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The relaxed stochastic maximum principle in optimal control of diffusions with controlled jumps

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Abstract. This paper is concerned with optimal control of systems driven by stochastic differential equations (SDEs), with jump processes, where the control variable appears in the drift and in the jump term. We study the relaxed problem, in which admissible controls are measure-valued processes and the state variable is governed by an SDE driven by a counting measure valued process called relaxed Poisson measure such that the compensator is a product measure. Under some conditions on the coefficients, we prove that every diffusion process associated to a relaxed control is a limit of a sequence of diffusion processes associated to strict controls. As a consequence, we show that the strict and the relaxed control problems have the same value function. Using similar arguments, we prove the existence of an optimal relaxed control. In a second step, we establish a maximum principle for this type of relaxed problem.

Key words: Stochastic control, Stochastic differential equation, jump process, optimal control, relaxed control - maximum principle.
AMS 2010 Mathematics Subject Classification : 93E20, 60H10

Résumé (French Abstract). L'objectif de cet article est l'étude du contrôle optimal de systèmes dirigés par des équations différentielles stochastiques (EDS), présentant des sauts, où le paramètre de contrôle apparaît aussi bien dans le drift que dans le terme de saut. Nous étudions le problème relaxé, dans lequel les contrôles admissibles sont des processus à valeurs mesures et la variable d'état est gouvernée par une EDS dirigée par une mesure de comptage appelée mesure de Poisson relaxée, dont le compensateur est une mesure produit. Sous certaines hypothèses sur les coefficients, nous montrons que tout processus de diffusion

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associé à un contrôle relaxé est limite d'une suite de diffusions associées à des contrôles stricts. Comme conséquence, nous établissons que les problèmes de contrôle strict et relaxé ont la même fonction de valeurs. En utilisant des arguments similaires, nous montrons l'existence d'un contrôle optimal relaxé. Dans une deuxième étape, nous démontrons un principe du maximum pour ce type de problème relaxé.

1. Introduction

We consider a control problem where the state variable is a solution of a stochastic differential equation (SDE), in which the control enters the drift and the jump term. More precisely the system evolves according to the SDE

$$\begin{cases} dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \int\limits_{\Gamma} f(t, x_{t^-}, \theta, u_t)\widetilde{N}(dt, d\theta) \\ x_0 = 0 \end{cases},$$

on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where b, σ, and f are given deterministic functions, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration governed by a standard Brownian motion B and an independent Poisson random measure N, whose compensator is given by $\upsilon(d\theta)dt$ and u stands for the control variable.

The expected cost to be minimized over the class of admissible controls is defined by

$$J(u) = E\left[g(x_T) + \int\limits_0^T h(t, x_t, u_t)dt\right].$$

A control process that solves this problem is called optimal. The strict control problem may fail to have an optimal solution, if we don't impose some kind of convexity assumption. In this case, we must embed the space of strict controls into a larger space that has nice properties of compactness and convexity. This space is that of probability measures on A, where A is the set of values taken by the strict control. These measure valued processes are called relaxed controls. The first existence result of an optimal relaxed control is proved by Fleming (1977), for the SDE's with uncontrolled diffusion coefficient and no jump term. For such systems of SDE's a maximum principle has been established in Bahlali *et al.* (2007, 2006); Mezerdi, and Bahlali (2002). For mean-field systems one can refer to Bahlali *et al.* (2014, 2017a,b). The case where the control variable appears in the diffusion coefficient has been solved in El-karaoui *et al.* (1987). The existence of an optimal relaxed control of SDE's, where the control variable enters in the jump term was derived by Kushner (2000). One can refer to mean-field control problems.

In this paper, we first show that under a continuity condition of the coefficients, each relaxed diffusion process with controlled jump is a strong limit of a sequence of diffusion processes associated with strict controls. The proof of this approximation result is based on Skorokhod selection theorem, and the tightness of the processes. Consequently, we show that the strict and the relaxed control problems have the same value function. Using the same techniques, we give another proof of the existence of an optimal relaxed control, based on the Skorokhod selection theorem.

The second main goal of this paper is to establish a Pontriagin maximum principle for the relaxed control problem. More precisely we derive necessary conditions for optimality satisfied by an optimal control. The proof is based on Pontriagin's maximum principle for nearly optimal strict controls and some stability results of trajectories and adjoint processes with respect to the control variable.

The rest of the paper is organized as follows : in section 2, we formulate the control problem, and introduce the assumptions of the model. Section 3 is devoted to the proof of the approximation and existence results. In the last section, we state and prove a maximum principle for our relaxed control problem, which is the main result of this paper.

2. Formulation of the problem

2.1. Strict control problem

We consider a control problem of systems governed by stochastic differential equations on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, such that $\mathcal{F}_0$ contains the $P-$null sets. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is generated by a standard Brownian motion B and an independent Poisson measure N, with compensator $\upsilon(d\theta)dt$, where the jumps are confined to a compact set Γ.And set

$$\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \upsilon(d\theta)dt$$

Consider a compact set A in $\mathbb{R}^k$ and let U the class of measurable, adapted processes $u : [0;T] \times \Omega \longrightarrow A$. For any $u \in U$, we consider the following stochastic differential equation (SDE)

$$\begin{cases} dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \int_{\Gamma} f(t, x_{t^-}, \theta, u_t)N(dt, d\theta) \\ x_0 = 0 \end{cases} \tag{1}$$

where

$$\begin{aligned} b &: [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n \\ \sigma &: [0;T] \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R}) \\ f &: [0;T] \times \mathbb{R}^n \times \Gamma \times A \longrightarrow \mathbb{R}^n \end{aligned}$$

are bounded, continuous functions. The expected cost is given by

$$J(u) = E\left[g(x_T) + \int_0^T h(t, x_t, u_t)dt\right] \tag{2}$$

where

$$\begin{aligned} g &: \mathbb{R}^n \longrightarrow \mathbb{R} \\ h &: [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R} \end{aligned}$$

be bounded and continuous functions. The strict optimal control problem is to minimize the functional $J(.)$ over U. A control that solves this problem is called optimal.

2.2. The relaxed control problem

The strict control problem, may fail to have an optimal solution. Then the space of strict controls must be injected into a wider space that has good properties of compactness and convexity. This space is that of probability measures on A, where A is the set of values taken by the strict control. These are called relaxed controls. For more details see Mezerdi, and Bahlali (2002); Mezerdi and Bahlali (2000) . The problem now is to define rigorously the dynamics associated to a relaxed control. More precisely, since the jump term is controlled, one has to define the concept of relaxed Poisson random measure. For this purpose, we follow closely Kushner and Dupuis (2001) page 357-365 and Kushner (2000), where the detailed proofs can be found.

Let us begin with a simple example. Suppose that u takes two values a_1 and a_2 such that

$$u^{\rho}(t) = \begin{cases} a_1, & t \in [k\rho; k\rho + \beta_1\rho] \\ a_2, & t \in [k\rho + \beta_1\rho; k\rho + \rho] \end{cases} \qquad k = 1, 2, ...$$

where $\rho > 0$, and $\beta_1 + \beta_2 = 1$.

Let x^{ρ} denotes the associated solution to (1). If we define $1_i^{\rho}(s)$ by

$$1_i^{\rho}(s) = \begin{cases} 1, & u^{\rho}(s) = a_i(s) \\ 0, & \text{otherwise} \end{cases},$$

then the SDE (1) takes the form

$$\begin{cases} dx_t^{\rho} = \sum\limits_{i=1}^{2} 1_i^{\rho}(t) b(t, x_t^{\rho}, a_i(t))dt + \sigma(t, x_t^{\rho})dB_t + \sum\limits_{i=1}^{2} \int\limits_{\Gamma} 1_i^{\rho}(t) f(t, x_{t^-}^{\rho}, \theta, a_i(t))N(dt, d\theta) \\ x_0^{\rho} = 0 \end{cases}$$

Let μ^{ρ} denotes the relaxed version of the control u^{ρ}, that is $\mu_t^{\rho}(da_i)dt = \delta_{u^{\rho}(t)}(da_i)dt$. It is easy to see that $1_i^{\rho}(s) = \mu_t^{\rho}(a_i)$ which converges weakly to $\mu_t(a_i) = \beta_i$, when $\rho \to 0$.

By the tightness of the set of jumps, we can fix a weakly convergent sub-sequence of the jumps, such that the limit satisfies the following SDE

$$\begin{cases} dx_t = \int\limits_{A} b(t, x_t, a_t)\mu_t(da)dt + \sigma(t, x_t)dB_t + \sum\limits_{i=1}^{2} \int\limits_{\Gamma} f(t, x_{t^-}, \theta, a_i(t))\overline{N_i}(dt, d\theta) \\ x_0 = 0 \end{cases}$$

where $\overline{N_i}, i = 1, 2$ are independent Poisson measures with compensator $\upsilon(d\theta)\beta_i dt$.

