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TENSOR PRODUCTS AND THE SPECTRAL CONTINUITY FOR k-QUASI-$*$-CLASS A OPERATORS

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ABSTRACT. An operator $T \in B(\mathcal{H})$ is called k-quasi-$*$-class A if $T^{*k}(|T^2| - |T^*|^2)T^k \geq 0$ for a positive integer k, which is a common generalization of $*$-class A and quasi-$*$-class A. In this paper, firstly we prove some inequalities of this class of operators; secondly we consider the tensor products for k-quasi-$*$-class A operators, giving a necessary and sufficient condition for $T \otimes S$ to be a k-quasi-$*$-class A operator when T and S are both non-zero operators; at last we prove that the spectrum is continuous on the class of all k-quasi-$*$-class A operators.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{C}$ be the set of complex numbers. Let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$, we shall write $\ker T$ and $\operatorname{ran} T$ for the null space and range of T respectively. Also let $\alpha(T) = \dim \ker T$, $\beta(T) = \dim \ker T^*$ and let $\sigma(T)$, $\sigma_a(T)$ denote the spectrum, approximate point spectrum of T. Let $p = p(T)$ be the ascent of T; i.e., the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist, we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T; i.e., the smallest nonnegative integer q such that $\operatorname{ran} T^q = \operatorname{ran} T^{q+1}$, and if such integer does not exist, we put $q(T) = \infty$. An operator $T \in B(\mathcal{H})$ is called upper (resp. lower) semi-Fredholm if $\operatorname{ran} T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). If $T \in B(\mathcal{H})$ is either an upper semi-Fredholm operator or a lower

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semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of a semi-Fredholm operator $T \in B(\mathcal{H})$, denoted by $\text{ind}(T)$, is given by the integer $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{H})$ are defined by $\sigma_e(T) = \{\lambda \in \mathcal{C} : T - \lambda \text{ is not Fredholm}\}$, $\sigma_w(T) = \{\lambda \in \mathcal{C} : T - \lambda \text{ is not Weyl}\}$, and $\sigma_b(T) = \{\lambda \in \mathcal{C} : T - \lambda \text{ is not Browder}\}$.

Let $\mathcal{H}$, $\mathcal{K}$ be complex Hilbert spaces and $\mathcal{H} \otimes \mathcal{K}$ the tensor product of $\mathcal{H}$, $\mathcal{K}$; i.e., the completion of the algebraic tensor product of $\mathcal{H}$, $\mathcal{K}$ with the inner product $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle$ for $x_1,\ x_2 \in \mathcal{H},\ y_1,\ y_2 \in \mathcal{K}$. Let $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$. $T \otimes S \in B(\mathcal{H} \otimes \mathcal{K})$ denotes the tensor product of T and S ; i.e., $(T \otimes S)(x \otimes y) = Tx \otimes Sy$ for $x \in \mathcal{H}$, $y \in \mathcal{K}$.

Recall that $T \in B(\mathcal{H})$ is called p-hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geq 0$ [10]; when $p = 1$, T is called hyponormal. And T is called paranormal if $\| Tx \|^2 \leq \| T^2x \| \| x \|$ for all $x \in \mathcal{H}$ [10, 11]. And T is called normaloid if $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$(equivalently, $\|T\| = r(T)$, the spectral radius of T). In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators (T is invertible and $\log T^*T \geq \log TT^*$), Furuta, Ito and Yamazaki [12] introduced a very interesting class of operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. Recently Duggal, Jeon and Kim [7] introduced $*$-class A(i.e., $|T^2| - |T^*|^2) \geq 0$) operators.

Definition 1.1. $T \in B(\mathcal{H})$ is called a k-quasi-$*$-class A operator for a positive integer k if

$$T^{*k}(|T^2| - |T^*|^2)T^k \geq 0.$$

when $k = 1$, called quasi-$*$-class A operator, see [25].

For more interesting properties on k-quasi-$*$-class A operators, see [7, 21, 22]. It is clear that

the class of $*$-class A operators $\subseteq$ the class of quasi-$*$-class A operators $\subseteq$ the class of k-quasi-$*$-class A operators

and

the class of k-quasi-$*$-class A operators $\subseteq$ the class of $(k + 1)$-quasi-$*$-class A operators.

We show that the inclusion relations above are strict, by an example which appeared in [13].

