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SOME NEW PERTURBATION RESULTS FOR GENERALIZED INVERSES OF CLOSED LINEAR OPERATORS IN BANACH SPACES

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Abstract. We consider the perturbation and expression for the generalized inverse and Moore–Penrose inverse of closed linear operator under a weaker perturbation condition. As a application, we also investigate the perturbation for the Moore–Penrose inverse of closed EP operator. Some new and interesting perturbation results and examples are obtained in this paper.

1. Introduction

It is well known that the perturbation analysis of Moore–Penrose inverses and generalized inverses in Hilbert spaces and Banach spaces is very important in practical applications in diverse fields like optimization, statistics, economics, games, programming, networks and so on [1, 3, 17, 19]. Many equivalent conditions for Moore–Penrose and generalized inverse to have the simplest expression $T^+(I + \delta TT^+)^{-1}$ have been obtained in the case of bounded operators [5, 6, 7, 8, 9, 10, 11, 12, 14, 20, 21, 22, 24, 25]. As everyone knows, a large number of the operators which arise naturally in applications (e.g mathematical physics, quantum mechanics and partial differential equations) are unbounded[18] and many of them have bounded inverses or bounded generalized inverses. So it is necessary to extend the results of bounded operators to the unbounded case. It is

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worthy to point out that the differential operators or partial differential operators are always unbounded closed linear operators.

In this paper, we explore the following general perturbation problem: Let T be a closed linear operator with a bounded generalized inverse T^+, what condition on the small perturbation δT can guarantee that the generalized inverse $(T+\delta T)^+$ exists and it has the simplest expression $T^+(I+\delta T T^+)^{-1}$? Such problems in the case of stable perturbation and in the case that the perturbation does not change the null space have been studied in [10, 12, 16, 20, 25]. It should be noted that the perturbation condition $a\|T^+\| + b\|TT^+\| < 1$, which implies $\|\delta T T^+\| < 1$, is always assumed[12, 20]. It is natural to ask whether this condition can be relaxed. Motivated by the idea in [4, 8, 25], we give a certain answer to the mentioned problem under a weaker perturbation condition. Utilizing this result, we consider the perturbation for the Moore–Penrose inverse of closed EP operator in Hilbert space. As an illustration, we give some examples of generalized inverses of closed operator and Moore–Penrose inverse of closed EP operator. Our results generalize and improve many well known results in this area.

2. Preliminaries

Let X and Y be Banach spaces. Let $L(X,Y)$, $C(X,Y)$ and $B(X,Y)$ denote the linear space of all linear operators, the homogeneous set of all closed linear operators with a dense domain and the Banach space of all bounded linear operators from X into Y, respectively. We write $C(X), B(X)$ as $C(X,X), B(X,X)$, respectively. For any $T \in L(X,Y)$, we denote by $D(T)$, $N(T)$ and $R(T)$ the domain, the null space and respectively, the range of T. The identity operator will be denoted by I.

To make precise what is meant by a "small" perturbation in the case of unbounded operators, we need the concept of $T-$boundedness[13].

Definition 2.1. [13] Let T and P be linear operators with the same domain space such that $D(T) \subset D(P)$ and

$$\|Pu\| \leq a\|u\| + b\|Tu\| \qquad (u \in D(T)),$$

where a, b are nonnegative constants, then we say P is relatively bounded with respect to T or simply T-bounded. The greatest low bound of all possible constants b is called the relative bound of P with respect to T or simply the T-bound.

Regarding the stability of closed operator, we have

Lemma 2.2. [13] *Let T and P be two linear operators from X into Y. Let P be $T-$bounded with the $T-$bound smaller than 1. Then $S = T + P$ is closable if and only if T is closable; in this case, the closures of T and S have the same domain. In particular S is closed if and only if T is.*

Let us introduce the concept of the generalized inverse for closed operator.

Definition 2.3. An operator $T \in C(X,Y)$ is said to possess a generalized inverse if there exists an operator $S \in B(Y,X)$ such that $R(S) \subset D(T)$ and (1) S is an inner inverse of T, i.e., $TSTx = Tx$, for all $x \in D(T)$; (2) S is an outer inverse of T, i.e., $STSy = Sy$, for all $y \in Y$; (3) ST is continuous.