Remark 1. Note that the previous type of approximation can be adapted to the case where the fractions of the intervals on which the a_i are used are time dependent in a non-anticipative way. in this case the compensator of $\overline{N_i}, i = 1, 2$, is the random and time varying quantity $\upsilon(d\theta)\mu_t(da_i)dt$. Moreover, the $\overline{N_i}, i = 1, 2$, would not be independent, but the martingales defined by

$$\int\limits_0^t \int\limits_{\Gamma} 1_i^{\rho}(s) f(s, x_{s^-}^{\rho}, \theta, a_i)N(ds, d\theta) - \int\limits_0^t \int\limits_{\Gamma} f(s, x_{s^-}^{\rho}, \theta, a_i)\upsilon(d\theta)\mu_s^{\rho}(da_i)ds$$

converge weakly to the processes

$$\int_0^t\int_\Gamma f(s,x_{s^-},\theta,a_i)\overline{N_i}(dt,d\theta)-\int_0^t\int_\Gamma f(s,x_{s^-},\theta,a_i)\upsilon(d\theta)\mu_s(da_i)ds$$

which are orthogonal $\mathcal{F}_t-$martingales.

The general case.

Let μ be the relaxed representation of an admissible control u, and let $A_0 \in \mathcal{B}(A)$ and $\Gamma_0 \in \mathcal{B}(\Gamma)$. Then define

$$N^\mu([0;t]\,,A_0,\Gamma_0) \equiv N^\mu(t,A_0,\Gamma_0) = \int_0^t\int_{\Gamma_0} 1_{A_0}(u(s))N(ds,d\theta),$$

the number of jumps of $\int_0^{\cdot}\int_{\Gamma_0}\theta N(ds,d\theta)$ on $[0;t]$ with values in Γ_0 and where $u(s)\in A_0$ at the jump times s.

Since $1_{A_0}(u(s)) = \mu_s(A_0)$, then the compensator of the counting measure valued process N^μ is $\upsilon(d\theta)\mu_t(da)dt = \mu_t \otimes \nu(da,d\theta)dt$. Moreover, for bounded and measurable real-valued functions $\varphi(.)$, the process

$$\int_0^t\int_\Gamma\int_A \varphi(s,x_{s^-},\theta,a)N^\mu(dt,d\theta,da)-\int_0^t\int_\Gamma\int_A \varphi(s,x_{s^-},\theta,a)\upsilon(d\theta)\mu_s(da)ds$$

is also an $\mathcal{F}_t-$martingale.

Definition 1. A relaxed Poisson measure N^μ is a counting measure valued process such that its compensator is the product measure of the relaxed control μ with the compensator υ of N, such that for any Borel set $\Gamma_0 \subset \Gamma$ and $A_0 \subset A$, the processes

$$\widetilde{N}^\mu(t,A_0,\Gamma_0) = N^\mu(t,A_0,\Gamma_0) - \mu(t,A_0)\nu(\Gamma_0)$$

are $\mathcal{F}_t-$martingales and are orthogonal for disjoint $\Gamma_0 \times A_0$.

Write the stochastic differential equation with controlled jumps in terms of relaxed Poisson measure as follows

$$\begin{cases} dx_t^\mu = \int_A b(t,x_t^\mu,a)\mu_t(da)dt + \sigma(t,x_t^\mu)dB_t + \int_A\int_\Gamma f(t,x_{t^-}^\mu,\theta,a)\widetilde{N}^\mu(dt,d\theta,da) \\ x_0^\mu = 0 \end{cases} \tag{3}$$

The expected cost associated to a relaxed control is defined as

$$J(\mu) = E\left[g(x_T^\mu) + \int_A\int_0^T h(t,x_t^\mu,a)\mu_t(da)dt\right]$$

Consider a sequence of random predictable measures $(\mu_s^n \otimes \nu)_n$ converging weakly to $\mu_s \otimes \nu$ on $[0;T] \times A \times \Gamma$ $P-$almost surely, then there exists a sequence of orthogonal martingale measures $\widetilde{N}^n$ defined on $\Omega \times [0;T] \times A \times \Gamma$ with compensator $\mu_s^n \otimes \nu(da, d\theta)ds$, such that for each bounded function φ

$$\int_0^t \int_A \int_\Gamma \varphi(s, x_{s-}^\mu, \theta, a)\widetilde{N}^n(ds, d\theta, da) \text{ converges to } \int_0^t \int_A \int_\Gamma \varphi(s, x_{s-}^\mu, \theta, a)\widetilde{N}^\mu(ds, d\theta, da)$$

3. Approximations and existence of a relaxed optimal control

3.1. Approximation of trajectories

In order for the relaxed control problem to be truly an extension of the strict one, the infimum of the expected cost for the relaxed controls must be equal to the infimum for the strict controls. This result is based on the approximation of a relaxed control by a sequence of strict controls, given by the next Lemma, which called chattering lemma

Lemma 1. *Let μ be a predictable process with values in the space $\mathcal{P}(A)$. Then there exists a sequence of predictable processes (u^n) with values in A such that*

$$\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t(da)dt \quad \textit{weakly}$$

Proof. see Fleming (1977)

The next theorem which is our main result in this section gives the stability of the stochastic differential equations with respect to the control variable, and that the two problems has the same infimum of the expected costs.

Theorem 1. *Let μ be a relaxed control, and let x^μ be the corresponding trajectory. We assume that we have strong uniqueness for the state equation. Then there exists a sequence (u^n) of strict controls such that*

$$\lim_{n\to\infty} E\left[\sup_{0\le t\le T} |x_t^n - x_t^\mu|^2\right] = 0.$$

$$\lim_{n\to\infty} J(u^n) = J(\mu) \tag{4}$$

where x^n denotes the trajectory associated with (u^n).

To prove Theorem (1), we need some results on the tightness of the processes

Lemma 2. *The family of relaxed controls $((\mu^n)_{n\ge 0}, \mu)$ is tight in $\mathcal{R}$ the space of probability measures on $[0;T] \times A$*

Proof. see Mezerdi, and Bahlali (2002)

Lemma 3. *The family of martingale measures $((\widetilde{N}^n)_{n\ge 0}, \widetilde{N}^\mu)$ is tight in the space $D_{S'}([0;T])$ of all cadlag mappings from $[0;T]$ with values in S' the topological dual of the Schwartz space S of rapidly decreasing functions.*

Proof. If we denote

$$Y_t^n = \int_0^t \int_{A\times\Gamma} \psi(t, x_{t^-}^n, \theta, a)\widetilde{N}^n(dt, d\theta, da)$$

and

$$Y_t = \int_{A\times\Gamma} \psi(t, x_{t^-}^\mu, \theta, a)\widetilde{N}^\mu(dt, d\theta, da),$$

and, let S, T two stopping times, such that $S \leq T \leq S + \theta$, then we have
$\forall n \in \mathbb{N}$, $\epsilon > 0$, $\exists m$ and $k > 0$, such that

$$n \geq m \quad P(\sup_{t\leq n} |Y_t^n| > k) \leq \frac{E\,|Y_t^n|^2}{k^2} \leq \varepsilon$$

and, for all $n \in \mathbb{N}$, for all $\epsilon > 0$. By the proposition (1) (See the appendix), we have

$$P(\sup_{t\in[S;T]} |Y_S^n - Y_T^n| \geq \eta) \leq \frac{\varepsilon}{\eta^2} + P(< Y^n >_T - < Y^n >_S \geq k)$$

Since $< Y^n >_T - < Y^n >_S \leq \omega(< Y^n >, \delta) = \sup_{|T-S|<\delta} |< Y^n >_T - < Y^n >_S|$, because $|T - S| \leq \delta$.

This implies that.

$$P(\sup_{t\in[S;T]} |Y_S^n - Y_T^n| \geq \eta) \leq \frac{\varepsilon}{\eta^2} + P(\omega(< Y^n >, \delta) \geq k),$$

by the $\mathcal{C}-$tightness of $< Y^n >$, we have

$$P(\omega(< Y^n >, \delta) \geq k) \leq \varepsilon.$$

Finally, we conclude that

$$\lim_{\delta\to 0} \limsup_n \sup_{S\leq T\leq S+\theta} P(\sup_{t\in[S;T]} |Y_S^n - Y_T^n| \geq \eta) = 0.$$

That is the Aldous conditions is fulfilled (See the appendix). Hence the sequence $(Y_t^n)_{n\geq 0}$ is tight. By the same method we can prove the tightness of (Y_t).

Lemma 4. *if x^n, and x are the solutions of (3) associated with μ^n and μ, respectively, then the family of processes (x^n, x) is tight in the $D([0;T], \mathbb{R}^d)$.*

Proof. By the same method in the proof of lemma (3).