Example 1.2. Given a bounded sequence of positive numbers $\{\alpha_i\}_{i=0}^{\infty}$. Let T be the unilateral weighted shift operator on l^2 with the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ by $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$, that is,

$$T = \begin{pmatrix} 0 & & & & \\ \alpha_0 & 0 & & & \\ & \alpha_1 & 0 & & \\ & & \alpha_2 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Straightforward calculations show that T is a k-quasi-$*$-class A operator if and only if $\alpha_n\alpha_{n+1} \geq \alpha_{n-1}^2$ for all $n \geq k$. So if $\alpha_k\alpha_{k+1} < \alpha_{k-1}^2$ and $\alpha_n\alpha_{n+1} \geq \alpha_{n-1}^2$ for all $n \geq k+1$, then T is a $(k+1)$-quasi-$*$-class A operator, but not a k-quasi-$*$-class A operator.

In this paper, firstly we prove some inequalities of this class of operators; secondly we consider the tensor products for k-quasi-$*$-class A operators, giving a necessary and sufficient condition for $T \otimes S$ to be a k-quasi-$*$-class A operator when T and S are both non-zero operators; at last we prove that the spectrum is continuous on the class of all k-quasi-$*$-class A operators.

2. Tensor products for k-quasi-$*$-class A operators

At first, we shall prove some inequalities of k-quasi-$*$-class A operators.

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be a k-quasi-$*$-class A operator for a positive integer k. Then the following assertions hold.*

(1) $\| T^{n+2}x \| \| T^n x \| \geq \| T^*T^n x \|^2$ *for all $x \in \mathcal{H}$ and all positive integers $n \geq k$.*

(2) *If $T^n = 0$ for some positive integer $n \geq k$, then $T^k = 0$.*

Proof. (1) Since it is clear that k-quasi-$*$-class A operators are $(k+1)$-quasi-$*$-class A operators, we only need to prove the case $n = k$. Since

$$\langle T^{*k}|T^*|^2T^kx, x\rangle = \langle T^{*k}TT^*T^kx, x\rangle = \| T^*T^kx \|^2,$$

and

$$\langle T^{*k}|T^2|T^kx, x\rangle = \langle |T^2|T^kx, T^kx\rangle \leq \| |T^2|T^kx \| \| T^kx \| = \| T^{k+2}x \| \| T^kx \| .$$

We have that

$$\| T^{k+2}x\| \|T^kx \| \geq \|T^*T^kx \|^2$$

for all $x \in \mathcal{H}$ for T is a k-quasi-$*$-class A operator.

(2) If $n = k$, it is obvious that $T^k = 0$. If $T^{k+1} = 0$, then $T^{k+2} = 0$. So we have that $T^*T^k = 0$ by (1). Hence we have that $\|T^kx\|^2 = \langle T^*T^kx, T^{k-1}x\rangle = 0$ for all $x \in \mathcal{H}$. So $T^k = 0$. The rest of the proof is similar.

□

Let $T \otimes S$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{K}$ for non-zero $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$. The operation of taking tensor products $T \otimes S$ preserves many properties of $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, but by no means all of them. For example the normaloid property is invariant under tensor products, the spectraloid property is not (see [23] pp. 623 and 631); and $T \otimes S$ is normal if and only if T and S are normal [16, 26]; however there exist paranormal operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ such that $T \otimes S$ is not paranormal [1]. Duggal [6] showed that for non-zero $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, $T \otimes S$ is p-hyponornal if and only if T, S are p-hyponornal. This result was extended to p-quasihyponormal operators, class A operators, log-hyponormal operators and class A(s,t) operators($(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$, s, $t > 0$) in [19, 20, 27], respectively. The following theorem gives a necessary and sufficient condition for $T \otimes S$ to be a k-quasi-$*$-class A operator when T and S are both non-zero operators, which is an extension of [7] Theorem 3.2.

Theorem 2.2. *Let $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ be non-zero operators. Then $T \otimes S \in B(\mathcal{H} \otimes \mathcal{K})$ is a k-quasi-$*$-class A operator if and only if one of the following assertions holds:*

(1) $T^k = 0$ *or* $S^k = 0$.
(2) T *and* S *are k-quasi-$*$-class A operators.*

Proof. It is clear that $T \otimes S$ is a k-quasi-$*$-class A operator if and only if

$$\begin{aligned}
&(T \otimes S)^{*k}(|(T \otimes S)^2| - |(T \otimes S)^*|^2)(T \otimes S)^k \geq 0\\
&\Longleftrightarrow T^{*k}(|T^2| - |T^*|^2)T^k \otimes S^{*k}|S^2|S^k + T^{*k}|T^*|^2T^k \otimes S^{*k}(|S^2| - |S^*|^2)S^k \geq 0\\
&\Longleftrightarrow T^{*k}|T^2|T^k \otimes S^{*k}(|S^2| - |S^*|^2)S^k + T^{*k}(|T^2| - |T^*|^2)T^k \otimes S^{*k}|S^*|^2S^k \geq 0.
\end{aligned}$$

Therefore the sufficiency is clear.

