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ON KILLING FIELDS PRESERVING MINIMAL FOLIATIONS OF POLYNOMIAL GROWTH AT MOST 2

By

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Abstract. Let $\mathscr{F}$ be a minimal foliation of a complete Riemannian manifold (M, g). Assume that the orthogonal distribution to $\mathscr{F}$ is also integrable. We show that if the growth of $\mathscr{F}$ is at most 2 then any Killing field with bounded length preserves the foliation $\mathscr{F}$.

1. Introduction

In the previous papers [5], [7], [8], it is shown that Killing fields preserve codimension-one totally geodesic foliations under some conditions. In [4], it is shown that on closed Riemannian manifolds, Killing fields preserve codimension-one minimal foliations with growth ≤ 1. Recently, Andrzejewski [2] shows that any Killing fields preserves minimal compact foliations if the orthogonal complement to $\mathscr{F}$ is also integrable by using the idea of Jacobi fields. In this paper, we extend these results to minimal foliations with growth ≤ 2 under some bounded conditions on Killing fields.

We shall give some definitions, preliminaries and the result in §2, and shall prove it in §3. A remark is given in §4.

2. Preliminaries and Result

In this paper, we work in the C^∞-category. In what follows, we always assume that foliations are p-dimensional, oriented and transversely oriented, and that the ambient manifolds are connected, oriented and of dimension $n = p + q \geq 2$, unless otherwise stated (see [6], [9], [11] for the generalities on foliations).

Let g be a Riemannian metric of M. Let $\mathscr{H}$ be the orthogonal bundle complementary to $\mathscr{F}$. We denote $g(,)$ by $\langle , \rangle$ and the Riemannian connection by

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∇. Orientations of M, $\mathscr{F}$ and $\mathscr{H}$ are related as follows: Let $\{E_1, E_2, \dots, E_p\}$ be an oriented local orthonormal frame of $\mathscr{F}$, and $\{X_1, X_2, \dots, X_q\}$ be an oriented local orthonormal frame of $\mathscr{H}$. Then the orientation of M coincides with the one given by $\{E_1, E_2, \dots, E_p, X_1, X_2, \dots, X_q\}$.

Now recall some preliminaries from the paper of Andrzejewski [2]. Let $\Gamma(\mathscr{F})$ and $\nabla^\top$ denote the set of all vector fields tangent to $\mathscr{F}$ and the induced connection on $\mathscr{F}$, respectively. Similarly, we define $\Gamma(\mathscr{H})$ and $\nabla^\perp$ on $\mathscr{H}$. For subbundles ξ, η of TM, denote by $L(\Gamma(\xi), \Gamma(\zeta))$ the set of all smooth linear transformations with the induced inner product.

Define the shape operator $A^V \in L(\Gamma(\mathscr{F}), \Gamma(\mathscr{F}))$ of $\mathscr{F}$ with respect to $V \in \Gamma(\mathscr{H})$ by

$$A^V(E) = -(\nabla_E V)^\top \quad \text{for } E \in \Gamma(\mathscr{F}).$$

Note that $\mathscr{F}$ is called *minimal* if $\mathrm{Tr}(A^V) = 0$ for all $V \in \Gamma(\mathscr{H})$. Regarding A as the mapping $A : \Gamma(\mathscr{H}) \to L(\Gamma(\mathscr{F}), \Gamma(\mathscr{F}))$, we define the transpose A^t of A by

$$\langle A^t(B), V\rangle(x) = \langle A^V, B\rangle(x) \quad \text{for } B \in L(\Gamma(\mathscr{F}), \Gamma(\mathscr{F})) \text{ and } x \in M,$$

and set

$$\hat{A} = A^t \circ A.$$

For $V \in \Gamma(\mathscr{H})$, define a new section $\nabla^{\perp^2} V$ in $\mathscr{H}$ by setting

$$\nabla^{\perp^2} V = \Sigma_{i=1}^p \nabla_{E_i}^\perp \nabla_{E_i}^\perp V - \nabla_{\nabla_{E_i}^\top E_i}^\perp V,$$

where $\{E_1, E_2, \dots, E_p\}$ is an orthonormal frame of $\mathscr{F}$. We also define $R(V)$ for $V \in \Gamma(\mathscr{H})$ by $R(V) = (\Sigma_{i=1}^p R(E_i, V)E_i)^\perp$, where $R(,)$ is the curvature tensor of ∇.

