} & 0 & 0 & \delta_{10} \\ \chi_{\nu_{11}} & 0 & 0 & \delta_{11} \\ \chi_{\nu_{12}} & 0 & 0 & \delta_{12} \\ \chi_{\nu_{13}} & \delta_{13} & \delta_{13} & \delta_{13} \\ \chi_{\nu_{14}} & \delta_{14} & \delta_{14} & \delta_{14} \\ \chi_{\nu_{15}} & \delta_{15} & \delta_{15} & \delta_{15} \\ \chi_{\nu_{16}} & \delta_{16} & \delta_{16} & \delta_{16} \end{array}$$

where $d^x_{i\alpha}$ are the generalized decomposition numbers of B_0 relative to x, $\delta_i=\pm 1$, $\chi_{\nu_1}=\chi_1=1_G$, $\{\chi_{\nu_2}, \cdots, \chi_{\nu_{16}}\}=\{\chi_2, \cdots, \chi_{16}\}$, $\{\phi^x_1=1_M, \phi^x_2, \phi^x_3\}=\mathrm{IBr}(b_x)$ and $M=C_G(x)$. Let $c=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ in $L_2(q)$. Then, by [10, Theorem 38.1] and Lemma 1.16, we may assume that

$$(6)\qquad \phi^x_2(c)=(-1+\sqrt{-q})/2 \quad\text{and}\quad \phi^x_3(c)=(-1-\sqrt{-q})/2.$$

By Lemma 6.6 and (5), $\{\chi_{\nu_1}, \cdots, \chi_{\nu_4}\}=\{\chi_1, \cdots, \chi_4\}$ and $\{\chi_{\nu_{13}}, \cdots, \chi_{\nu_{16}}\}=\{\chi_{13}, \cdots, \chi_{16}\}$. We may assume that $\chi_5\in\{\chi_{\nu_5}, \cdots, \chi_{\nu_8}\}$. By Lemma 6.6, $\chi_5|_H=\chi_6|_H=\chi_7|_H$. Thus, by (5) and (6), we get that χ_6 and χ_7 are both in $\{\chi_{\nu_5}, \cdots, \chi_{\nu_8}\}$. Similarly, none of $\{\chi_9, \chi_{10}, \chi_{11}\}$ are in $\{\chi_{\nu_5}, \cdots, \chi_{\nu_8}\}$. Hence, by (2), (5) and (6), we know $\{\chi_{\nu_5}, \cdots, \chi_{\nu_8}\}=\{\chi_5, \cdots, \chi_8\}$. Thus, $\{\chi_{\nu_9}, \cdots, \chi_{\nu_{12}}\}=\{\chi_9, \cdots, \chi_{12}\}$. Hence we may assume that $\chi_{\nu_i}=\chi_i$ for all $i=1, \cdots, 16$. Therefore, by Lemma 6.6,

$$\begin{array}{ccc} \phi^x_1 & \phi^x_2 & \phi^x_3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}$$

(7)
$$(d_{i\alpha}^x)_{i,\alpha}=\begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1. \end{matrix}$$

Next, we want to know the generalized decomposition numbers $d_{i\alpha}^z$ of B_0 relative to z. Let $L=C_G(z)$. As for x, $e(L)=3$ and $l(B_0(L))=3$. Since $N_L(P)=\langle t, C_L(P)\rangle$ and $z^t=z$, we get from Proposition 6.1 and Theorem 2.4(1) that $k(B_0(L))=k(b_z)=16$. By Theorem 2.4(2) and Lemmas 6.6 and 6.5, L is solvable, so that $b_z\cong B_0(P\cdot Z_3)$ from Proposition 6.1 where $P\cdot Z_3$ is a semi-direct product of its normal subgroup P by Z_3 and it is not the direct product $P\times Z_3$. Thus, by Lemma 6.5,

(8)
$$(d_{\nu_i,\alpha}^z)_{i,\alpha}=\begin{matrix} & \phi_1^z & \phi_2^z & \phi_3^z \\ 1_G=\chi_{\nu_1} & 1 & 0 & 0 \\ \chi_{\nu_2} & 0 & \delta_2 & 0 \\ \chi_{\nu_3} & 0 & 0 & \delta_3 \\ \chi_{\nu_4} & \delta_4 & \delta_4 & \delta_4 \\ \chi_{\nu_5} & \delta_5 & 0 & 0 \\ \chi_{\nu_6} & 0 & \delta_6 & 0 \\ \chi_{\nu_7} & 0 & 0 & \delta_7 \\ \chi_{\nu_8} & \delta_8 & \delta_8 & \delta_8 \\ \chi_{\nu_9} & \delta_9 & 0 & 0 \\ \chi_{\nu_{10}} & 0 & \delta_{10} & 0 \\ \chi_{\nu_{11}} & 0 & 0 & \delta_{11} \\ \chi_{\nu_{12}} & \delta_{12} & \delta_{12} & \delta_{12} \\ \chi_{\nu_{13}} & \delta_{13} & 0 & 0 \\ \chi_{\nu_{14}} & 0 & \delta_{14} & 0 \\ \chi_{\nu_{15}} & 0 & 0 & \delta_{15} \\ \chi_{\nu_{16}} & \delta_{16} & \delta_{16} & \delta_{16} \end{matrix}$$