We always denote the generalized inverse of T by T^+. Concerning the existence and properties on generalized inverses of closed operator, we can see [17].

Proposition 2.4. [17] *$T \in C(X, Y)$ has a generalized inverse $T^+ \in B(Y, X)$ if and only if $N(T)$ and $R(T)$ have topological complements in X and Y, i.e.,*

$$X = N(T) \oplus \overline{R(T^+)} \quad \text{and} \quad Y = R(T) \oplus N(T^+).$$

Further, it follows from the Closed Graph Theorem that TT^+ is a projector onto $R(T)$ along $N(T^+)$ such that $R(TT^+) = R(T)$ and $N(TT^+) = N(T^+)$. By (3) in Definition 2.3, we know that T^+T can be extended to a projector onto $\overline{R(T^+)}$ with the null space $N(T)$.

Definition 2.5. Let X and Y be two Hilbert spaces and $T \in C(X, Y)$. If the topological decompositions in Proposition 2.4 are orthogonal, i.e.,

$$X = N(T) \dot{+} \overline{R(T^+)} \quad \text{and} \quad Y = R(T) \dot{+} N(T^+),$$

where $\dot{+}$ denotes the orthogonal direct sum, then the corresponding generalized inverse is usually called the Moore–Penrose inverse.

The Moore–Penrose inverse of T is always denoted by $T^\dagger$. If the Moore–Penrose inverse $T^\dagger$ commutes with T, then T is called an EP (Equal Projections) operator which has many nice properties[2, 15].

Definition 2.6. $T \in C(X)$ is said to be an EP operator if T has a bounded Moore–Penrose inverse $T^\dagger$ and $T^\dagger$ satisfies $T^\dagger T = TT^\dagger$ on $D(T)$.

It is easy to see that if T has a bounded inverse, then it is an EP operator.

3. Main results

Unless other specified, X and Y denote two Banach spaces and let $T \in C(X, Y)$ with a generalized inverse $T^+ \in B(Y, X)$. Let $\delta T \in L(X, Y)$ be T–bounded with T–bound $b < 1$ and satisfy

$$\|\delta TT^+ y\| \le \lambda_1 \|y\| + \lambda_2 \|(I + \delta TT^+)y\| \quad (y \in Y), \tag{3.1}$$

where $\lambda_1, \lambda_2 \in [0, 1)$. We start our investigation with the following lemmas, which are preparation for the proof of our main results.

Lemma 3.1. [4] *Let X be a Banach space and $P \in B(X)$. If there exist two constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Px\| \le \lambda_1 \|x\| + \lambda_2 \|(I + P)x\| \qquad (x \in X),$$

then $I + P : X \to X$ is bijective and its inverse $(I + P)^{-1} \in B(X)$.

Lemma 3.2. *The operator $\overline{T} = T + \delta T$ is closed, $I + \delta TT^+$ is invertible and*

$$B = T^+(I + \delta TT^+)^{-1} : \ Y \to X$$

is an outer inverse of $\overline{T}$.

Proof. It follows from Lemma 2.2 that $\overline{T}$ is closed and for all $y \in Y$, by (3.1),

$$\|\delta TT^+y\| \le (\lambda_1 + \lambda_2)\|y\|/(1-\lambda_2),$$

which means that δTT^+ is bounded. By Lemma 3.1, $I + \delta TT^+$ is invertible and $(I+\delta TT^+)^{-1} \in B(Y)$. Hence $B = T^+(I+\delta TT^+)^{-1} : Y \to X$ is a bounded linear operator. To the end, we need to show that B is an outer inverse of $\overline{T}$, i.e., $B\overline{T}B = B$ on Y. Indeed, $R(B) = R(T^+) \subset D(T) = D(\overline{T})$ and

$$(I+\delta TT^+)TT^+ = TT^+ + \delta TT^+ = (T+\delta T)T^+ = \overline{T}T^+.$$

Therefore,

$$\begin{aligned} B\overline{T}B &= T^+(I+\delta TT^+)^{-1}\overline{T}T^+(I+\delta TT^+)^{-1} \\ &= T^+TT^+(I+\delta TT^+)^{-1} \\ &= T^+(I+\delta TT^+)^{-1} = B. \end{aligned}$$

□

Lemma 3.3. *The operator $I + T^+\delta T : D(T) \to D(T)$ is bijective and*

$$B = T^+(I+\delta TT^+)^{-1} = (I+T^+\delta T)^{-1}T^+ : Y \to X$$

is a bounded operator with $R(B) = R(T^+)$ and $N(B) = N(T^+)$.