Proof (Proof of theorem 1). (a)- Let μ be a relaxed control, then by the Lemma 1, there exists a sequence (u^n) such that $\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t(da)dt$ in $\mathcal{R}$, $P - a.s.$. Let x^n, and x are the solutions of (3) associated with μ^n and μ, respectively. Suppose that the result of theorem (1) is false, then there exists $\gamma > 0$ such that

$$\inf_n E\left[|x_t^n - x_t^\mu|^2\right] \geq \gamma \tag{5}$$

According to Lemmas (2), (3) and (4), the family of processes

$$\beta^n = (\mu^n, \mu, x^n, x, \widetilde{N}^n, \widetilde{N}^\mu)$$

is tight in the space

$$(\mathcal{R} \times \mathcal{R}) \times (D \times D) \times (D_{S'} \times D_{S'}).$$

Then, by the Skorokhod selection theorem, there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ and a sequence $\widehat{\beta^n} = (\widehat{\mu^n}, \widehat{\upsilon^n}, \widehat{x^n}, \widehat{y^n}, \widehat{\widetilde{N}^n}, \widehat{\widetilde{M}^n})$ defined on it such that

(b)- For each $n \in \mathbb{N}$, the laws of β^n and $\widehat{\beta^n}$ coincide,

(c)- there exists a sub-sequence $(\widehat{\beta^{n_k}})$ of $(\widehat{\beta^n})$ which converges to $\widehat{\beta}$, $\widehat{P} - a.s$ on the space $(\mathcal{R} \times \mathcal{R}) \times (D \times D) \times (D_{S'} \times D_{S'})$, where $\widehat{\beta} = (\widehat{\mu}, \widehat{\upsilon}, \widehat{x}, \widehat{y}, \widehat{\widetilde{N}^\mu}, \widehat{\widetilde{M}^\mu})$.

By the uniform integrability, we have

$$\gamma \leq \liminf_n E\left[\sup_{0\leq t\leq T} |x_t^n - x_t^\mu|^2\right] = \liminf_n \widehat{E}\left[\sup_{0\leq t\leq T} \left|\widehat{x_t^n} - \widehat{y_t^n}\right|^2\right] = \widehat{E}\left[\sup_{0\leq t\leq T} |\widehat{x}_t - \widehat{y}_t|^2\right]$$

where $\widehat{E}$ is the expectation with respect to $\widehat{P}$. We see that $\widehat{x_t^n}$ and $\widehat{y_t^n}$ satisfy the following equations

$$\begin{cases} d\widehat{x_t^n} = \int\limits_A b(s, \widehat{x_t^n}, a)\widehat{\mu_s^n}(da)ds + \sigma(s, \widehat{x_t^n})dB_s + \int\limits_A\int\limits_\Gamma f(s, \widehat{x_{t^-}^n}, \theta, a)\widehat{\widetilde{N}^n}(ds, d\theta, da) \\ \widehat{x_0^n} = 0 \end{cases}$$

$$\begin{cases} d\widehat{y_t^n} = \int\limits_A b(s, \widehat{y_t^n}, a)\widehat{\upsilon_s^n}(da)ds + \sigma(s, \widehat{y_t^n})dB_s + \int\limits_A\int\limits_\Gamma f(s, \widehat{y_{t^-}^n}, \theta, a)\widehat{\widetilde{M}^n}(ds, d\theta, da) \\ \widehat{y_0^n} = 0, \end{cases}$$

using the fact that $(\widehat{\beta^n})$ converges to $\widehat{\beta}$, $\widehat{P} - a.s$, it holds that $(\widehat{x_t^n})$ and $(\widehat{y_t^n})$ converge respectively to $\widehat{x}_t$ and $\widehat{y}_t$, which satisfy

$$\begin{cases} d\widehat{x}_t = \int\limits_A b(t, \widehat{x}_t, a)\widehat{\mu}_t(da)dt + \sigma(t, \widehat{x}_t)dB_t + \int\limits_A\int\limits_\Gamma f(t, \widehat{x_{t^-}}, \theta, a)\widehat{\widetilde{N}^\mu}(dt, d\theta, da) \\ \widehat{x_0} = 0 \end{cases}$$

$$\begin{cases} d\widehat{y}_t = \int\limits_A b(t, \widehat{y}_t, a)\widehat{\upsilon}_t(da)dt + \sigma(t, \widehat{y}_t)dB_t + \int\limits_A\int\limits_\Gamma f(t, \widehat{y_{t^-}}, \theta, a)\widehat{\widetilde{M}^\mu}(dt, d\theta, da) \\ \widehat{y}_0 = 0, \end{cases}$$

By the Lemma 1, the sequence (μ^n, μ) converges to (μ, μ) in $\mathcal{R}^2$. Moreover

$$law(\mu^n,\mu) = law(\widehat{\mu^n},\widehat{\upsilon^n}),$$

$$(\widehat{\mu^n},\widehat{\upsilon^n}) \longrightarrow (\widehat{\mu},\widehat{\upsilon}),\ \widehat{P}-a.s\ \ in\ \mathcal{R}^2,$$

if n tends to ∞. Hence, $law(\widehat{\mu},\widehat{\upsilon}) = law(\mu,\mu)$, then $\widehat{\mu}=\widehat{\upsilon}$, $\widehat{P}-a.s.$ By the same method we can prove that $\widehat{\widetilde{N}^{\mu}}(ds,d\theta,da) = \widehat{\widetilde{M}^{\mu}}(ds,d\theta,da)$, $\widehat{P}-a.s.$ It follows that $\widehat{x_t}=\widehat{y_t}$, $\widehat{P}-a.s$, by the uniqueness of solution, which is a contradiction (5).

(d)- By using the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |J(u^n)-J(\mu)| &\leq C\left(E\left|g(x_T^n)-g(x_T^{\mu}|^2)\right)^{\frac{1}{2}} \\ &+ CE\left|\int_0^t\int_A h(s,x_s^n,a)\mu_s^n(da)ds - \int_0^t\int_A h(s,x_s^n,a)\mu_s(da)ds\right| \\ &+ C\int_0^t\left(E\left|h(s,x_s^n,u)-h(s,x_s^{\mu},u)\right|^2\right)^{\frac{1}{2}}ds \end{aligned}$$

The first and the third terms in the right hand side converge to 0 because g and h are Lipshitz continuous in x, and the fact that

$$\lim_{n\to\infty} E\left[\left|x_t^n - x_t^{\mu}\right|^2\right] = 0.$$

Since h is bounded and continuous in a , an application of the dominated convergence theorem allows us to conclude that the second term in the right hand side tends to 0 .

3.2. Existence of an optimal relaxed control

We show in this section that there exists an optimal solution for the relaxed control problem, the proof is based on Skorokhod selection theorem and some results of tightness.

Theorem 2. *Under the continuity of the coefficients b, σ, f, g, and h, the relaxed control problem admits an optimal relaxed control.*

Proof. Let (x^n,μ^n) be a minimizing sequence for the cost function $J(\mu)$, that is

$$\lim_{n\to\infty} J(\mu^n) = \inf_{\mu\in\mathcal{R}} J(\mu)$$

where x^n is the solution of (3),corresponding to μ^n.

According to Lemmas (2), (3), and (4) the family of processes $\beta^n = (\mu^n, x^n, \widetilde{N^n})$ is tight in the space $(\mathcal{R}, D, D_{S'})$, by the Skorokhod selection theorem, there exist a probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$ and a sequence $\widehat{\beta^n} = (\widehat{\mu^n},\widehat{x^n},\widehat{\widetilde{N^n}})$ defined on it such that

1. For each $n\in\mathbb{N}$, the laws of β^n and $\widehat{\beta^n}$ coincide,

2. there exists a sub-sequence $(\widehat{\beta^{n_k}})$ of $(\widehat{\beta^n})$ which converges to $\widehat{\beta}$, $\widehat{P}-a.s$ on the space $\mathcal{R}\times D\times D_{S'}$, where $\widehat{\beta}=(\widehat{\mu},\widehat{x},\widehat{\widehat{N}})$.it holds that $\widehat{x^n}\ \underrightarrow{proba}\ \widehat{x}$ then, we have

$$\begin{aligned}\left|J(\widehat{\mu^{n_k}})-J(\widehat{\mu})\right| &\leq E\left|g(\widehat{x^{n_k}_T})-g(\widehat{x_T})\right| \\ &+E\left|\int_0^T\int_A h(t,\widehat{x^{n_k}_t},a)\widehat{\mu^{n_k}_t}(da)dt-\int_0^T\int_A h(t,\widehat{x}_t,a)\widehat{\mu^{n_k}_t}(da)dt\right| \\ &+E\left|\int_0^T\int_A h(t,\widehat{x}_t,a)\widehat{\mu^{n_k}_t}(da)dt-\int_0^T\int_A h(t,\widehat{x}_t,a)\widehat{\mu}_t(da)dt\right|\end{aligned}$$

then

$$\begin{aligned}\left|J(\widehat{\mu^{n_k}})-J(\widehat{\mu})\right| &\leq E\left|g(\widehat{x^{n_k}_T})-g(\widehat{x_T})\right| \\ &+E\int_0^T\left|h(t,\widehat{x^{n_k}_t},u^{n_k}_t)-h(t,\widehat{x}_t,u^{n_k}_t)\right|dt \\ &+E\left|\int_0^T\int_A h(t,\widehat{x}_t,a)\widehat{\mu^{n_k}_t}(da)dt-\int_0^T\int_A h(t,\widehat{x}_t,a)\widehat{\mu}_t(da)dt\right|\end{aligned}$$

The first and second terms in the right-hand side converge to 0, because h and g are bounded and continuous functions with respect to x. using the convergence of $(\widehat{\mu^{n_k}_t})_n$ to $\widehat{\mu}_t$, and the dominated convergence theorem to conclude that the last term tends to 0. Hence

$$\inf_{\mu\in\mathcal{R}}J(\mu)=\lim_{n\to\infty}J(\mu^n)=\lim_{n\to\infty}J(\widehat{\mu^n})=\lim_{n\to\infty}J(\widehat{\mu^{n_k}})=J(\widehat{\mu})$$

then $\widehat{\mu}$ is an optimal control.