where $\delta_i=\pm 1$, $\chi_{\nu_1}=\chi_1=1_G$, $\{\chi_{\nu_2}, \cdots, \chi_{\nu_{16}}\}=\{\chi_2, \cdots, \chi_{16}\}$ and $\{\phi_1^z, \phi_2^z, \phi_3^z\}=\mathrm{IBr}(b_z)$. Clearly, $\phi_1^z(1)=\phi_2^z(1)=\phi_3^z(1)=1$. Hence, by Lemma 6.6 and (8),

$$\{\chi_{\nu_4}, \chi_{\nu_8}, \chi_{\nu_{12}}, \chi_{\nu_{16}}\}=\{\chi_4, \chi_8, \chi_{12}, \chi_{16}\}$$
$$\{\delta_4, \delta_8, \delta_{12}, \delta_{16}\}=\{1, 1, 1, -1\}. \tag{9}$$

By Lemma 6.6, $\chi_i(z)=\chi_{i+1}(z)=\chi_{i+2}(z)=1$ for $i=1, 5, 9$. So it follows from (4), (7), (8) and [10, Theorem 63.3] that $\delta_4=\delta_8=\delta_{12}=1$ and $\delta_{16}=-1$. Thus, again by (4), (7), (8) and [10, Theorem 63.3],

$$(d_{i\alpha}^z)_{i,\alpha}=\begin{array}{rrr} \phi_1^z & \phi_2^z & \phi_3^z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{array} \tag{10}$$

for suitable indexing. By (2), (3), (10) and [10, Theorem 63.3],

$$D=\begin{array}{c|ccccccccc} \chi_{13} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \chi_{14} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \chi_{15} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \chi_{16} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}.$$

This completes the proof of (i).

(ii) Since $b_0\cong B_0(S)$, as in the proof of (i) we get

(11)
$$D=\begin{array}{ccccccccc}
1&0&0& & & & & & \\
0&1&0& &\mathbf{0}& & &\mathbf{0}& \\
0&0&1& & & & & & \\
1&1&1& & & & & & \\
 &?& & &?& & &\mathbf{0}& \\
1&1&1&1&1&1& & & \\
 &?& & &\mathbf{0}& & &?& \\
1&1&1& & & &1&1&1\\
 &?& & &?& & &?& \\
1&1&1&1&1&1&1&1&1
\end{array}$$

and

(12)
$$\begin{array}{ll}
d_{5\lambda}+d_{6\lambda}+d_{7\lambda}=1 & \text{for } \lambda=1,\cdots,6\\
d_{9\lambda}+d_{10,\lambda}+d_{11,\lambda}=1 & \text{for } \lambda=1,2,3,7,8,9\\
d_{13,\lambda}+d_{14,\lambda}+d_{15,\lambda}=1 & \text{for } \lambda=1,\cdots,9.
\end{array}$$

As in the proof of (i), we have

(13)
$$(d_{i1}^{xz})_i=\begin{array}{r}
1\\1\\1\\-1\\-1\\-1\\-1\\1\\-1\\-1\\-1\\1\\1\\1\\1\\-1,
\end{array}
\quad
(d_{i\alpha}^{x})_{i,\alpha}=\begin{array}{rrr}
\phi_1^x&\phi_2^x&\phi_3^x\\
1&0&0\\
1&0&0\\
1&0&0\\
-1&0&0\\
1&1&0\\
1&1&0\\
1&1&0\\
-1&-1&0\\
1&0&1\\
1&0&1\\
1&0&1\\
-1&0&-1\\
1&1&1\\
1&1&1\\
1&1&1\\
-1&-1&-1,
\end{array}
\quad
(d_{i\alpha}^{z})_{i,\alpha}=\begin{array}{rrr}
\phi_1^z&\phi_2^z&\phi_3^z\\
1&0&0\\
0&1&0\\
0&0&1\\
1&1&1\\
-1&0&0\\
0&-1&0\\
0&0&-1\\
-1&-1&-1\\
-1&0&0\\
0&-1&0\\
0&0&-1\\
-1&-1&-1\\
1&0&0\\
0&1&0\\
0&0&1\\
1&1&1
\end{array}$$

where d_{i1}^{xz}, $d_{i\alpha}^{x}$, $d_{i\alpha}^{z}$, ϕ_α^x and ϕ_α^z are the same as in (i). By Lemmas 6.6 and 1.16, we know the degrees of all χ_i and ϕ_α. Thus, by (11), (12), (13) and [10, Theorem 63.3], we may assume

(14)
$$D=\begin{array}{c|cccccc|c} \chi_5 & 1 & 0 & 0 & 1 & 0 & 0 & \\ \chi_6 & 0 & 1 & 0 & 0 & 1 & 0 & \mathbf{0} \\ \chi_7 & 0 & 0 & 1 & 0 & 0 & 1 & \\ \chi_8 & 1 & 1 & 1 & 1 & 1 & 1 & \end{array}.$$