To prove the necessary. Suppose that $T \otimes S$ is a k-quasi-$*$-class A operator. Let $x \in \mathcal{H}$ and $y \in \mathcal{K}$ be arbitrary. Then we have

$$\langle T^{*k}(|T^2| - |T^*|^2)T^k x, x\rangle\langle S^{*k}|S^2|S^k y, y\rangle + \langle T^{*k}|T^*|^2T^k x, x\rangle\langle S^{*k}(|S^2| - |S^*|^2)S^k y, y\rangle \geq 0. \tag{2.1}$$

It suffices to prove that if (1) does not hold, then (2) holds. Suppose that $T^k \neq 0$ and $S^k \neq 0$. To the contrary, assume that T is not a k-quasi-$*$-class A operator, then there exists $x_0 \in \mathcal{H}$ such that

$$\langle T^{*k}(|T^2| - |T^*|^2)T^k x_0, x_0\rangle = \alpha < 0 \text{ and } \langle T^{*k}|T^*|^2T^k x_0, x_0\rangle = \beta > 0.$$

From (2.1) we have

$$\alpha\langle S^{*k}|S^2|S^k y, y\rangle + \beta\langle S^{*k}(|S^2| - |S^*|^2)S^k y, y\rangle \geq 0$$

for all $y \in \mathcal{K}$, that is,

$$(\alpha + \beta)\langle S^{*k}|S^2|S^k y, y\rangle \geq \beta\langle S^{*k}|S^*|^2S^k y, y\rangle \tag{2.2}$$

for all $y \in \mathcal{K}$. Therefore S is a k-quasi-$*$-class A operator. We have

$$\langle S^{*k}|S^*|^2S^k y, y\rangle = \|S^*S^k y\|^2 \text{ and } \langle S^{*k}|S^2|S^k y, y\rangle \leq \|S^{k+2}y\|\|S^k y\|.$$

So we have

$$(\alpha + \beta)\|S^{k+2}y\|\|S^k y\| \geq \beta\|S^*S^k y\|^2 \tag{2.3}$$

for all $y \in \mathcal{K}$ by (2.2). Because S is a k-quasi-$*$-class A operator, from [21] Lemma 2.1 we can write $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$ on $\mathcal{K} = \overline{\operatorname{ran}(S^k)} \bigoplus \ker S^{*k}$, where S_1 is a $*$-class A operator (hence it is normaloid by [21] Theorem 2.6). By (2.3) we have

$$(\alpha+\beta)\|S_1^2\eta\|\|\eta\| \geq \beta\|S^*\eta\,\|^2 \geq \beta\|S_1^*\eta\,\|^2 \text{ for all } \eta \in \overline{\operatorname{ran}(S^k)}.$$

So we have

$$(\alpha+\beta)\|S_1\,\|^2 = (\alpha+\beta)\|S_1^2\,\| \geq \beta\|S_1^*\,\|^2 = \beta\|S_1\,\|^2,$$

where equality holds since S_1 is normaloid.

This implies that $S_1 = 0$. Since $S^{k+1}y = S_1 S^k y = 0$ for all $y \in \mathcal{H}$, we have $S^{k+1} = 0$. Since S is a k-quasi-$*$-class A operator, by Theorem 2.1 (ii) we have that $S^k = 0$. This contradicts the assumption $S^k \neq 0$. Hence T must be a k-quasi-$*$-class A operator. A similar argument shows that S is also a k-quasi-$*$-class A operator. The proof is complete.

□

3. Spectral continuity for k-quasi-$*$-class A operators

Let $\{\tau_n\}$ be a sequence of compact subsets of $\mathcal{C}$. Then its limit inferior is defined by

$$\lim\inf\{\tau_n\} = \{\lambda \in \mathcal{C} : \text{there exists } \ \lambda_n \in \tau_n \ \text{ such that } \ \lambda_n \longrightarrow \lambda\}$$

and its limit superior is defined by

$$\lim\sup\{\tau_n\} = \{\lambda \in \mathcal{C} : \text{there exists } \ \lambda_{n_k} \in \tau_{n_k} \ \text{ such that } \ \lambda_{n_k} \longrightarrow \lambda\}.$$

If $\lim\inf\{\tau_n\} = \lim\sup\{\tau_n\}$, then $\lim\{\tau_n\}$ is defined by this common limit. A map p, defined on $B(\mathcal{H})$, whose values are compact subsets of $\mathcal{C}$, is said to be upper (resp. lower) semi-continuous at T, if $T_n \longrightarrow T$ then $\lim\sup p(T_n) \subset p(T)$ (resp. $p(T) \subset \lim\inf p(T_n)$. If p is both upper and lower semi-continuous at T, then it is said to be continuous at T and in this case $\lim p(T_n) = p(T)$.