Now define the Jacobi operator $J : \Gamma(\mathscr{H}) \to \Gamma(\mathscr{H})$ by

$$J(V) = -\nabla^{\perp^2} V + R(V) - \hat{A}(V).$$

A normal section $V \in \Gamma(\mathscr{H})$ is called *a Jacobi field* of $\mathscr{F}$ if $J(V) = 0$ on M. It is known that if Z is a Killing field, then $Z^\perp$ is a Jacobi field (cf. [10]). The notion of Jacobi field is closely related to the fact that a vector field Z preserves $\mathscr{F}$ from the following results (Propositions 2.7 and 2.8 in [2]).

Proposition A1. *Let $\mathscr{F}$ be a minimal foliation of a manifold M with the integrable orthogonal distribution. If a vector field X on M is foliation preserving, i.e., maps leaves to leaves, then $X^\perp$ is a Jacobi field.*

Proposition A2. *Let $\mathscr{F}$ be a minimal compact foliation of a manifold M with the integrable orthogonal distribution. If a vector field X on M satisfies $J(X^{\perp}) = 0$, that is, $X^{\perp}$ is a Jacobi field, then X is foliation preserving. In particular, every Killing field preserves $\mathscr{F}$.*

For codimension-one minimal foliations, the following is proved in [4].

Theorem C. *Let $\mathscr{F}$ be a codimension-one minimal foliation of a closed manifold M. If every leaf of $\mathscr{F}$ has polynomial growth of first order, then every Killing field preserves $\mathscr{F}$.*

Our result is an extension of these results.

Theorem. *Let $\mathscr{F}$ be a minimal foliation of a complete Riemannian manifold (M, g) with the integrable orthogonal distribution. If every leaf of $\mathscr{F}$ has polynomial growth of at most second order, then every Killing field with the bounded length preserves $\mathscr{F}$.*

3. Proof of Theorem

For $V \in \Gamma(\mathscr{H})$, define a mapping α_V by

$$\alpha_V(E) = [V, E]^{\perp} \quad \text{for } E \in \Gamma(\mathscr{F}).$$

It is easy to see that $\alpha_V = 0$ if and only if V preserves $\mathscr{F}$. By Lemma 4.2 in [2], we have the following relation between the Jacobi operator and α_V.

Lemma. *Let $\mathscr{F}$ be a minimal foliation of a manifold M and the orthogonal distribution is integrable. Then we have the following formula*

$$\langle J(V), W\rangle = \langle \alpha_V, \alpha_W\rangle + \mathrm{div}_L(\alpha_V^t(W)) \quad \text{for } V, W \in \Gamma(\mathscr{H}),$$

where α_V^t is the transpose of α_V.

(Proof of Theorem.) Let Z a Killing field of M. Set $V = Z^{\perp}$. Firstly, note that V is a Jacobi field of $\mathscr{F}$ (cf. [2], [10]). Thus, by Lemma,

$$0 = \langle J(V), V\rangle = \langle \alpha_V, \alpha_V\rangle + \mathrm{div}_L(\alpha_V^t(V)).$$

In order to prove that Z preserves $\mathscr{F}$, it is sufficient to show that $\alpha_V = 0$.

Let L be a leaf of $\mathscr{F}$. If L is a closed leaf, then

$$\int_L \langle \alpha_V, \alpha_V \rangle = -\int_L \mathrm{div}_L(\alpha_V^t(V)) = 0.$$

Thus, $\alpha_V = 0$ and this completes the proof.

Now assume L is a non-compact leaf with the growth $\mathrm{gr}(L) \leq 2$. Fix $x \in L$. Then, by definition,

$$\mathrm{vol}(D(r)) \leq ar^2 + b \quad (r \geq 0),$$

for some positive constants a and b, where $D(r) = \{y \in L \mid d_L(x, y) \leq r\}$. Set $f(r) = \int_{D(r)} \langle \alpha_V, \alpha_V \rangle$ and $v(r) = \mathrm{vol}(D(r))$. It is known that $f(r)$ and $v(r)$ are locally Lipschitz, and thus, a.e. differentiable on $r > 0$. By integrating the equality $\langle \alpha_V, \alpha_V \rangle = -\mathrm{div}_L(\alpha_V^t(V))$ over $D(r)$, we have

$$\int_{D(r)} \langle \alpha_V, \alpha_V \rangle = -\int_{D(r)} \mathrm{div}_L(\alpha_V^t(V)) = -\int_{\partial D(r)} \langle \alpha_V^t(V), \nu \rangle,$$