Similarly, we may assume

(15)
$$D=\begin{array}{c|ccc|c|ccc} \chi_9 & & & & & 1 & 0 & 0 \\ \chi_{10} & & ? & & \mathbf{0} & 0 & 1 & 0 \\ \chi_{11} & & & & & 0 & 0 & 1 \\ \chi_{12} & 1 & 1 & 1 & & 1 & 1 & 1 \end{array}.$$

So, by (11), (12), (13), (14), (15) and [10, Theorem 63.3],

(16)
$$D=\begin{array}{c|ccc|cccccc} \chi_{13} & & & & 1 & 0 & 0 & 1 & 0 & 0 \\ \chi_{14} & & ? & & 0 & 1 & 0 & 0 & 1 & 0 \\ \chi_{15} & & & & 0 & 0 & 1 & 0 & 0 & 1 \\ \chi_{16} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}.$$

Thus, considering the degrees of χ_i and ϕ_α, by (11)-(16) we get the following six cases:

χ_9	1	0	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1
χ_{10}	0	1	0	0	0	1	1	0	0	1	0	0	0	0	1	0	1	0
χ_{11}	0	0	1	0	1	0	0	0	1	0	1	0	1	0	0	1	0	0
χ_{12}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_{13}	1	0	0	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1
χ_{14}	0	1	0	0	0	1	1	0	0	1	0	0	0	0	1	0	1	0
χ_{15}	0	0	1	0	1	0	0	0	1	0	1	0	1	0	0	1	0	0
χ_{16}	1	1	1,	1	1	1,	1	1	1,	1	1	1,	1	1	1,	1	1	1.
	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3

Thus, for suitable indexing of χ_i and ϕ_α, we obtain (ii).

The following theorem is one of the main results of this section.

THEOREM 6.8. *Let* $\bar{G}=G/O(G)$ *and* $S=O'(\bar{G})$. *If* G *is nonsolvable,* $e=9$ *and* $e(S)\neq 9$, *then we have the following.*

(i) $S\cong Z_2\times Z_2\times L_2(q)$ *for some* $q>3$ *with* $q\equiv 3$ *or* 5 (mod 8).

(ii) *For any subnormal subgroup* $\bar{L}$ *of* $\bar{G}$ *of odd index with* $e(\bar{L})=9$, $B_0\cong B_0(\bar{L})$.

PROOF. We may assume $O(G)=1$ by Lemma 1.1, so that $S=O'(G)$.

(i) Since G is nonsolvable, $e(S)=3$. Thus, we get (i) from Proposition 1.10 and Lemma 1.12.

(ii) Firstly, we want to prove that

$$(*)\qquad \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \text{ is an odd prime and} \\ e(L)=9 \text{ and if } H \text{ is a normal subgroup of } L \text{ such that } |L:H| \text{ is an odd} \\ \text{prime and } e(H)=3, \text{ then } B_0\cong B_0(L). \end{cases}$$

Let $b_0=B_0(L)$. By Lemma 3.1, $k(B_0)=k(b_0)=16$ and $l(B_0)=l(b_0)=9$. We may write $O'(G)=S=\langle x, y\rangle\times L_2(q)$ and $P=\langle x, y, z, w\rangle$ where $\langle z, w\rangle$ is a Sylow 2-subgroup of $L_2(q)$. By the proof of Lemma 6.6, we may assume $x^s=x$, $y^s=y$, $z^s=w$, $w^s=zw$, $x^t=y$, $y^t=xy$, $z^t=z$ and $w^t=w$ where $s, t\in N_L(P)=\langle s, t, C_L(P)\rangle$. So $N_G(P)=\langle s, t, C_G(P)\rangle$. By the proof of Lemma 3.1, $l(b_{xz})=1$. Thus, by Lemma 1.9, $\chi_i(xz)=\pm 1$ for all $\chi_i\in \mathrm{Irr}(B_0)$. By Lemma 6.6, we know the values $\tilde{\chi}_j(1)$ and $\tilde{\chi}_j(xz)$ for all $\tilde{\chi}_j\in \mathrm{Irr}(b_0)$. Using this, if $\tilde{\chi}, \tilde{\chi}'\in \mathrm{Irr}(b_0)$ and $\tilde{\chi}(1)=\tilde{\chi}'(1)$, then $\tilde{\chi}(xz)=\tilde{\chi}'(xz)=\pm 1$. Hence it follows from Corollary 1.8 that $B_0\cong b_0$. Thus we get (*). On the other hand, G/S is solvable from [12, Theorem]. Hence we can verify (ii) by repeating the above way. This completes the proof.

PROPOSITION 6.9. *Let D be the decomposition matrix of B_0, and let $S=O'(G/O(G))$. If $e=9$, then we have following.*

(i) *When G is solvable,*

$$D=\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}.$$

(ii) *When G is nonsolvable and $e(S)=9$, we know D from Theorem* 3.4(2) *and Lemma* 1.16(ii).