Remark 2. From the previous results, we see that the relaxed model is a true extension of the strict one, because the infimum of the two cost functions are equal, and the relaxed model have an optimal solution.

4. Maximum principle for relaxed control problems

Our main goal in this section is to establish optimality necessary conditions for relaxed control problems, where the system is described by a SDE driven by a relaxed Poisson measure which is a martingale measure, of the form (3) and the admissible controls are measure-valued processes which called relaxed controls. The proof is based on the chattering lemma, and using Ekeland's variational principle, we derive necessary conditions of near optimality satisfied by a sequence of strict controls. By using stability properties of the state equations and adjoint processes, we obtain the maximum principle for our relaxed problem.

Throughout this section the following additional assumptions will be required.

($\mathbf{H}_1$) The maps b, σ, f and h are continuously differentiable with respect to x, and g is continuously differentiable in x .

($\mathbf{H}_2$) σ_x, f_x and g_x are bounded and b_x, h_x are bounded uniformly in u.

Under the above hypothesis, (1) has a unique strong solution and the cost functional (2) is well defined from U into $\mathbb{R}$.

4.1. The maximum principle for strict control

The purpose of this subsection is to derive optimality necessary conditions, satisfied by an optimal strict control. The proof is based on the strong perturbation of the optimal control u^*, which defined by :

$$u^h = \begin{cases} \nu & \text{if } \ t \in [t_0; t_0 + h] \\ u^* & \quad \text{otherwise} \end{cases}$$

where $0 \leq t_0 < T$ is fixed, h is sufficiently small, and ν is an arbitrary $A-$valued $\mathcal{F}_{t_0}-$measurable random such that $E\left|\nu\right|^2 < \infty$. Let x_t^h denotes the trajectory associated with u^h, then

$$\begin{cases} x_t^h = x_t^* \quad ; \ t \leq t_0 \\ dx_t^h = b(t, x_t^h, \nu)dt + \sigma(t, x_t^h)dB_t + \int\limits_\Gamma f(t, x_{t-}^h, \theta, \nu)\widetilde{N}(dt, d\theta) \quad ; t_0 < t < t_0 + h \\ dx_t^h = b(t, x_t^h, u^*)dt + \sigma(t, x_t^h)dB_t + \int\limits_\Gamma f(t, x_{t-}^h, \theta, u^*)\widetilde{N}(dt, d\theta) \quad ; t_0 + h < t < T \end{cases}$$

We first have

Lemma 5. *Under assumptions (H_1)-(H_2), we have*

$$\lim_{h \to 0} E\left[\sup_{t\in[t_0;T]} \left|x_t^h - x_t^*\right|^2\right] = 0 \tag{6}$$

Proof. For $t \in [t_0; t_0 + h]$, we get by standard arguments from stochastic calculus

$$\begin{aligned} \left|x_t^h - x_t^*\right|^2 \leq M \int\limits_{t_0}^{t} \left|x_s^h - x_s^*\right|^2 ds \\ + M \int\limits_{t_0}^{t} \left|\nu - u_s^*\right|^2 ds \\ + 3\left|M_t\right|^2 \end{aligned} \tag{7}$$

where $M_t = \int_{t_0}^{t} \left[\sigma(s, x_s^h) - \sigma(s, x_s^*)\right] dB_s + \int_{t_0}^{t}\int_{\Gamma} \left[f(s, x_{s^-}^h, \theta, u_s^h) - f(s, x_s^*, \theta, u_s^*)\right] \widetilde{N}(ds, d\theta)$

Let us take care to the last term, since σ, f, are continuous and $\int_{\Gamma} \upsilon(d\theta) < \infty$

$$|M_t|^2 \leq K\int_{t_0}^{t} \left|x_s^h - x_s^*\right|^2 ds + K\int_{t_0}^{t} |\nu - u_s^*|^2 ds \tag{8}$$

Replacing (8) in (7), and take the supremum and the expectation we get

$$\begin{aligned} E\left[\sup_{t\in[t_0;t_0+h]} \left|x_t^h - x_t^*\right|^2\right] &\leq C \sup_{s\in[t_0;t_0+h]} \int_{t_0}^{t} E\left[\left|x_s^h - x_s^*\right|^2\right] ds \\ &\quad + CE\left[\int_{t_0}^{t_0+h} |\nu - u_s^*|^2 ds\right] \end{aligned}$$

We can deduce by the standard arguments that,

$$E\int_{t_0}^{t_0+h} \left[\left|x_s^h - x_s^*\right|^2\right] ds \leq K \int_{t_0}^{t_0+h} E\left|\nu - u_s^*\right|^2 ds,$$

then,

$$E\left[\sup_{t\in[t_0;t_0+h]} \left|x_t^h - x_t^*\right|^2\right] \leq K \int_{t_0}^{t_0+h} E\left|\nu - u_s^*\right|^2 ds. \tag{9}$$

We next have for $t \in [t_0 + h; T]$,

$$\left|x_t^h - x_t^*\right|^2 \leq M\left|x_{t_0+h}^h - x_{t_0+h}^*\right|^2 + M\int_{t_0+h}^{t} \left|x_s^h - x_s^*\right|^2 ds.$$

Hence

$$E\left[\sup_{t\in[t_0+h;T]} \left|x_t^h - x_t^*\right|^2\right] \leq ME\left|x_{t_0+h}^h - x_{t_0+h}^*\right|^2 + ME\int_{t_0+h}^{T} \left|x_s^h - x_s^*\right|^2 ds.$$

We have

$$E\int_{t_0+h}^{T} \left|x_s^h - x_s^*\right|^2 ds \leq KE\left|x_{t_0+h}^h - x_{t_0+h}^*\right|^2,$$

then,

$$E\left[\sup_{t\in[t_0+h;T]} \left|x_t^h - x_t^*\right|^2\right] \leq KE\left|x_{t_0+h}^h - x_{t_0+h}^*\right|^2 \tag{10}$$

From (9) and (10), letting h tend to 0, we obtain (6).

Since u^* is optimal, then

$$J(u^*) \leq J(u^h) = J(u^*) + h \left.\frac{dJ(u^h)}{dh}\right|_{h=0} + \circ(h)$$

Thus a necessary condition for optimality is that

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} \geq 0$$

Let us compute this derivative. Note that the following properties holds, because $b(t,x,u)$, $h(t,x,u)$ and $f(t,x_{t^-},\theta,u)$ are sufficiently integrable

$$\frac{1}{h}\int_t^{t+h} E\left[|k(s,x_s,u_s) - k(t,x_t,u_t)|^2\right] \underrightarrow{h \to 0}\ 0\ dt - a.e \tag{11}$$

$$\frac{1}{h}\int_\Gamma\int_t^{t+h} E\left[|f(s,x_{s^-},\theta,u_s) - f(t,x_{t^-},\theta,u_t)|^2\right] \upsilon(d\theta) \underrightarrow{h \to 0}\ 0\ dt - a.e \tag{12}$$

where k stands for b or h. Choose t_0 such that (11) and (12) holds, then we have

Corollary 1. *Under assumptions (H_1)-(H_3), one has*

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[g_x(x_T^*)z_T + \varsigma_T\right] \tag{13}$$

where

$$\begin{cases} d\varsigma_t = h_x(t,x_t^*,u_t^*)z_t dt & t_0 \leq t \leq T \\ \varsigma_{t_0} = h(t_0,x_{t_0}^*,\nu) - h(t_0,x_{t_0}^*,u_{t_0}^*) \end{cases}$$

and the process z is the solution of the linear SDE

$$dz_t = \begin{cases} b_x(t,x_t^*,u_t^*)z_t dt + \sigma_x(t,x_t^*)z_t dB_t + \int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)z_{t^-}\widetilde{N}(dt,d\theta);\ t_0 \leq t \leq T \\ z_{t_0} = \left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right]. \end{cases} \tag{14}$$

From (H_2) the variational equation (14) has a unique solution.