For every $T \in B(\mathcal{H})$, $\sigma(T)$ is a compact subset of $\mathcal{C}$. The function σ viewed as a function from $B(\mathcal{H})$ into the set of all compact subsets of $\mathcal{C}$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel [3] have carried out a detailed study of spectral continuity in $B(\mathcal{H})$. Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators in [9, 17]. It has been proved that σ is continuous in the set of normal operators and hyponormal operators in [15]. And this result has been extended to quasi-hyponormal operators by S. V. Djordjević in [4], to p-hyponormal operators by Hwang and Lee in [18], to (p,k)-quasihyponormal, M-hyponormal, $*$-paranormal and paranormal operators by Duggal, Jeon and Kim in [8], and to quasi-class (A,k) operators by Gao and Fang in [14]. In the following, we extend this result to k-quasi-$*$-class A operators.

In the following, we prove that spectrum σ is continuous on the set of all k-quasi-$*$-class A operators.

Lemma 3.1. *Let T be a k-quasi-$*$-class A operator for a positive integer k. Then the following assertions hold:*

(1) *If T is quasinilpotent (i.e., $\sigma(T) = \{0\}$), then T is nilpotent.*
(2) *For every non-zero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $\mathcal{H} = \ker(T-\lambda) \bigoplus (\ker(T-\lambda))^\perp$ is: $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.*

Proof. Suppose T is a k-quasi-$*$-class A operator for a positive integer k. (1) holds by [22] Corollary 2.2. If $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have that $\ker(T-\lambda)$ reduces T by [21] Lemma 2.5. So we have that $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $\mathcal{H} = \ker(T-\lambda) \bigoplus (\ker(T-\lambda))^\perp$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$. □

The Berberian extension theorem shows that given an operator $T \in B(\mathcal{H})$, there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an isometric $*$-isomorphism $T \to T^\circ \in B(\mathcal{K})$ preserving order such that $\sigma(T) = \sigma(T^\circ)$ and $\sigma_p(T^\circ) = \sigma_a(T^\circ) = \sigma_a(T)$. See details in the following lemma:

Lemma 3.2. [2]. *Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$ and a map $\varphi : B(\mathcal{H}) \longrightarrow B(\mathcal{K})$ such that*

(1) *φ is a faithful $*$-representation of the algebra $B(\mathcal{H})$ on $\mathcal{K}$, i.e., $\varphi(S+T) = \varphi(S) + \varphi(T)$, $\varphi(\lambda T) = \lambda\varphi(T)$, $\varphi(ST) = \varphi(S)\varphi(T)$, $\varphi(T^*) = (\varphi(T))^*$, $\varphi(I) = I$ and $\|\varphi(T)\| = \|T\|$ for any $S, T \in B(\mathcal{H})$ and $\lambda \in \mathcal{C}$.*
(2) *$\varphi(A) \geq 0$ for any $A \geq 0$ in $B(\mathcal{H})$.*
(3) *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(\mathcal{H})$.*

Theorem 3.3. *The spectrum σ is continuous on the set of k-quasi-$*$-class A operators for a positive integer k.*

Proof. Suppose T is a k-quasi-$*$-class A operator for a positive integer k. Let $\varphi : B(\mathcal{H}) \longrightarrow B(\mathcal{K})$ be Berberian's faithful $*$-representation of Lemma 3.2. In the following, we shall show that $\varphi(T)$ is also a k-quasi-$*$-class A operator for a positive integer k. In fact, since T is a k-quasi-$*$-class A operator, we have $T^{*k}(|T^2| - |T^*|^2)T^k \geq 0$. Hence we have

$$\begin{aligned} &(\varphi(T))^{*k}(|(\varphi(T))^2| - |\varphi(T)^*|^2)(\varphi(T))^k \\ &= \varphi(T^{*k}(|T^2| - |T^*|^2)T^k) \text{ by Lemma 3.2 (1)} \\ &\geq 0 \text{ by Lemma 3.2 (2).} \end{aligned}$$