(iii) *When G is nonsolvable and $e(S)=3$, we know D from Theorem* 6.8 *and Proposition* 6.7.

REMARK 1. There is a finite group G with an elementary abelian Sylow 2-subgroup P of order 16 such that $e(G)=9$ and $O'(G/O(G))\cong Z_2\times Z_2\times L_2(q)$ for $q>3$ with $q\equiv 3$ or 5 (mod 8). Let $\langle z, w\rangle$ be a Sylow 2-subgroup of $L_2(q)$, and let $S=\langle x, y\rangle\times L_2(q)$ and $P=\langle x, y, z, w\rangle$ where $\langle x, y\rangle$ is elementary abelian of order 4. There is an automorphism r of $\langle x, y\rangle$ with $x^r=y$ and $y^r=xy$. We can consider that $r\in\mathrm{Aut}(S)$ if its we consider that r is trivial on $L_2(q)$. So there is a semi-direct product G of its normal subgroup S by $\langle r\rangle$. Then, $e(G)=9$ and $O'(G)=S=Z_2\times Z_2\times L_2(q)$.

The next theorem is one of the main results of this section.

THEOREM 6.10. *If G is nonsolvable and $e=15$, then $B_0\cong B_0(SL(2, 16))$.*

PROOF. By Lemma 1.1, we may assume $O(G)=1$. Let $S=O'(G)$. Since G is nonsolvable, it follows from Proposition 1.10 and Lemma 1.12 that $e(S)\neq 1$ and $e(S)\neq 5$. So that $e(S)=3$ or 15. Firstly, suppose $e(S)=3$. By Proposition 1.10 and Lemma 1.12, $S\cong Z_2\times Z_2\times L_2(q)$ for some $q>3$ with $q\equiv 3$ or 5 (mod 8). Thus, there is an involution $x\in P\cap Z(S)$. We can write $N_S(P)=\langle s, C_S(P)\rangle$ for some $s\in N_S(P)$. Thus, $x^s=x$. Since $e=15$, we can write $N_G(P)=\langle t, C_G(P)\rangle$ for some $t\in N_G(P)$. Since $N_S(P)/C_S(P)$ can be considered as a subgroup of $N_G(P)/C_G(P)$ through the canonical monomorphism, we get that $s\equiv t^{5i}$ (mod $C_G(P)$) for some integer i with $i\not\equiv 0$ (mod 3). Thus, $x=x^{t^{5i}}$. This is a contradiction. Hence $e(S)=15$, so that $S\cong SL(2, 16)$ from Proposition 1.10 and Lemma 1.12.

We prove $B_0\cong B_0(S)$ by induction on $|G|$. Let $G\neq S$. Since G/S is solvable by [12, Theorem], G has a normal subgroup H of odd prime index with $S\subseteq H$. Let $b_0=B_0(H)$, and let z be an involution in P. Since $b_0\cong B_0(S)$ by induction, we get $k(b_0)=16$ and

$$(*)\qquad \tilde{\chi}_i(1)=\begin{cases} 1 & \text{for } i=1\\ 15 & \text{for } i=2,\cdots,9\\ 17 & \text{for } i=10,\cdots,16,\end{cases}\qquad \tilde{\chi}_i(z)=\begin{cases} 1 & \text{for } i=1\\ -1 & \text{for } i=2,\cdots,9\\ 1 & \text{for } i=10,\cdots,16\end{cases}$$

using [10, Theorem 38.2], where $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{16}\}=\mathrm{Irr}(b_0)$. Since all involutions in P are G-conjugate, $P\cap G'=P$ by [10, Theorem 18.4]. Thus, $k'(B_0)$ is odd from Proposition 1.5. Now, we want to claim that $k(B_0)=16$. If $k'(B_0)=1$, we get from Propositions 1.5 and 1.6 that $k(B_0)=16$. Suppose $k(B_0)\neq 16$. Since $e=15$,

$l(b_z)=1$. Thus, by Lemma 1.15(2), $k(B_0)=8$. So that $k'(B_0)=3$, 5 or 7. Let $\{\chi_1, \cdots, \chi_8\}=\mathrm{Irr}(B_0)$.

Case 1. $k'(B_0)=7$: We may assume $\chi_1(1)=\cdots=\chi_7(1)=1$ and $\chi_8(1)>1$. By Clifford's theorem, Proposition 1.3 and (*), we have $\chi_1|_H=\cdots=\chi_7|_H=\tilde{\chi}_1$. Thus, by Proposition 1.4, $(\chi_8|_H, \tilde{\chi}_j)\neq 0$ for $j=2, \cdots, 16$. Then we have a contradiction from Clifford's theorem and (*) by considering the degrees of $\tilde{\chi}_j$.