To prove the corollary (1) we need the following estimates.

Lemma 6. *Under assumptions (H_1)-(H_3), it holds that*

$$\lim_{h\to 0} E\left[\left|\frac{x_t^h - x_t^*}{h} - z_t\right|^2\right] = 0.$$

and

$$\lim_{h\to 0} E\left[\left|\frac{1}{h}\int_{t_0}^T \left[(h(t,x_t^*,u_t^h) - (h(t,x_t^*,u_t^*)\right] - \varsigma_T\right|^2\right] = 0.$$

Poof. Let

$$y_t^h = \frac{x_t^h - x_t^*}{h} - z_t$$

Then, we have for $t \in [t_0;t_0+h]$,

$$\begin{aligned} dy_t^h &= \frac{1}{h}\left[b(t, x_t^* + h(y_t^h + z_t), \nu) - b(t, x_t^*, u_t^*) - hb_x(t, x_t^*, u_t^*)z_t\right] dt \\ &+ \frac{1}{h}\left[\sigma(t, x_t^* + h(y_t^h + z_t)) - \sigma(t, x_t^*) - h\sigma_x(t, x_t^*)z_t\right] dB_t \\ &+ \frac{1}{h}\int_\Gamma \left[f(t, x_{t^-}^* + h(y_{t^-}^h + z_{t^-}), \theta, \nu) - f(t, x_{t^-}^*, \theta, u_t^*) - hf_x(t, x_{t^-}^*, \theta, u_t^*)z_{t^-}\right] \widetilde{N}(dt, d\theta) \end{aligned}$$

and

$$y_{t_0}^h = -\left[b(t_0, x_{t_0}^*, \nu) - b(t_0, x_{t_0}^*, u_{t_0}^*)\right].$$

Hence, we have

$$\begin{aligned} y_{t_0+h}^h &= \frac{1}{h}\int_{t_0}^{t_0+h} \left[b(t, x_t^* + h(y_t^h + z_t), \nu) - b(t, x_t^*, \nu)\right] dt + \frac{1}{h}\int_{t_0}^{t_0+h} \left[b(t, x_t^*, \nu) - b(t, x_{t_0}^*, \nu)\right] dt \\ &+ \frac{1}{h}\int_{t_0}^{t_0+h} \left[b(t, x_{t_0}^*, \nu) - b(t_0, x_{t_0}^*, \nu)\right] dt + \frac{1}{h}\int_{t_0}^{t_0+h} \left[b(t_0, x_{t_0}^*, u_{t_0}^*) - b(t, x_t^*, u_t^*)\right] dt \\ &+ \frac{1}{h}\int_{t_0}^{t_0+h} \left[\sigma(t, x_t^* + h(y_t^h + z_t)) - \sigma(t, x_t^*)\right] dB_t \\ &+ \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Gamma \left[f(t, x_{t^-}^* + h(y_{t^-}^h + z_{t^-}), \theta, \nu) - f(t, x_{t^-}^*, \theta, \nu)\right] \widetilde{N}(dt, d\theta) \\ &+ \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Gamma \left[f(t, x_{t^-}^*, \theta, \nu) - f(t, x_{t_0^-}^*, \theta, \nu)\right] \widetilde{N}(dt, d\theta) \\ &+ \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Gamma \left[f(t, x_{t_0^-}^*, \theta, \nu) - f(t_0, x_{t_0^-}^*, \theta, \nu)\right] \widetilde{N}(dt, d\theta) \\ &+ \frac{1}{h}\int_{t_0}^{t_0+h}\int_\Gamma \left[f(t_0, x_{t_0^-}^*, \theta, \nu) - f(t_0, x_{t_0^-}^*, \theta, u_{t_0}^*)\right] \widetilde{N}(dt, d\theta) \end{aligned}$$

$$+\ \frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})-f(t,x^*_{t^-},\theta,u^*_t)\right]\widetilde{N}(dt,d\theta)$$
$$-\int_{t_0}^{t_0+h} b_x(t,x^*_t,u^*_t)z_t dt-\int_{t_0}^{t_0+h}\sigma_x(t,x^*_t)z_t dB_t-\int_{t_0}^{t_0+h}\int_{\Gamma} f_x(t,x^*_{t^-},\theta,u^*_t)z_t\widetilde{N}(dt,d\theta).$$

Then

$$\begin{aligned}
E\left|y^h_{t_0+h}\right|^2 &\leq C[E\sup_{t_0\leq t\leq t_0+h}\left|x^h_t-x^*_t\right|^2+\sup_{t_0\leq t\leq t_0+h}E\left|b(t,x^*_{t_0},\nu)-b(t_0,x^*_{t_0},\nu)\right|^2 dt\\
&+\frac{1}{h}E\int_{t_0}^{t_0+h}\left|b(t_0,x^*_{t_0},u^*_{t_0})-b(t,x^*_t,u^*_t)\right|^2 dt+E\sup_{t_0\leq t\leq t_0+h}\left|x^*_t-x^*_{t_0}\right|^2\\
&+E\int_{t_0}^{t_0+h}\int_{\Gamma}\left|\nu-u^*_{t_0}\right|^2\upsilon(d\theta)dt+E\int_{t_0}^{t_0+h}\left|z_t\right|^2 dt\\
&+\sup_{t_0\leq t\leq t_0+h}E\int_{\Gamma}\left|f(t,x^*_{t_0^-},\theta,\nu)-f(t_0,x^*_{t_0^-},\theta,\nu)\right|^2\upsilon(d\theta)\\
&+\frac{1}{h}E\int_{t_0}^{t_0+h}\int_{\Gamma}\left|f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})-f(t,x^*_{t^-},\theta,u^*_t)\right|^2\upsilon(d\theta)dt.
\end{aligned}\tag{15}$$

By Lemma (5), and the properties (11) and (12), it is easy to see that $E\left|y^h_{t_0+h}\right|^2$ tends to 0 as $h\to 0$.

For $t\in[t_0+h;T]$, we denote $x^{h,\lambda}_t=x^*_t+\lambda h(y^h_t+z_t)$, then y^h_t satisfies the following SDE

$$\begin{aligned}
dy^h_t &= \frac{1}{h}\left[b(t,x^*_t+h(y^h_t+z_t),u^*_t)-b(t,x^*_t,u^*_t)\right]dt+\frac{1}{h}\left[\sigma(t,x^*_t+h(y^h_t+z_t))-\sigma(t,x^*_t)\right]dB_t\\
&+\frac{1}{h}\int_{\Gamma}\left[f(t,x^*_{t^-}+h(y^h_{t^-}+z_{t^-}),\theta,u^*_t)-f(t,x^*_{t^-},\theta,u^*_t)\right]\widetilde{N}(dt,d\theta)\\
&-b_x(t,x^*_t,u^*_t)z_t dt-\sigma_x(t,x^*_t)z_t dB_t\\
&-\int_{\Gamma} f_x(t,x^*_{t^-},\theta,u^*_t)z_t\widetilde{N}(dt,d\theta)
\end{aligned}$$

then

$$\begin{aligned}
y^h_t &= y^h_{t_0+h}+\int_{t_0+h}^{t}\int_0^1 b_x(s,x^{h,\lambda}_s,u^*_s)y^h_s d\lambda ds+\int_{t_0+h}^{t}\int_0^1\sigma_x(s,x^{h,\lambda}_s)y^h_s d\lambda dB_s\\
&+\int_0^1\int_{t_0+h}^{t}\int_{\Gamma} f_x(s,x^{h,\lambda}_s,\theta,u^*_s)y^h_s d\lambda\widetilde{N}(ds,d\theta)+\rho^h_t
\end{aligned}$$

where

$$\rho_t^h = \int_{t_0+h}^{t}\int_0^1 b_x(s,x_s^{h,\lambda},u_s^*)z_s d\lambda ds + \int_{t_0+h}^{t}\int_0^1 \sigma_x(s,x_s^{h,\lambda})z_s d\lambda dB_s$$

$$+ \int_{t_0+h}^{t}\int_0^1\int_\Gamma f_x(s,x_s^{h,\lambda},\theta,u_s^*)z_s d\lambda \widetilde{N}(ds,d\theta)$$

$$- \int_{t_0+h}^{t} b_x(s,x_s^*,u_s^*)z_s ds - \int_{t_0+h}^{t} \sigma_x(s,x_s^*)z_s dB_s - \int_{t_0+h}^{t}\int_\Gamma f_x(s,x_{s^-}^*,\theta,u_s^*)z_s \widetilde{N}(ds,d\theta).$$