Proof. It follows from Lemma 3.2 that $I + \delta TT^+ : Y \to Y$ is invertible and $B = T^+(I+\delta TT^+)^{-1} : Y \to X$ is well defined. We first prove that $I + T^+\delta T : D(T) \to D(T)$ is injective. In fact, if $x \in D(T)$ satisfies $(I+T^+\delta T)x = 0$, then $x = -T^+\delta Tx \in R(T^+)$ and $(I+T^+\delta T)T^+\delta Tx = -(I+T^+\delta T)x = 0$. This implies that $T^+(I+\delta TT^+)\delta Tx = 0$ and $y \triangleq (I+\delta TT^+)\delta Tx \in N(T^+)$. Then $y = (I+\delta TT^+)y$ and so $y = (I+\delta TT^+)^{-1}y = \delta Tx$. Thus $x = -T^+\delta Tx = -T^+y = 0$. Next, we shall show that $I + T^+\delta T : D(T) \to D(T)$ is surjective, i.e., for all $y \in D(T)$, we need to find a $x \in D(T)$ such that $(I+T^+\delta T)x = y$. Noting

$$\begin{aligned} B\delta Ty + T^+\delta TB\delta Ty &= (I+T^+\delta T)B\delta Ty = (I+T^+\delta T)T^+(I+\delta TT^+)^{-1}\delta Ty \\ &= T^+(I+\delta TT^+)(I+\delta TT^+)^{-1}\delta Ty = T^+\delta Ty, \end{aligned}$$

we have $(I+T^+\delta T)(I-B\delta T)y = y$ which implies that $x \triangleq (I-B\delta T)y$ satisfies $(I+T^+\delta T)x = y$. Therefore, $I+T^+\delta T : D(T) \to D(T)$ is bijective. By $(I+T^+\delta T)T^+ = T^+(I+\delta TT^+)$, we can get that B is a bounded operator. Obviously, we can see $R(B) = R(T^+)$ and $N(B) = N(T^+)$. □

We shall give the main theorem of this paper.

Theorem 3.4. *Let X and Y be two Banach spaces. Let $T \in C(X,Y)$ with a generalized inverse $T^+ \in B(Y,X)$. Let $\delta T \in L(X,Y)$ be T–bounded with T–bound $b < 1$ and satisfy (3.1), then the following statements are equivalent:*

(1) $$B = T^+(I+\delta TT^+)^{-1} = (I+T^+\delta T)^{-1}T^+ : \ Y \to X$$
is a generalized inverse of $\overline{T} = T + \delta T$;

(2) $R(\overline{T}) \cap N(T^+) = \{0\}$;

(3) $Y = R(\overline{T}) \oplus N(T^+)$;

(4) $X = N(\overline{T}) \oplus \overline{R(T^+)}$;

(5) $X = N(\overline{T}) + \overline{R(T^+)}$;

(6) $(I+\delta TT^+)^{-1}R(\overline{T}) = R(T)$;
(7) $(I+\delta TT^+)^{-1}\overline{T}N(T) \subset R(T)$;
(8) $(I+T^+\delta T)^{-1}N(T) = N(\overline{T})$. *In this case, $R(\overline{T})$ is closed and*

$$\|B - T^+\| \leq \|T^+\| \cdot \|(I+\delta TT^+)^{-1}\| \cdot \|\delta TT^+\|.$$

Proof. Obviously, we can see (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7).