Case 2. $k'(B_0)=5$: We may assume $\chi_i(1)=1$ for $i=1, \cdots, 5$ and $\chi_j(1)>1$ for $j=6, 7, 8$. As in Case 1 we know $\chi_1|_H=\cdots=\chi_5|_H=\tilde{\chi}_1$. Since $k(B_0)\neq k(b_0)$, $B_0\not\cong b_0$. So that we get from Proposition 1.6 that $G\neq VH$ where V is a subgroup of G with $C_G(P)=P\times V$. Since $k'(B_0)=5$, $|G:H|=5$ by Proposition 1.5. So, by Clifford's theorem and Proposition 1.4,

$$\chi_6|_H=\tilde{\chi}_2+\cdots+\tilde{\chi}_6, \quad \chi_7|_H=\tilde{\chi}_7+\cdots+\tilde{\chi}_{11},$$
$$\chi_8|_H=\tilde{\chi}_{12}+\cdots+\tilde{\chi}_{16}$$

for suitable indexing of $\tilde{\chi}_2, \cdots, \tilde{\chi}_{16}$. Hence we have a contradiction from Clifford's theorem and (*) by considering the degrees of $\tilde{\chi}_j$

Case 3. $k'(B_0)=3$: Let $\chi_i(1)=1$ for $i=1, 2, 3$ and $\chi_j(1)>1$ for $j=4, \cdots, 8$. As in Case 2, $|G:H|=3$. Then, by Proposition 1.4, for suitable indexing of $\tilde{\chi}_2, \cdots, \tilde{\chi}_{16}$, we get

$$\chi_4|_H=\tilde{\chi}_2+\tilde{\chi}_3+\tilde{\chi}_4, \quad \chi_5|_H=\tilde{\chi}_5+\tilde{\chi}_6+\tilde{\chi}_7, \quad \chi_6|_H=\tilde{\chi}_8+\tilde{\chi}_9+\tilde{\chi}_{10}$$
$$\chi_7|_H=\tilde{\chi}_{11}+\tilde{\chi}_{12}+\tilde{\chi}_{13}, \quad \chi_8|_H=\tilde{\chi}_{14}+\tilde{\chi}_{15}+\tilde{\chi}_{16}$$

Then we have a contradiction as in Case 2.

Thus, $k(B_0)=16$. Let $\{\chi_1, \cdots, \chi_{16}\}=\mathrm{Irr}(B_0)$. Since $l(b_z)=1$, $\chi_i(z)=\pm 1$ for $i=1, \cdots, 16$ from Lemma 1.9. Thus, we know from Clifford's theorem, Proposition 1.3 and (*) that $\chi_i|_H\in\mathrm{Irr}(b_0)$ for all $i=1, \cdots, 16$. Hence, by Proposition 1.4, we may assume that $\chi_i|_H=\tilde{\chi}_i$ for all $i=1, \cdots, 16$. This shows $k'(B_0)=1$. So that $B_0\cong b_0$ from Propositions 1.5 and 1.6. This completes the proof of the theorem.

Proposition 6.11. *If $e=15$, then there is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$\begin{array}{cccccc}
1_G & 1 & & & & \\
 & & \delta_2 & & \mathbf{0} & \\
 & & & \cdot & & \\
 & & & & \cdot & \\
 & & \mathbf{0} & & & \cdot \\
 & & & & & \delta_{15} \\
 & \delta_{16} & \delta_{16} & \cdot & \cdot & \cdot\ \delta_{16}
\end{array}$$

where $\delta_i = \pm 1$.

PROOF. The proof is similar to that of Proposition 5.2 (cf. the case when $e=7$ in Proposition 5.2).

LEMMA 6.12. *If* $e=21$, *then there is an involution* $z \in P$ *and there are two elements* $s, t \in N_G(P)$ *such that* $N_G(P) = \langle s, t, C_G(P) \rangle$, $z^s = z$ *and* $z^t = z$.

PROOF. Firstly, we want to prove that

(*) there is an involution $u \in P$ and there are two elements $s, t \in N_G(P)$ such that $N_G(P) = \langle s, t, C_G(P) \rangle$, s and t have orders 3 and 7 modulo $C_G(P)$ respectively, and $u^s = u$.

We may assume $O(G)=1$ by the proof of Lemma 1.1. Since $S=O'(G)$ is normal in G, $e(S)=1, 7$ or 21. When $e(S)=7$ or 21, we get (*) from Proposition 1.10 and Lemma 1.12. Assume $e(S)=1$. Then P is normal in G and $|G:P|=21$. We can write $G=\langle s, t, P \rangle$ for $s, t \in G$ such that s and t have orders 3 and 7 modulo P, respectively. Clearly, there is an involution $y \in P$ with $y^t = y$. Suppose $x^s \neq x$ for all involutions $x \in P$. Then, $e(C_G(y))=7$. By Proposition 5.2 and [10, Lemma 66.1], the Cartan matrix of b_y has 2 as an elementary divisor of multiplicity 6. Thus, by [5, (7G)], [18, Proposition 1.2] and [10, Theorem 65.4], we get $l(B_0) \geqq 7$ since all involutions in G are conjugate. On the other hand, $l(B_0)=5$ since G/P is noncyclic of order 21 (cf. Lemma 1.13). This is a contradiction. Hence we obtain (*).