Hence

$$E\left|y_t^h\right|^2 \leq E\left|y_{t_0+h}^h\right|^2 + KE\int_{t_0+h}^{t}\left|\int_0^1 b_x(s,x_s^{h,\lambda},u_s^h)y_s^h d\lambda\right|^2 ds + KE\int_{t_0+h}^{t}\left|\int_0^1 \sigma_x(s,x_s^{h,\lambda})y_s^h d\lambda\right|^2 ds$$

$$+ KE\int_{t_0+h}^{t}\int_\Gamma\left|\int_0^1 f_x(s,x_s^{h,\lambda},\theta,u_s^h)y_s^h d\lambda\right|^2 \upsilon(d\theta)ds + KE\left|\rho_t^h\right|^2$$

Since b_x, σ_x, and f_x are bounded, then

$$E\left|y_t^h\right|^2 \leq E\left|y_{t_0+h}^h\right|^2 + CE\int_0^t \left|y_s^h\right|^2 ds + KE\left|\rho_t^h\right|^2$$

We conclude by the continuity of b_x, σ_x and f_x, and the dominated convergence that $\lim_{h\to 0}\rho_t^h = 0$. Hence by the Gronwall lemma, and (15) we get

$$\lim_{h\to 0}\sup_{t_0+h\leq t\leq T} E\left|y_t^h\right|^2 = 0.$$

The second estimate is proved in a similar way. ■

We use the same notations as in the proof of Lemma (6), to prove corollary 1.

Proof (proof of corollary 1). We have by the definition of J that

$$\frac{1}{h}\left[J(u^h) - J(u^*)\right] = \frac{1}{h}\left[E\left[g(x_T^h) - g(x_T^*)\right] + \int_{t_0}^{T}\left[h(t,x_t^h,u_t^h) - h(t,x_t^*,u_t^*)\right]dt\right].$$

Then,

$$\frac{1}{h}\left[J(u^h) - J(u^*)\right] = E\left[\int_0^1 g_x(x_T^{h,\lambda})(\frac{x_T^h - x_T^*}{h})d\lambda + \frac{1}{h}\int_{t_0}^{T}\left[h(t,x_t^h,u_t^h) - h(t,x_t^*,u_t^*)\right]dt\right].$$

From Lemma (6), we obtain (13) by letting h tend to 0.

Let us introduce the adjoint process. We proceed as in Bensoussan (1983) and Oksendal and Sulem (2005). Let $\varphi(t,\tau)$ be the solution of the linear equation

$$\begin{cases} d\varphi(t,\tau) = b_x(t,x_t^*,u_t^*)\varphi(t,\tau) + \sigma_x(t,x_t^*)\varphi(t,\tau)dB_t \\ \qquad + \int\limits_\Gamma f_x(t,x_{t-}^*,\theta,u_t^*)\varphi(t^-,\tau)\widetilde{N}(dt,d\theta) \\ \varphi(\tau,\tau) = I_d \end{cases} \qquad 0 \leq \tau \leq t \leq T$$

This equation is linear with bounded coefficients. Hence it admits a unique strong solution. Moreover, the process φ is invertible, with an inverse ψ satisfying suitable integrability conditions.

From Ito's formula, we can easily check that $d(\varphi(t,\tau)\psi(t,\tau)) = 0$, and $\varphi(\tau,\tau)\psi(\tau,\tau) = I_d$, where ψ is the solution of the following equation

$$\begin{cases} d\psi(t,\tau) = \left\{ \begin{array}{c} \sigma_x(t,x_t^*)\psi(t,\tau)\sigma_x(t,x_t^*) - b_x(t,x_t^*,u_t^*)\psi(t,\tau) \\ -\int\limits_\Gamma f_x(t,x_{t-}^*,\theta,u_t^*)\psi(t^-,\tau)\upsilon(d\theta) \end{array} \right\} dt \\ \qquad\qquad -\sigma_x(t,x_t^*)\psi(t,\tau)dB_t \\ -\psi(t^-,\tau)\int\limits_\Gamma \left(f_x(t,x_{t-}^*,\theta,u_t^*) + I_d\right)^{-1} f_x(t,x_{t-}^*,\theta,u_t^*)N(dt,d\theta) \\ \psi(\tau,\tau) = I_d \end{cases} \qquad 0 \leq \tau \leq t \leq T$$

If $\tau = 0$ we simply write $\varphi(t,0) = \varphi_t$ and $\psi(t,0) = \psi_t$.

By the uniqueness property, it is easy to check that

$$z_t = \varphi(t,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right].$$

Then, (13) will become

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[\begin{array}{l} \int\limits_{t_0}^{T} h_x(t,x_t^*,u_t^*)\varphi(t,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right]dt \\ +g_x(x_T^*)\varphi(T,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right] \\ +\left[h(t_0,x_{t_0}^*,\nu) - h(t_0,x_{t_0}^*,u_{t_0}^*)\right] \end{array}\right] \tag{16}$$

Now, if we define the adjoint process by

$$p_t = y_t\psi_t^*$$

where

$$\begin{aligned} y_t &= E\left[g_x(x_T^*)\varphi_T^* + \int\limits_t^T h_x(s,x_s^*,u_s^*)\varphi_s^* dt \diagup \mathcal{F}_t\right] \\ &= E\left[X \diagup \mathcal{F}_t\right] - \int\limits_0^t h_x(s,x_s^*,u_s^*)\varphi_s^* dt \end{aligned}$$

with

$$X = g_x(x_T^*)\varphi_T^* + \int_0^T h_x(s, x_s^*, u_s^*)\varphi_s^* dt.$$

It follows that

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[p_t\left[b(t_0, x_{t_0}^*, \nu) - b(t_0, x_{t_0}^*, u_{t_0}^*)\right] + \left[h(t_0, x_{t_0}^*, \nu) - h(t_0, x_{t_0}^*, u_{t_0}^*)\right]\right].$$

Define the Hamiltonian H from $[0;T] \times \mathbb{R}^n \times A \times \mathbb{R}^n$ into $\mathbb{R}$ by

$$H(t, x, u, p) = h(t, x_t, u_t) + pb(t, x_t, u_t). \tag{17}$$

We get from optimality of u^* that

$$E\left[H(t_0, x_{t_0}, \nu, p_{t_0}) - H(t_0, x_{t_0}, u_{t_0}^*, p_{t_0})\right] \geq 0.dt_0 - a.e.$$

By the Ito representation theorem Ikeda and Watanabe (2014), there exists two processes $Q \in \mathcal{M}^2$ and $R \in \mathcal{L}^2$ satisfying

$$E\left[X \diagup \mathcal{F}_t\right] = E\left[X\right] + \int_0^t Q_s dB_s + \int_0^t \int_\Gamma R_s(\theta)\widetilde{N}(ds, d\theta).$$

Hence,

$$y_t = E\left[X\right] - \int_0^t h_x(s, x_s^*, u_s^*)\varphi_s ds + \int_0^t Q_s dB_s + \int_0^t \int_\Gamma R_s(\theta)\widetilde{N}(ds, d\theta)$$

Let

$$\begin{aligned} q_t &= Q_t\psi_t - p_t\sigma_x(t, x_t^*) \\ r_t(\theta) &= R_t(\theta)\psi_t\left(f_x(t, x_{t-}^*, \theta, u_t^*) + I_d\right)^{-1} + p_t\left[\left(f_x(t, x_{t-}^*, \theta, u_t^*) + I_d\right) - I_d\right] \end{aligned}$$

The above discussion will allow us to introduce the next theorem which is the main result of this subsection.

Theorem 3 (maximum principle for strict control). *Let u^* be the optimal strict control minimizing the cost $J(.)$ over U, and denote by x^* the corresponding optimal trajectory. Then there exists a unique triple of square integrable adapted processes (p^n, q^n, r^n) which is the unique solution of the linear backward SDE*

$$\begin{cases} dp_t = -\begin{bmatrix} h_x(t, x_t^*, u_t^*) + p_t b_x(t, x_t^*, u_t^*) + q_t\sigma_x(t, x_t^*) \\ + \int_\Gamma r_t(\theta) f(t, x_{t-}^*, \theta, u_t^*)\upsilon(d\theta). \end{bmatrix} dt \\ \qquad\qquad + q_t^* dB_t + \int_\Gamma r_t(\theta)\widetilde{N}(dt, d\theta) \\ p_T^* = g_x(x_T^*) \end{cases} \tag{18}$$

such that for all $\nu \in U$ the following inequality holds

$$E\left[H(t, x_t^*, \nu, p_t) - H(t, x_t^*, u_t^*, p_t)\right] \geq 0.dt - a.e.,$$

where the Hamiltonian H is defined by (17).