(2) $\Rightarrow$ (1). It follows from Lemmas 3.2 and 3.3 that B is well defined and it is an outer inverse of $\overline{T}$ with $N(B) = N(T^+)$ and $R(B) = R(T^+)$. We claim that B is an inner inverse of $\overline{T}$ and $B\overline{T}$ is bounded. In fact, for any $x \in D(\overline{T})$, we have $B(\overline{T}x - \overline{T}B\overline{T}x) = B\overline{T}x - B\overline{T}B\overline{T}x = 0$, which implies $\overline{T}x - \overline{T}B\overline{T}x \in N(B)$. Thus

$$\overline{T}x - \overline{T}B\overline{T}x \in R(\overline{T}) \cap N(B) = R(\overline{T}) \cap N(T^+) = \{0\}.$$

i.e., $\overline{T}B\overline{T}x = \overline{T}x$. Since $T^+T = B(I+\delta TT^+)T = BT + B\delta TT^+T$ and $\delta T(I - T^+T)$ is bounded, we get that

$$\begin{aligned} B\overline{T} &= BT + B\delta T = T^+T - B\delta TT^+T + B\delta T \\ &= T^+T + B\delta T(I - T^+T) \end{aligned}$$

is bounded. Therefore, B is a generalized inverse of $\overline{T}$.

(3) $\Rightarrow$ (6). Since B is an outer inverse of $\overline{T}$ with $N(B) = N(T^+)$, $\overline{T}B : Y \to Y$ is the projector of Y onto $R(\overline{T}B)$ and

$$Y = R(\overline{T}B) \oplus N(\overline{T}B) = R(\overline{T}B) \oplus N(B) = R(\overline{T}B) \oplus N(T^+).$$

By $Y = R(\overline{T}) \oplus N(T^+)$ and $R(\overline{T}B) \subset R(\overline{T})$, we can get $R(\overline{T}B) = R(\overline{T})$. Then for all $x \in D(T)$, there exists a $y \in Y$ satisfying $\overline{T}x = \overline{T}By$. Hence

$$\begin{aligned} & (I+\delta TT^+)^{-1}\overline{T}x \\ =\ & (I+\delta TT^+)^{-1}\overline{T}T^+(I+\delta TT^+)^{-1}y \\ =\ & (I+\delta TT^+)^{-1}(I+\delta TT^+)TT^+(I+\delta TT^+)^{-1}y \\ =\ & TT^+(I+\delta TT^+)^{-1}y \in R(T). \end{aligned}$$

Thus $(I+\delta TT^+)^{-1}R(\overline{T}) \subset R(T)$. On the other hand,

$$(I+\delta TT^+)R(T) = (I - TT^+ + \overline{T}T^+)R(T) = \overline{T}T^+R(T) \subset R(\overline{T}).$$

(5) $\Rightarrow$ (7). For all $x \in N(T) \subset X$, we can set $x = x_1 + x_2$, where $x_1 \in N(\overline{T})$, $x_2 \in \overline{R(T^+)}$. Then $x_2 = x - x_1 \in D(T)$ and $x_2 - T^+Tx_2 \in \overline{R(T^+)} \cap N(T) = \{0\}$, i.e., $x_2 = T^+Tx_2$. Thus

$$\begin{aligned} (I+\delta TT^+)^{-1}\overline{T}x &= (I+\delta TT^+)^{-1}\overline{T}x_2 = (I+\delta TT^+)^{-1}\overline{T}T^+Tx_2 \\ &= (I+\delta TT^+)^{-1}(I+\delta TT^+)TT^+Tx_2 = Tx_2 \in R(T). \end{aligned}$$

(7) $\Rightarrow$ (1). As in (2) $\Rightarrow$ (1), we only show B is an inner inverse of $\overline{T}$. Indeed, for all $x \in D(\overline{T})$,

$$\begin{aligned} & (I - TT^+)(I+\delta TT^+)^{-1}\overline{T}T^+Tx \\ =\ & (I - TT^+)(I+\delta TT^+)^{-1}(T+\delta T)T^+Tx \\ =\ & (I - TT^+)(I+\delta TT^+)^{-1}(I+\delta TT^+)Tx = 0 \end{aligned}$$

and by (7), $(I+\delta TT^+)^{-1}\overline{T}(I-T^+T)x \in R(T)$ and

$$\begin{aligned}(\overline{T}-\overline{T}B\overline{T})x &= (I-\overline{T}B)\overline{T}x = (I+\delta TT^+ - \overline{T}T^+)(I+\delta TT^+)^{-1}\overline{T}x \\ &= (I-TT^+)(I+\delta TT^+)^{-1}\overline{T}(T^+T+I-T^+T)x \\ &= (I-TT^+)(I+\delta TT^+)^{-1}\overline{T}(I-T^+T)x = 0.\end{aligned}$$