Next, we prove the lemma. There is an involution $z \in P$ with $z^t = z$. By (*), there are other two involutions $v, w \in P$ such that $v^s = v$, $w^s = w$ and u, v, w are all distinct. It suffices to show $z \in \{u, v, w\}$. Suppose $z \notin \{u, v, w\}$. Since $z^s \neq z$, we know that $\{1, u, v, w, z, uz, vz, wz\}$ is the set of all representatives of $\langle s \rangle$-conjugate classes of P. Since $u \neq z$, we get $u^t \neq u$. Thus, $\{1, u, uz, z\}$ is the set of all representatives of $\langle t \rangle$-conjugate classes of P. Hence, by elementary calculation, we get that $v \in \{uz, u^t z, \cdots, u^{t^6} z\}$. Hence no two elements in $\{u, v, z\}$ are conjugate in G. These show that all G-conjugate classes of P are $\{1\}$, $\{z\}$ $\{u, u^t, \cdots, u^{t^6}\}$ and $\{uz, u^t z, \cdots, u^{t^6} z\}$. Thus, $z^s = z$. This is a contradiction. This completes the proof.

LEMMA 6.13. *If* $e=21$, *then* $k(B_0)=16$ *and* $l(B_0)=5$.

PROOF. Let s, t and z be the same as in Lemma 6.12. Hence s and t have orders 3 and 7 modulo $C_G(P)$, respectively. There is an involution $x \in P$ with $x^s = x$ and $x^t \neq x$. Thus, $\{1, x, xz, z\}$ is the set of all representatives of conjugate classes of G of 2-elements by [10, Lemma 18.5]. We may assume $O(G)=1$ by Lemma 1.1. By Z^*-theorem [10, Theorem 67.1], $z\in Z(G)$. These imply from [10, Theorems 68.4 and 65.4] that $k(B_0)=2l(B_0)+l(b_x)+l(b_{xz})$. Since $e(C_G(x))=3$, $l(b_x)=3$ by Theorem 2.4(1). Similarly, $l(b_{xz})=3$. Since $z\in Z(G)$, as in the proof of Lemma 2.2 we get from Lemma 4.1 and Proposition 1.2 that $l(B_0)=5$, so that $k(B_0)=16$.

LEMMA 6.14. *Assume $e=21$, $O(G)=1$, $O'(G)=Z_2\times SL(2, 8)$ and G has a normal subgroup H of odd prime index with $e(H)=7$. Then for any involution z in $SL(2, 8)$, we have*

$$\chi_1(1)=\cdots=\chi_6(1)=1,\quad \chi_7(1)=\cdots=\chi_{12}(1)=7,$$

$$\chi_{13}(1)=\chi_{14}(1)=21,\quad \chi_{15}(1)=\chi_{16}(1)=27,$$

$$\chi_1(z)=\cdots=\chi_6(z)=1,\quad \chi_7(z)=\cdots=\chi_{12}(z)=-1,$$

$$\chi_{13}(z)=\chi_{14}(z)=-3,\quad \chi_{15}(z)=\chi_{16}(z)=3$$

where $\{\chi_1=1_G, \chi_2, \cdots, \chi_{16}\}=\mathrm{Irr}(B_0)$.

PROOF. Let $b_0=B_0(H)$, $S=O'(G)=\langle w\rangle\times SL(2, 8)$ and $P=\langle w, x, y, z\rangle$ where $\langle x, y, z\rangle$ is a Sylow 2-subgroup of $SL(2, 8)$. As in the proof of Lemma 5.3, $G/H=\langle rH\rangle$ for some $r\in N_G(P)$, $C_G(P)=C_H(P)$ and $|G:H|=3$. We can write $N_S(P)=\langle t, C_S(P)\rangle$ for some $t\in N_S(P)$. Since $P\cap Z(S)=\langle w\rangle$, it follows from Lemma 6.12 and Z^*-theorem [10, Theorem 67.1] that $w\in Z(G)$. Then, by the proof of Lemma 6.13, we may assume that $z^t\neq z$, $z^r=z$ and $l(b_z)=3$. By the proof of Lemma 2.2, $l(B_0(C_S(z)))=1$. So that $C_S(z)$ is 2-nilpotent from [10, Corollary 65.3]. Hence $C_G(z)$ is solvable. Since $e(C_G(z))=3$, by Proposition 6.1, $b_z\cong B_0(P\cdot Z_3)$ where $P\cdot Z_3$ is a semi-direct product of its normal subgroup P by Z_3 and it is not the direct product $P\times Z_3$. Then, by Lemma 6.5, we know the generalized decomposition numbers of B_0 relative to z. This implies

$$(*)\qquad \chi_i(z)=\begin{cases} \pm 1 & \text{for } i=1, \cdots, 12 \\ \pm 3 & \text{for } i=13, \cdots, 16 \end{cases}$$

for suitable indexing of $\chi_2, \cdots, \chi_{16}$. By Lemma 2.3, $b_0\cong B_0(S)$. So, by [10, Theorem 38.2],