4.2. The maximum principle for near optimal controls

In this subsection, we establish necessary conditions of near optimality satisfied by a sequence of nearly optimal strict controls. This result is based on Ekeland's variational principle, which is given by the following Lemma

Lemma 7 (Ekeland's variational principle). *Let (E, d) be a complete metric space and $f : E \to \overline{\mathbb{R}}$ be lower semi-continuous and bounded from below. Given $\varepsilon > 0$, suppose $u^\varepsilon \in E$ satisfies $f(u^\varepsilon) \leq \inf(f) + \varepsilon$. Then for any $\lambda > 0$, there exists $\nu \in E$ such that*

- $f(\nu) \leq f(u^\varepsilon)$
- $d(u^\varepsilon, \nu) \leq \lambda$
- $f(\nu) \leq f(\omega) + \frac{\varepsilon}{\lambda} d(\omega, \nu)$ *for all* $\omega \neq \nu$.

To apply Ekeland's variational principle, we have to endow the set U of strict controls with an appropriate metric. For any u and $\nu \in U$, we set

$$d(u, \nu) = P \otimes dt \{(\omega, t) \in \Omega \times [0; T] ; u(t, \omega) \neq \nu(t, \omega)\}$$

where $P \otimes dt$ is the product measure of P with the Lebesgue measure dt.

Remark 3. It is easy to see that (U, d) is a complete metric space, and it well known that the cost functional J is continuous from U into $\mathbb{R}$. For more detail see Mezerdi (1988).

Now, let $\mu^* \in \mathcal{R}$ be an optimal relaxed control and denote by x^{μ^*} the trajectory of the system controlled by μ^*. From Lemma (1), there exists a sequence (u^n) of strict controls such that

$$\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t^*(da)dt \quad \text{weakly}$$

and

$$\lim_{n\to\infty} E\left[\left|x_t^n - x_t^{\mu^*}\right|^2\right] = 0$$

where x^n is the solution of (3) corresponding to μ^n.

According to the optimality of μ^* and lemm(7), there exists a sequence (ε_n) of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$J(u^n) = J(\mu^n) \leq J(\mu^*) + \varepsilon_n = \inf_{u \in U} J(u) + \varepsilon_n$$

a suitable version of Lemma (7) implies that, given any $\varepsilon_n > 0$, there exists $u^n \in U$ such that

$$J(u^n) \leq J(u) + \varepsilon_n d(u^n, u), \forall u \in U \tag{19}$$

Let us define the perturbation

$$u^{n,h} = \begin{cases} \nu & \text{if } t \in [t_0; t_0 + h] \\ u^n & \text{otherwise} \end{cases}$$

From (19) we have

$$0 \leq J(u^{n,h}) - J(u^n) + \varepsilon_n d(u^{n,h}, u^n)$$

Using the definition of d it holds that

$$0 \leq J(u^{n,h}) - J(u^n) + \varepsilon_n Ch \tag{20}$$

where C is a positive constant. Now, we can introduce the next theorem which is the main result of this section.

Theorem 4. *For each $\varepsilon_n > 0$, there exists $(u^n) \in U$ such that there exists a unique triple of square integrable adapted processes (p^n, q^n, r^n) which is the solution of the backward SDE*

$$\begin{cases} dp_t^n = -\begin{bmatrix} h_x(t, x_t^n, u_t^n) + p_t^n b_x(t, x_t^n, u_t^n) + q_t^n \sigma_x(t, x_t^n) \\ +\displaystyle\int_\Gamma r_t^n(\theta) f(t, x_{t-}^n, \theta, u_t^n) \upsilon(d\theta). \end{bmatrix} dt \\ \qquad\qquad +q_t^n dB_t + \displaystyle\int_\Gamma r_t^n(\theta)\widetilde{N}(dt, d\theta) \\ p_T^n = g_x(x_T^n) \end{cases} \tag{21}$$

such that for all $\nu \in U$

$$E\left[H(t, x_t^n, \nu, p_t^n) - H(t, x_t^n, u_t^n, p_t^n)\right] + C\varepsilon_n \geq 0.dt - a.e. \tag{22}$$

where C is a positive constant.

Proof. From the inequality (20), we use the same method as in the previous subsection, we obtain (22).

4.3. The relaxed stochastic maximum principle

Now, we can introduce the next theorem, which is the main result of this section

Theorem 5 (The relaxed stochastic maximum principle). *Let μ^* be an optimal relaxed control minimizing the functional J over $\mathcal{R}$, and let $x_t^{\mu^*}$ be the corresponding optimal trajectory. Then there exists a unique triple of square integrable and adapted processes $(p^{\mu^*}, q^{\mu^*}, r^{\mu^*})$ which is the solution of the backward SDE*

$$\begin{cases} dp_t^{\mu^*} = -\begin{bmatrix} \displaystyle\int_A h_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + \int_A p_t^{\mu^*} b_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + q_t^{\mu^*}\sigma_x(t, x_t^{\mu^*}) \\ +\displaystyle\int_A\int_\Gamma r_t^{\mu^*}(\theta) f(t, x_{t-}^{\mu^*}, \theta, a)\mu_t^* \otimes \upsilon(da, d\theta). \end{bmatrix} dt \\ \qquad\qquad +q_t^{\mu^*} dB_t + \displaystyle\int_\Gamma r_t^{\mu^*}(\theta)\widetilde{N}^{\mu^*}(dt, d\theta, da) \\ p_T^{\mu_t^*} = g_x(x_T^{\mu_t^*}) \end{cases} \tag{23}$$

such that for all $\nu \in U$

$$E\left[H(t, x_t^{\mu^*}, \nu_t, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.)) - \int_\Gamma H(t, x_t^{\mu^*}, a, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.))\mu_t^*(da)\right] \geq 0 \ dt - a.e \tag{24}$$

The proof of this theorem is based on the following Lemma.

Lemma 8. *Let* $(p^{\,n}, q^{\,n}, r^{\,n})$ *and* $(p^{\,\mu^*}, q^{\,\mu^*}, r^{\,\mu^*})$, *be the solutions of (21) and (23), respectively. Then we have*

$$lim_{n\to\infty}\left[E\left|p^n - p^{\mu^*}\right|^2 + E\int_t^T \left|q^n - q^{\mu^*}\right|^2 ds + E\int_t^T\int_\Gamma \left|r^n - r^{\mu^*}\right|^2 \upsilon(d\theta)ds\right] = 0.$$

To prove the Lemma (8), we need to state and prove the stability theorem of BSDEs with jumps. Note that this theorem is proved by Hu and Peng (1997) in the case without jump.

4.3.1. Stability theorem for BSDE's with jump

Let us denote by $M^2(0,T;\mathbb{R}^m)$ the subset of $L^2(\Omega\times[0;T], dP\times dt;\mathbb{R}^m)$ consisting of $\mathcal{F}_t-$progressively measurable processes. consider the following BSDE's with jump depending on a parameter n.

$$p_t^n = p_T^n + \int_t^T F^n(s, p_s^n, q_s^n, r_s^n)ds - \int_t^T q_s^n dB_s - \int_t^T\int_\Gamma r_s^n(\theta)N^n(ds, d\theta) \qquad t\in[0;T].$$

Using the linearity of the last adjoint equation, it is not difficult to check that:

1. For any n, $(p,q,r)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}$, $F^n(.,p,q,r)\in M^2(0,T;\mathbb{R}^m)$ and $p_T^n\in L^2(\Omega,\mathcal{F}_t,P,\mathbb{R}^m)$,
2. There exists a constant $C_0>0$ such that
$$\begin{aligned}&|F^n(s,p_1,q_1,r_1) - F^n(s,p_2,q_2,r_2)|\\ &\le C_0\left(|p_1-p_2| + |q_2-q_2| + \int_\Gamma |r_1-r_2|\,\upsilon(d\theta)\right) \quad P.a.s\ \ a.e\ \ t\in[0;T],\end{aligned}$$
3. $E\left(|p_T^n - p_T^*|^2\right)\overrightarrow{n\to\infty}\,0,$
4. $\forall t\in[0;T]$,
$$\lim_{n\to\infty} E\left[\left|\int_t^T \left(F^n(s,p_s^*,q_s^*,r_s^*) - F^*(s,p_s^*,q_s^*,r_s^*)\right)ds\right|^2\right] = 0$$

Theorem 6 (Stability theorem for BSDE's with jumps). *Let* $(p^{\,n}, q^{\,n}, r^{\,n})$ *and* $(p^{\,*}, q^{\,*}, r^{\,*})$, *be the solutions of (21) and (23), respectively. Then we have*

$$\lim_{n\to\infty} E\left[|p^n - p^*|^2 + \int_t^T |q^n - q^*|^2\,ds + \int_t^T\int_\Gamma |r^n - r^*|^2\,\upsilon(d\theta)ds\right] = 0.$$