So we have that its Berberian extension $T^\circ = \varphi(T)$ is also a k-quasi-$*$-class A operator for a positive integer k. By Lemma 3.1 we have that T belongs to the set $\mathcal{C}(i)$ (see definition in [8]). So we have that the spectrum σ is continuous on the set of k-quasi-$*$-class A operators for a positive integer k by [8] Theorem 1.1. This completes the proof. □

Corollary 3.4. *The Weyl spectrum σ_w is continuous if and only if the Browder spectrum σ_b is continuous on the set of k-quasi-$*$-class A operators for a positive integer k.*

Proof. Suppose T is a k-quasi-$*$-class A operator for a positive integer k. By Theorem 3.3 and [24] Theorem 2.2, we have that Browder's theorem holds for T. Hence Corollary 3.4 holds by the remark of [24] or the equivalence between (ii) and (iii) of [5] Theorem 2.2. □

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References

1. T. Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged) **33** (1972), 169–178.
2. S.K. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc. **13** (1962), 111–114.
3. J.B. Conway and B.B. Morrel, *Operators that are points of spectral continuity*, Integral Equation Operator Theory **2** (1979), 174–198.
4. S. V. Djordjević, *Continuity of the essential spectrum in the class of quasihyponormal operators*, Vesnik Math. **50** (1998), 71–74.
5. S.V. Djordjević and Y.M. Han, *Browder's theorem and spectral continuity*, Glasgow Math. J. **42** (2000), 479–486.
6. B.P. Duggal, *Tensor products of operators-strong stability and p-hyponormality*, Glasg. Math. J. **42** (2000), 371–381.
7. B.P. Duggal, I.H. Jeon and I.H. Kim, *On $*$-paranormal contractions and properties for $*$-class A operators*, Linear Algebra Appl. **436** (2012), 954–962.
8. B.P. Duggal, I.H. Jeon and I.H. Kim, *Continuity of the spectrum on a class of upper triangular operator matrices*, J. Math. Anal. Appl. **370** (2010), 584–587.
9. D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), 4153–4174.
10. T. Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.
11. T. Furuta, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.
12. T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several classes*, Sci. Math. **1** (1998), no. 3, 389–403.
13. F.G. Gao, and X.C. Fang, *On k-quasi-class A operators*, J. Inequal. Appl. **2009** (2009), Article ID 921634, 1–10.
14. F.G. Gao, and X.C. Fang, *Generalized Weyl's theorem and spectral continuity for quasi-class (A, k) operators*, Acta Sci. Math. (Szeged) **78** (2012), 241–250.
15. P.R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1982.
16. J.C. Hou, *On tensor products of operators*, Acta Math. Sin. (N. S.) **9** (1993), 195–202.
17. I.S. Hwang and W.Y. Lee, *On the continuity of spectra of Toeplitz operators*, Arch. Math. **70** (1998), 66–73.
18. I.S. Hwang and W.Y. Lee, *The spectrum is continuous on the set of p-hyponormal operators*, Math. Z. **235** (2000), 151–157.
19. I.H. Jeon and B.P. Duggal, *On operators with an absolute value condition*, J. Korean. Math. Soc. **41** (2004), 617–627.

20. I.H. Kim, *Tensor products of log-hyponormal operators*, Bull. korean math. Soc. **42** (2005), 269–277.
21. S. Mecheri, *Isolated points of spectrum of k-quasi-$*$-class A operators*, Studia Math. **208** (2012), 87–96.
22. S. Mecheri, *On operators satisfying an inequality*, J. Inequal. Appl. **2012** 2012:244 doi:10.1186/1029-242X-2012-244.
23. T. Saitô, *Hyponormal Operators and Related Topics*, Lecture Notes in Mathematics, **247**, Springer-Verlag, 1971.
24. S. Sánchez-Perales and S. V. Djordjević, *Continuity of spectra and compact perturbations*, Bull. Korean Math. Soc. **48** (2011), 1261–1270.
25. J.L. Shen, F. Zuo and C.S. Yang, *On operators satisfying $T^*|T^2|T \geq T^*|T^*|^2T$*, Acta Math. Sinica (English Ser.) **26** (2010), 2109–2116.
26. J. Stochel, *Seminormality of operators from their tensor products*, Proc. Amer. Math. Soc. **124** (1996), 435–440.
27. K. Tanahashi, *Tensor products of log-hyponormal operators and of class $A(s,t)$ operators*, Glasgow Math. J. **46** (2004), 91–95.

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