$$(**)\qquad \begin{aligned} &\tilde{\chi}_1(1)=\tilde{\chi}_2(1)=1,\quad \tilde{\chi}_3(1)=\cdots\tilde{\chi}_{10}(1)=7,\\ &\tilde{\chi}_{11}(1)=\cdots=\tilde{\chi}_{16}(1)=9,\\ &\tilde{\chi}_1(z)=\tilde{\chi}_2(z)=1,\quad \tilde{\chi}_3(z)=\cdots=\tilde{\chi}_{10}(z)=-1,\\ &\tilde{\chi}_{11}(z)=\cdots=\tilde{\chi}_{16}(z)=1 \end{aligned}$$

where $\{\tilde{\chi}_1=1_H, \tilde{\chi}_2, \cdots, \tilde{\chi}_{16}\}=\mathrm{Irr}(b_0)$. We can write $C_G(P)=P\times V$. By Theorem 2.4 and Lemma 6.13, we get $l(B_0)\neq l(b_0)=l(B_0(S))$. Hence $B_0\not\cong b_0$, so that $VH\neq G$ by Proposition 1.6. Thus, $|G:VH|=|G:H|=3$. Hence, by Proposition 1.5, $k'(B_0)$ is divisible by 3. Since $|G:H|=3$, it follows from Frobenius reciprocity, Proposition 1.3 and (**) that $k'(B_0)\leqq 6$. By observing the conjugate classes of G of 2-elements, we know $P\cap G'\neq P$ from [10, Theorem 18.4]. Hence $|G:VG'|$ is divisible by 2. Thus, $k'(B_0)=|G:VG'|=6$ from Proposition 1.5. Then, by (*) and (**), we may assume that

$$\chi_1|_H=\chi_2|_H=\chi_3|_H=\tilde{\chi}_1,\quad \chi_4|_H=\chi_5|_H=\chi_6|_H=\tilde{\chi}_2.$$

Similarly, we may assume that

$$\chi_{13}|_H=\tilde{\chi}_5+\tilde{\chi}_6+\tilde{\chi}_7,\quad \chi_{14}|_H=\tilde{\chi}_8+\tilde{\chi}_9+\tilde{\chi}_{10},$$

$$\chi_{15}|_H=\tilde{\chi}_{11}+\tilde{\chi}_{12}+\tilde{\chi}_{13},\quad \chi_{16}|_H=\tilde{\chi}_{14}+\tilde{\chi}_{15}+\tilde{\chi}_{16}.$$

Hence we may assume that

$$\chi_7|_H=\chi_8|_H=\chi_9|_H=\tilde{\chi}_3,\quad \chi_{10}|_H=\chi_{11}|_H=\chi_{12}|_H=\tilde{\chi}_4.$$

Therefore the lemma is proved by (**).

Now, we state the next theorem which is one of the main results of this section.

THEOREM 6.15. *Let* $\bar{G}=G/O(G)$ *and* $S=O'(\bar{G})$. *If* G *is nonsolvable,* $e=21$ *and* $e(S)\neq 21$, *then we have the following.*

(i) $S\cong Z_2\times SL(2, 8)$.

(ii) *For any subnormal subgroup* $\bar{L}$ *of* $\bar{G}$ *of odd index with* $e(\bar{L})=21$, $B_0\cong B_0(\bar{L})$.

PROOF. We can assume $O(G)=1$ by Lemma 1.1. Hence $S=O'(G)$.

(i) By Lemma 1.13, $e(S)=7$. Hence, by Proposition 1.10 and Lemma 1.12, $S\cong Z_2\times SL(2, 8)$.

(ii) Firstly, we want to show that

(*) if L is a normal subgroup of G such that $|G:L|$ is an odd prime and $e(L)=21$, and if H is a normal subgroup of L such that $|L:H|$ is an odd prime and $e(H)=7$, then $B_0\cong B_0(L)$.

Let $b_0=B_0(L)$. By Lemma 6.13, $k(B_0)=k(b_0)=16$ and $l(B_0)=l(b_0)=5$. Let $S=O'(G)=\langle w\rangle\times SL(2, 8)$ and $P=\langle w, x, y, z\rangle$ where $\langle x, y, z\rangle$ is a Sylow 2-subgroup of $SL(2, 8)$. As in the proof of Lemma 6.14,

(**)
$$\chi_i(z)=\begin{cases} \pm 1 & \text{for } i=1, \cdots, 12 \\ \pm 3 & \text{for } i=13, \cdots, 16 \end{cases}$$

where $\{\chi_1, \cdots, \chi_{16}\}=\mathrm{Irr}(B_0)$. Let $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_{16}\}=\mathrm{Irr}(b_0)$. By Lemma 6.14, we may assume that

(***)
$$\begin{gathered} \tilde{\chi}_1(1)=\cdots=\tilde{\chi}_6(1)=1, \quad \tilde{\chi}_7(1)=\cdots=\tilde{\chi}_{12}(1)=7, \\ \tilde{\chi}_{13}(1)=\tilde{\chi}_{14}(1)=21, \quad \tilde{\chi}_{15}(1)=\tilde{\chi}_{16}(1)=27, \\ \tilde{\chi}_1(z)=\cdots=\tilde{\chi}_6(z)=1, \quad \tilde{\chi}_7(z)=\cdots=\tilde{\chi}_{12}(z)=-1, \\ \tilde{\chi}_{13}(z)=\tilde{\chi}_{14}(z)=-3, \quad \tilde{\chi}_{15}(z)=\tilde{\chi}_{16}(z)=3. \end{gathered}$$

Thus, as in the proof of Theorem 5.4, by (**), (***), Clifford's theorem and Proposition 1.4, we may assume that $\chi_i|_L=\tilde{\chi}_i$ for all $i=1, \cdots, 16$. Hence we get $B_0\cong b_0$ by Corollary 1.7. This proves (*). Since G/S is solvable by [12, Theorem], we can verify (ii).