(7) $\Rightarrow$ (8). For any $x \in N(\overline{T})$, then $(I+T^+\delta T)x = x - T^+Tx \in N(T)$. Hence $(I+T^+\delta T)N(\overline{T}) \subseteq N(T)$. Conversely, if $x \in N(T)$, then by (7), there exists a $y \in X$ such that $\overline{T}x = (I+\delta TT^+)Ty = \overline{T}T^+Ty$, i.e., $x - T^+Ty \in N(\overline{T})$. Hence

$$(I+T^+\delta T)(x-T^+Ty) = (I-T^+T)(x-T^+Ty) = x.$$

This implies $N(T) \subseteq (I+T^+\delta T)N(\overline{T})$. Thus, by Lemma 3.3 , $(I+T^+\delta T)^{-1}N(T) = N(\overline{T})$.

(8) $\Rightarrow$ (2). Let $y \in R(\overline{T}) \cap N(T^+)$, then there exists $x \in X$ such that $y = \overline{T}x$ and $T^+\overline{T}x = 0$. Hence

$$T(I+T^+\delta T)x = Tx + TT^+\delta Tx = Tx + TT^+\overline{T}x - TT^+Tx = 0,$$

which implies $(I+T^+\delta T)x \in N(T)$. By (8), $x \in N(\overline{T})$ and so $y = \overline{T}x = 0$.

In this case, $R(\overline{T}) = R(\overline{T}B) = N(I-\overline{T}B)$ is closed and

$$\begin{aligned}\|B-T^+\| &= \|T^+(I+\delta TT^+)^{-1} - T^+\| \\ &\leq \|T^+\|\cdot\|(I+\delta TT^+)^{-1} - I\| \\ &\leq \|T^+\|\cdot\|(I+\delta TT^+)^{-1}\|\cdot\|\delta TT^+\|.\end{aligned}$$

□

Remark 3.5. (1) In Theorem 3.4, we assume the condition T–bound $b < 1$ to guarantee that $\overline{T}$ is closed and the inequality (3.1) to guarantee that $I+\delta TT^+$ is invertible. The role of these two conditions are independent.

(2) If $a\|T^+\| + b\|TT^+\| < 1$, then $\|\delta TT^+\| < 1$. Let $\lambda_1 = \|\delta TT^+\|$ and $\lambda_2 = 0$ in Theorem 3.4, then we can obtain the previous results in [12, 20, 25]. Also, Theorem 3.4 extends the main results in [5, 6, 7, 8, 9, 11, 14, 21, 24] to the case of closed linear operators.

Corollary 3.6. *If $N(T) \subset N(\delta T)$ or $R(\delta T) \subset R(T)$, then*

$$B = T^+(I+\delta TT^+)^{-1} = (I+T^+\delta T)^{-1}T^+ :\ Y \to X$$

is a generalized inverse of $\overline{T} = T + \delta T$.

Proof. If $N(T) \subset N(\delta T)$, then $N(T) \subset N(\overline{T})$. By condition (7) in Theorem 3.4, B is a generalized inverse of $\overline{T}$. If $R(\delta T) \subset R(T)$, then $R(\overline{T}) \subset R(T)$. By $R(T) \cap N(T^+) = \{0\}$ and (2) in Theorem 3.4, we can get what we desired. □

Remark 3.7. Corollary 3.6 is a generalization of main results in [20, 22, 25]. It should be noted that the proof in [20, 25] relies heavily on the condition $N(T) \subset N(\delta T)$ and their methods cannot be used to deal with the range preserving perturbations.