where ν is the outward unit normal vector to $\partial D(r) \subset L$. It follows that

$$\begin{aligned} \int_{D(r)} |\alpha_V|^2 &\leq \int_{\partial D(r)} |\langle \alpha_V^t(V), \nu \rangle| = \int_{\partial D(r)} |\langle \alpha_V(\nu), V \rangle| \\ &\leq \int_{\partial D(r)} C|\alpha_V| \leq C\sqrt{\int_{\partial D(r)} 1}\sqrt{\int_{\partial D(r)} |\alpha_V|^2}, \end{aligned}$$

where $|V| \leq |Z| \leq C < +\infty$ by the boundedness assumption on Z. As $f'(r) = \int_{\partial D(r)} |\alpha_V|^2$ and $v'(r) = \int_{\partial D(r)} 1$, we have

$$f(r)^2 \leq C^2 v'(r) f'(r).$$

Assume that $|\alpha_V|(x) \neq 0$. Then $f(r) > 0$ for $r > 0$. As $v'(r) > 0$, we have

$$\frac{1}{v'(r)} \leq \frac{C^2 f'(r)}{f(r)^2} = \left(-\frac{C^2}{f(r)}\right)'.$$

Integrating this on $[r, R]$ with $0 < r < R$, we get

$$\int_r^R \frac{1}{v'(r)}\, dr \leq \frac{C^2}{f(r)} - \frac{C^2}{f(R)}.$$

The inequality

$$\left(\int_r^R dr\right)^2 = \left(\int_r^R \sqrt{v'(r)}\sqrt{\frac{1}{v'(r)}}\, dr\right)^2 \leq \left(\int_r^R v'(r)\, dr\right)\left(\int_r^R \frac{1}{v'(r)}\, dr\right)$$

implies

$$\frac{(R-r)^2}{v(R)-v(r)} \le \int_r^R \frac{1}{v'(r)}\,dr.$$

It follows that

$$\frac{(R-r)^2}{v(R)-v(r)} \le \int_r^R \frac{1}{v'(r)}\,dr \le \frac{C^2}{f(r)} - \frac{C^2}{f(R)}.$$

Letting $R = 2r$, we have

$$\frac{r^2}{4ar^2+b} \le \frac{r^2}{v(2r)} \le \frac{r^2}{v(2r)-v(r)} \le \frac{C^2}{f(r)} - \frac{C^2}{f(2r)}.$$

As $f'(r) \ge 0$, if $f(r)$ is bounded above with $f(r) \to C_0$ as $r \to \infty$, then the above inequality implies

$$0 < \frac{1}{8a} \le \frac{r^2}{4ar^2+b} \le \frac{C^2}{f(r)} - \frac{C^2}{f(2r)} \to \frac{C^2}{C_0} - \frac{C^2}{C_0} = 0 \quad (\text{as } r \to \infty),$$

which is a contradiction. If $f(r)$ tends to the infinity as $r \to \infty$, then we have

$$0 < \frac{1}{8a} \le \frac{r^2}{4ar^2+b} \le \frac{C^2}{f(r)} - \frac{C^2}{f(2r)} \to 0 \quad (\text{as } r \to \infty),$$

which is also a contradiction. Therefore we have $\alpha_V = 0$ on L. As this holds on every leaf of $\mathscr{F}$, it follows that V preserves $\mathscr{F}$. This completes the proof.

4. A Concluding Remark

As a corollary to our theorem, we get the following.

Corollary. *Let $\mathscr{F}$ be a minimal foliation of the Euclidean space (E^n, g_0) and assume that the orthogonal distribution $\mathscr{H}$ is integrable. If the growth of $\mathscr{F}$ is at most* 2, *then $\mathscr{F}$ and $\mathscr{H}$ are totally geodesic foliations.*

(Proof of Corollary.) By our theorem, any parallel vector field preserves $\mathscr{F}$. This means that $\mathscr{F}$ is invariant under any parallel transformation of E^n, which implies that any leaf of $\mathscr{F}$ is a p-plane, thus, totally geodesic. By the result of Abe [1], as $\mathscr{H}$ is integrable, $\mathscr{H}$ is also totally geodesic.

This gives a proof of the famous Bernstein's Theorem: Any smooth function $E^2 \to R$ with the minimal graph is linear.

However, as this does not hold for higher dimensions (cf. [3]), the growth condition of our theorem can not be removed without any additional conditions.

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