REMARK 2. There is a finite group G with elementary abelian Sylow 2-subgroups of order 16 such that $e(G)=21$ and $O'(G/O(G))\cong Z_2\times SL(2, 8)$. We know it as in Remark 1 of § 2.

PROPOSITION 6.16. *If $e=21$, then there is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$\begin{array}{cccccc}
1_G & 1 & 0 & 0 & 0 & 0 \\
 & \delta_2 & 0 & 0 & 0 & 0 \\
 & 0 & \delta_3 & 0 & 0 & 0 \\
 & 0 & \delta_4 & 0 & 0 & 0 \\
 & 0 & 0 & \delta_5 & 0 & 0 \\
 & 0 & 0 & \delta_6 & 0 & 0 \\
 & 0 & 0 & 0 & \delta_7 & 0 \\
 & 0 & 0 & 0 & \delta_8 & 0 \\
 & 0 & 0 & 0 & 0 & \delta_9 \\
 & 0 & 0 & 0 & 0 & \delta_{10} \\
 & \delta_{11} & 0 & 0 & \delta_{11} & \delta_{11} \\
 & \delta_{12} & 0 & 0 & \delta_{12} & \delta_{12} \\
 & 0 & \delta_{13} & 0 & \delta_{13} & \delta_{13} \\
 & 0 & \delta_{14} & 0 & \delta_{14} & \delta_{14} \\
 & 0 & 0 & \delta_{15} & \delta_{15} & \delta_{15} \\
 & 0 & 0 & \delta_{16} & \delta_{16} & \delta_{16}
\end{array}$$

where $\delta_i = \pm 1$.

PROOF. We can verify the proposition as in Proposition 5.2.

References

[1] Alperin, J. L., Isomorphic blocks. J. Algebra **43** (1976), 694-698.
[2] Bender, H., On groups with abelian Sylow 2-subgroups. Math. Z. **117** (1970), 164-176.
[3] Brauer, R., Some applications of the theory of blocks of characters of finite groups I-IV. J. Algebra **1** (1964), 152-167; ibid. **1** (1964), 307-334; ibid. **3** (1966), 225-255; ibid. **17** (1971), 489-521.
[4] Brauer, R., On blocks and sections in finite groups I-II. Amer. J. Math. **89** (1967), 1115-1136; ibid. **90** (1968), 895-925.
[5] Brauer, R., Defect groups in the theory of representations of finite groups. Illinois J. Math. **13** (1969), 53-73.
[6] Brauer, R., On 2-blocks with dihedral defect groups. Symposia Mathematica **13** (1974), 367-393.
[7] Brauer, R. and Nesbitt, C., On the modular characters of groups. Ann. of Math. (2) **42** (1941), 556-590.
[8] Curtis, C. W. and Reiner, I., Representation theory of finite groups and associative algebras. Interscience, New York, 1962.
[9] Dade, E. C., Blocks with cyclic defect groups. Ann. of Math. (2) **84** (1966), 20-48.
[10] Dornhoff, L., Group representation theory (parts A and B). Dekker, New York, 1971-72.
[11] Feit, W., Representations of finite groups (mimeographed notes). Yale University, Connecticut, 1969.
[12] Feit, W. and Thompson, J. G., Solvability of groups of odd order. Pacific J. Math. **13** (1963), 775-1029.
[13] Fong, P., On decomposition numbers of J_1 and $R(q)$. Symposia Mathematica **13** (1974), 415-422.
[14] Gorenstein, D., Finite groups. Harper and Row, New York, 1968.
[15] Huppert, B., Endliche Gruppen I. Springer, Berlin, 1967.
[16] Janko, Z., A new finite simple group with abelian Sylow 2-subgroups and its characterization. J. Algebra **3** (1966), 147-186.
[17] Janko, Z. and Thompson, J. G., On a class of finite simple groups of Ree. J. Algebra **4** (1966), 274-292.
[18] Olsson, J. B., On 2-blocks with quaternion and quasidihedral defect groups. J. Algebra **36** (1975), 212-241.
[19] Reynolds, W. F., A block correspondence and isometries of group characters. Math. Z. **113** (1970), 1-16.
[20] Walter, J. H., The characterization of finite groups with abelian Sylow 2-subgroups. Ann. of Math. (2) **89** (1969), 405-514.
[21] Ward, H. N., On Ree's series of simple groups. Trans. Amer. Math. Soc. **121** (1966), 62-89.

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