In the following, we shall consider the perturbation of Moore–Penrose inverse.

Theorem 3.8. *Let X and Y be two Hilbert spaces. Let $T \in C(X,Y)$ with the Moore–Penrose inverse $T^\dagger \in B(Y,X)$. Let $\delta T \in L(X,Y)$ be T–bounded with T–bound $b<1$ and satisfy*

$$\|\delta T T^\dagger y\| \le \lambda_1 \|y\| + \lambda_2 \|(I+\delta T T^\dagger)y\| \quad (y \in Y), \tag{3.2}$$

where $\lambda_1, \lambda_2 \in [0,1)$. Then

$$B = T^\dagger (I+\delta T T^\dagger)^{-1} = (I + T^\dagger \delta T)^{-1} T^\dagger : \ Y \to X$$

is the Moore–Penrose inverse of $\overline{T} = T + \delta T$ if and only if

$$R(\overline{T}) = R(T) \quad \textit{and} \quad N(\overline{T}) = N(T).$$

Proof. If $R(\overline{T}) = R(T)$ and $N(\overline{T}) = N(T)$, then by Corollary 3.6, B is a generalized inverse of $\overline{T}$ and so

$$X = N(\overline{T}) \oplus \overline{R(B)} \quad \text{and} \quad Y = R(\overline{T}) \oplus N(B). \tag{3.3}$$

Since $X = N(T) \dot{+} \overline{R(T^\dagger)}$, $Y = R(T) \dot{+} N(T^\dagger)$, $R(T^\dagger) = R(B)$ and $N(T^\dagger) = N(B)$, we get

$$X = N(\overline{T}) \dot{+} \overline{R(B)} \quad \text{and} \quad Y = R(\overline{T}) \dot{+} N(B),$$

i.e., the topological decomposition in (3.3) is orthogonal. Thus B is the Moore–Penrose inverse of $\overline{T}$. Conversely, if B is the Moore–Penrose inverse of $\overline{T}$, then

$$X = N(\overline{T}) \dot{+} \overline{R(B)} \quad \text{and} \quad Y = R(\overline{T}) \dot{+} N(B),$$

i.e., $N(\overline{T}) = R(B)^\perp$ and $R(\overline{T}) = N(B)^\perp$. Noting $N(T) = R(T^\dagger)^\perp$, $R(T) = N(T^\dagger)^\perp$, $R(T^\dagger) = R(B)$ and $N(T^\dagger) = N(B)$, we can get $R(\overline{T}) = R(T)$ and $N(\overline{T}) = N(T)$. □

Remark 3.9. From Theorem 3.8, we can get Theorem 3.1 in [7] and Theorem 3.2 in [12]. It should be noted that our proof is straightforward and concise.

In the following, we shall consider the stable perturbation for the Moore–Penrose inverse of closed EP operator. The next lemma is inspired from [2, 15].

Lemma 3.10. *Let X be a Hilbert space and $T \in C(X)$. Then T is an EP operator if and only if*

$$X = N(T) \dot{+} R(T).$$

Furthermore, if T is Moore–Penrose invertible, then T is an EP operator if and only if

$$N(T) = N(T^\dagger) \quad \textit{and} \quad R(T) = \overline{R(T^\dagger)}.$$

Proof. If $X = N(T) \dot{+} R(T)$, then T has a bounded Moore–Penrose inverse $T^\dagger \in B(X)$ such that $N(T) = N(T^\dagger)$ and $R(T) = \overline{R(T^\dagger)}$. Hence for all $x \in D(T)$, $T^\dagger T x \in R(T^\dagger) \subset R(T)$, $x - T^\dagger T x \in N(T) = N(T^\dagger)$ and so $TT^\dagger(T^\dagger T x) = T^\dagger T x$, $TT^\dagger (x - T^\dagger T x) = 0$. Thus $T^\dagger T x = TT^\dagger x$, i.e, $T^\dagger T = TT^\dagger$ on $D(T)$. Therefore, T is an EP operator. Conversely, if T is an EP operator, then

$$X = N(T) \dot{+} \overline{R(T^\dagger)} = R(T) \dot{+} N(T^\dagger)$$

and $TT^\dagger = T^\dagger T$ on $D(T)$. Hence, $TT^\dagger$ is exactly the unique extended orthogonal projector of $T^\dagger T$ onto $\overline{R(T^\dagger)}$ with the null space $N(T)$. While $TT^\dagger$ itself is the orthogonal projector onto $R(T)$ with the null space $N(T^\dagger)$, we get $R(T) = \overline{R(T^\dagger)}$ and $N(T) = N(T^\dagger)$. Therefore, $X = N(T)\dot{+}R(T)$. □