Proof. We proceed as in Hu and Peng (1997). Let $\widehat{p^n} = p^n - p^*$, $\widehat{q^n} = q^n - q^*$, $\widehat{r^n} = r^n - r^*$ and $\widehat{p^n_T} = p^n_T - p^*_T$, then

$$\widehat{p^n_t} + \int_t^T \widehat{q^n_s} dB_s + \int_t^T\int_\Gamma \widehat{r^n_s}\widetilde{N}(dt,d\theta) = \widehat{p^n_T} + \int_t^T \left[F^n(s,p^n_s,q^n_s,r^n_s) - F^n(s,p^*_s,q^*_s,r^*_s)\right] ds$$
$$+ \int_t^T \left[F^n(s,p^*_s,q^*_s,r^*_s) - F^*(s,p^*_s,q^*_s,r^*_s)\right] ds$$

Taking the square and the expectation, we get

$$\begin{aligned}
&E\left|\widehat{p^n_t}\right|^2 + \int_t^T \left|\widehat{q^n_s}\right|^2 ds + \int_t^T\int_\Gamma \left|\widehat{r^n_s}\right|^2 \upsilon(d\theta)ds \\
&\leq 2E\left|\alpha^n_t\right|^2 \\
&+ 2E\left(\int_t^T \left[F^n(s,p^n_s,q^n_s,r^n_s) - F^n(s,p^*_s,q^*_s,r^*_s)\right] ds\right)^2 \\
&\leq 2E\left|\alpha^n_t\right|^2 + 2(T-t)E\int_t^T \left|F^n(s,p^n_s,q^n_s,r^n_s) - F^n(s,p^*_s,q^*_s,r^*_s)\right|^2 ds
\end{aligned}$$

with

$$\alpha^n_t = \widehat{p^n_T} + \int_t^T \left[F^n(s,p^*_s,q^*_s,r^*_s) - F^*(s,p^*_s,q^*_s,r^*_s)\right] ds$$

Because of the assumption 2

$$E\left|\widehat{p^n_t}\right|^2 + E\int_t^T \left|\widehat{q^n_s}\right|^2 ds + E\int_t^T\int_\Gamma \left|\widehat{r^n_s}\right|^2 \upsilon(d\theta)ds \leq 2E\left|\alpha^n_t\right|^2 + 2(T-t)C_0E\left|\widehat{p^n_t}\right|^2$$
$$+ 2(T-t)C_0E\left[\int_t^T \left|\widehat{q^n_s}\right|^2 ds + \int_t^T\int_\Gamma \left|\widehat{r^n_s}\right|^2 \upsilon(d\theta)ds\right].$$

For $t \in [T-\varepsilon; T]$ with $\varepsilon = \dfrac{1}{4C_0}$

$$\begin{aligned}
&E\left|\widehat{p^n_t}\right|^2 + \int_t^T \left|\widehat{q^n_s}\right|^2 ds + \int_t^T\int_\Gamma \left|\widehat{r^n_s}\right|^2 \upsilon(d\theta)ds \\
&\leq 2E\left|\alpha^n_t\right|^2 + \frac{1}{2}E\int_t^T \left[\left|\widehat{p^n_s}\right|^2 + \left|\widehat{q^n_s}\right|^2 + \int_\Gamma \left|\widehat{r^n_s}\right|^2 \upsilon(d\theta)\right] ds.
\end{aligned}$$

Hence

$$E\left|\widehat{p_t^n}\right|^2 + \frac{1}{2}E\int_t^T \left|\widehat{q_s^n}\right|^2 ds + \frac{1}{2}E\int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds$$

$$\leq 2E\left|\alpha_t^n\right|^2 + \frac{1}{2}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds.$$

Then we have

$$E\left|\widehat{p_t^n}\right|^2 \leq \frac{2}{3}E\left|\alpha_t^n\right|^2 + \frac{1}{6}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds \tag{25}$$

$$E\int_t^T \left|\widehat{q_s^n}\right|^2 ds \leq \frac{4}{3}E\left|\alpha_t^n\right|^2 + \frac{1}{3}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds \tag{26}$$

$$E\int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds \leq \frac{4}{3}E\left|\alpha_t^n\right|^2 + \frac{1}{3}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds \tag{27}$$

Now, for apply the Gronwall lemma we need to prove that $\lim\limits_{n\to\infty} E\left|\alpha_t^n\right|^2 = 0$, we have

$$E\left|\alpha_t^n\right|^2 \leq 2E\left|p_T^n - p_T^*\right|^2 + 2(T-t)C_0E\int_t^T \left|F^n(s,p_s^*,q_s^*,r_s^*) - F^*(s,p_s^*,q_s^*,r_s^*)\right|^2 ds.$$

By the assumptions 3 and 4, we deduce that $\lim\limits_{n\to\infty} E\left|\alpha_t^n\right|^2 = 0$.

By the Gronwall lemma, we can deduce that $\lim\limits_{n\to\infty} E\left|\widehat{p_t^n}\right|^2 = 0$, hence $\lim\limits_{n\to\infty} E\int_t^T \left|\widehat{q_s^n}\right|^2 ds = 0$ and $\lim\limits_{n\to\infty} E\int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds = 0$.

We can use the same argument to prove that the above convergence holds on $[T-2\delta;T-\delta]$, $[T-3\delta;T-2\delta]$ ....This complete the proof.

To prove the Lemma (8), it is sufficient to show that the coeffiecients of our BSDE verify the assumptions of stability theorem (6) :

Proof (Proof of Lemma 8). By the continuity of the derivatives of the coefficients, and the fact that $\lim\limits_{n\to\infty} E\left|x_T^n - x_T^{\mu^*}\right|^2 = 0$, we can deduce that

$$\lim_{n\to\infty} E\left[\left|\int_t^T \left(F^n(s,p_s^{\mu^*},q_s^{\mu^*},r_s^{\mu^*}) - F^{\mu^*}(s,p_s^{\mu^*},q_s^{\mu^*},r_s^{\mu^*})\right) ds\right|^2\right] = 0$$

and

$$\lim_{n\to\infty} E\left(\left|p_T^n - p_T^{\mu^*}\right|^2\right) = 0,$$

and, by the boundedness of b_x, σ_x, and f, we can easily check that there exists a constant $C_0 > 0$ such that

$$\left|F^n(s, p_s^n, q_s^n, r_s^n) - F^n(s, p_s^{\mu^*}, q_s^{\mu^*}, r_s^{\mu^*})\right| \leq C_0\left(\left|p_s^n - p_s^{\mu^*}\right| + \left|q_s^n - q_s^{\mu^*}\right| + \int_\Gamma \left|r_s^n - r_s^{\mu^*}\right| \upsilon(d\theta)\right)$$

$$P.a.s \quad a.e \quad t \in [0;T],$$

where

$$F^n(s, X, Y, Z) = h_x + X b_x + Y\sigma_x + \int_\Gamma Z f \upsilon(d\theta)$$

and

$$F^{\mu^*}(s, X, Y, Z) = \int_A h_x(a)\mu_t^*(da) + \int_A X b_x(a)\mu_t^*(da) + Y\sigma_x + \int_\Gamma Z f(\theta, a)\mu_t^* \otimes \upsilon(da, d\theta).$$

This complete the proof.

Proof (Proof of Theorem 5). The result is proved by passing to the limit in inequality (22), and using Lemma (8), we get easily the inequality (24)

5. Appendix

Lemma 9 (Skorokhod selection theorem). *Ikeda and Watanabe (2014) Let (E, ρ) be a complete separable metric space, and let P and P_n, $n = 1, 2....$ be probability measures on $(E, \mathcal{B}(E))$, such that (P_n) converges weakly to to P. Then, on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, there exist E-valued random variables $x^n, n = 1, 2....$, and x such that*

- $P = \widetilde{P}_{x}$,
- $P_n = \widetilde{P}_{x^n}, n = 1, 2....$,
- $x^n \longrightarrow_{n\to\infty} x$, $\widetilde{P} - a.s.$

Lemma 10 (Aldous criterion of tightness). *Aldous, (1989) Let (x^n) be a sequence of càdlag processes, suppose that for each n, x^n is defined on a filtred probability space $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_t, P^n)$, and the two following conditions holds*

- (x^n_t) is tight on $\mathbb{R}, \forall t$
- $\forall \varepsilon > 0, \forall \eta > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, for all stopping time S, T, such that $S < T < S + \delta$, we have

$$\lim_{\delta\to 0} \limsup_{n} \sup_{S<T<S+\delta} P(\sup_{t\leq n} |Y^n_S - Y^n_T| \geq \eta) = 0.$$

Then, (x^n) is tight on $D(\mathbb{R})$.

Proposition 1. *(Jacod and Shiryaev (1987)) Let x be a cadlag square integrable martingale and let $< x >$ its predictable " crochet ". If $S \leq T$ two finite stopping times, then*

$$P(\sup_{t\in[S;T]} |x^n_S - x^n_T| \geq \eta) \leq \frac{\varepsilon}{\eta^2} + P(< x^n >_T - < x^n >_S \geq k)$$

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