Theorem 3.11. *Let X be a Hilbert space. Let $T \in C(X)$ be an EP operator with the Moore–Penrose inverse $T^\dagger \in B(X)$. Let $\delta T \in L(X)$ be T-bounded with T-bound $b < 1$ and satisfy (3.2), then $\overline{T} = T + \delta T$ is an EP operator and*

$$B = T^\dagger(I + \delta T T^\dagger)^{-1} = (I + T^\dagger \delta T)^{-1}T^\dagger : \ Y \to X$$

is its Moore–Penrose inverse if and only if $R(\overline{T}) = R(T)$ and $N(\overline{T}) = N(T)$.

Proof. It follows from Theorem 3.8 that we can obtain the necessity. Conversely, if $R(\overline{T}) = R(T)$ and $N(\overline{T}) = N(T)$, then by Theorem 3.8, B is the Moore–Penrose of $\overline{T}$. From Lemmas 3.3 and 3.10, we have

$$N(\overline{T}) = N(T) = N(T^\dagger) = N(B)$$

and

$$R(\overline{T}) = R(T) = \overline{R(T^\dagger)} = \overline{R(B)}.$$

Hence by Lemma 3.10 again, $\overline{T}$ is an EP operator. □

Theorem 3.12. *Under the assumptions of Theorem 3.11, if $\overline{T} = T + \delta T$ is an EP operator, then the following statements are equivalent:*
(1) $B = T^\dagger(I + \delta T T^\dagger)^{-1}$ *is the Moore–Penrose inverse of* $\overline{T}$*;*
(2) $R(\overline{T}) = R(T)$*;*
(3) $N(\overline{T}) = N(T)$.

Proof. Since both T and $\overline{T}$ are EP operators, $N(T) = R(T)^\perp$, $R(T) = N(T)^\perp$, $N(\overline{T}) = R(\overline{T})^\perp$ and $R(\overline{T}) = N(\overline{T})^\perp$. By Theorem 3.11, we can get what we desired. □

Remark 3.13. It seems that Theorems 3.11 and 3.12 are new even in the case of bounded linear operators.

To illustrate our theorems, we give two examples.

Example 3.14. Let $X = C[0, 1]$ be the Banach space of all continuous functions on the interval $[0, 1]$. Define the linear operator T by

$$Tf = f'' \quad (f \in C^2[0, 1])$$

and δT by

$$\delta T f = -f' \quad (f \in C^1[0, 1]),$$

then T and δT are densely defined closed operators and δT is $T-$bounded with $T-$bound 0[13]. It can be verified that $R(T) = R(\delta T) = X$, $N(T) = \{c_1 x + c_2 :$ c_1 and c_2 are constant numbers$\}$ and $N(\delta T) = \{$constant functions$\}$. We also can see

$$X = N(T) \oplus \overline{X_1},$$

where $X_1 = \{f \in C[0, 1] : f(0) = f'(0) = 0\}$ is a subspace of X. Hence T has a bounded generalized inverse $T^+ : X \to X_1 \subset X$ defined by

$$(T^+ f)(x) = \int_0^x (\int_0^t f(s)\mathrm{d}s)\mathrm{d}t \qquad (f \in X).$$

Thus

$$(\delta TT^+f)(x) = -\int_0^x f(s)\mathrm{d}s \qquad (f \in X).$$

It is easy to verify $\|\delta TT^+\| = 1$. Let $\alpha \in (\frac{1}{4}, \ln\frac{4}{3})$, then $0 < (1-\alpha)e^{\alpha} < 1$ and $0 < \alpha e^{\alpha} < 1$. In the following, we shall show

$$\|\delta TT^+f\| \le (1-\alpha)e^{\alpha}\|f\| + \alpha e^{\alpha}\|(I+\delta TT^+)f\| \qquad \forall f \in X.$$

In fact, we put $F(x) = \int_0^x f(s)\mathrm{d}s$ and it suffices to show

$$\max_{0\le x\le 1}|F(x)| \le (1-\alpha)e^{\alpha}\max_{0\le x\le 1}|f(x)| + \alpha e^{\alpha}\max_{0\le x\le 1}|f(x)-F(x)|.$$

Set

$$G(x) = e^{-\alpha x}F(x),$$

then for all $x \in [0,1]$, we have

$$\begin{aligned} G(x) &= G(x) - G(0) = G'(\xi)x \qquad\qquad \xi \in [0,x] \\ &= e^{-\alpha\xi}[f(\xi) - \alpha F(\xi)]x \\ &= \alpha x e^{-\alpha\xi}[f(\xi) - F(\xi)] + (1-\alpha)xe^{-\alpha\xi}f(\xi) \end{aligned}$$

and so

$$F(x) = \alpha x e^{\alpha(x-\xi)}[f(\xi) - F(\xi)] + (1-\alpha)xe^{\alpha(x-\xi)}f(\xi).$$

Hence for all $x \in [0,1]$,

$$|F(x)| \le \alpha e^{\alpha}\max_{0\le x\le 1}|f(x)-F(x)| + (1-\alpha)e^{\alpha}\max_{0\le x\le 1}|f(x)|.$$

Thus

$$\max_{0\le x\le 1}|F(x)| \le \alpha e^{\alpha}\max_{0\le x\le 1}|f(x)-F(x)| + (1-\alpha)e^{\alpha}\max_{0\le x\le 1}|f(x)|.$$

Therefore, by $R(T) = R(\overline{T})$ and Theorem 3.4 (or Corollary 3.6), $\overline{T} = T + \delta T$ has a bounded generalized inverse $T^+(I+\delta TT^+)^{-1}$. That is, for all $f \in C[0,1]$, the differential equation

$$y'' - y' = f$$

has a solution $y = T^+(I+\delta TT^+)^{-1}f$.

Remark 3.15. It should be noted that $N(T) \not\subseteq N(\delta T)$ and $N(T) \neq N(\overline{T})$ in Example 3.14. And we also can verify

$$\begin{aligned} [T^+(I+\delta TT^+)^{-1}f](x) &= \int_0^x [e^{-t}\int_0^t e^s f(s)\mathrm{d}s]\mathrm{d}t \\ &= \int_0^x (1-e^{s-x})f(s)\mathrm{d}s, \end{aligned}$$

which exactly is a bounded generalized inverse of $\overline{T}$.

Example 3.16. Let

$$L^2[0,1] = \{f,\ f \text{ measurable complex} - \text{valued on } [0,1], \int_{[0,1]}|f(x)|^2dx < \infty\ \}$$

be the Hilbert space with the inner product

$$\langle f, g\rangle = \int_{[0,1]} \overline{f(x)} g(x) dx, \qquad f, g \in L^2[0,1].$$

Set $t : [0,1] \to C$ by

$$t(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{\sqrt{x}} & 0 < x \leq 1 \end{cases}$$

and define the maximal operator of multiplication T by t on $L^2[0,1]$, i.e.,

$$Tf = tf, \quad \text{for } f \in D(T) = \{f \in L^2[0,1],\ tf \in L^2[0,1]\},$$

then T is a densely defined closed linear operator[23]. Since $|t(x)| \geq 1$, $\forall x \in [0,1]$, $R(T) = L^2[0,1]$ and T has a bounded inverse $T^{-1} : L^2[0,1] \to L^2[0,1]$ defined by

$$T^{-1} g = t_1 g, \qquad \forall g \in L^2[0,1],$$

where

$$t_1(x) = \begin{cases} 1 & x = 0 \\ \sqrt{x} & 0 < x \leq 1. \end{cases}$$

Therefore, T is a closed EP operator on $L^2[0,1]$.

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