



IMPROVEMENT OF JENSEN–STEFFENSEN’S INEQUALITY FOR SUPERQUADRATIC FUNCTIONS

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This paper is dedicated to Professor Lars-Erik Persson on the occasion of his 65th birthday.

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ABSTRACT. In this paper, improvements for superquadratic functions of Jensen–Steffensen’s and related inequalities are discussed. For superquadratic functions which are not convex we get inequalities analog to Jensen–Steffensen’s inequality for convex functions. For superquadratic functions which are convex (including many useful functions), we get improvements and extensions of Jensen–Steffensen’s inequality and related inequalities.

1. INTRODUCTION AND PRELIMINARIES

In this paper we refine results derived from Jensen–Steffensen’s inequality and its extension for superquadratic functions. These functions were introduced in [4] and [5] and dealt with in numerous papers (see for example [9]).

Jensen–Steffensen’s inequality states that if $f : I \rightarrow \mathbb{R}$ is convex, then

$$f\left(\frac{1}{A_n} \sum_{i=1}^n a_i x_i\right) \leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i)$$

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holds, where I is an interval in \mathbb{R} , $\mathbf{x} = (x_1, \dots, x_n)$ is any monotonic n -tuple in I^n and $\mathbf{a} = (a_1, \dots, a_n)$ is real n -tuple that satisfies

$$\begin{aligned} 0 \leq A_j \leq A_n, \quad j = 1, \dots, n, \quad A_n > 0, \\ A_j = \sum_{i=1}^j a_i, \quad \bar{A}_j = \sum_{i=j}^n a_i, \quad j = 1, \dots, n, \end{aligned} \quad (1.1)$$

(see [11, p. 57]).

In [10], J. Pečarić proved a refinement of Slater's inequality established in [12]. J. Pečarić proved that under the same conditions leading to Jensen–Steffensen's inequality, together with

$$\sum_{i=1}^n a_i f'(x_i) \neq 0 \quad \text{and} \quad M = \frac{\sum_{i=1}^n a_i x_i f'(x_i)}{\sum_{i=1}^n a_i f'(x_i)} \in I, \quad (1.2)$$

the inequality

$$\sum_{i=1}^n a_i f(x_i) \leq A_n f(M)$$

holds.

Now we quote some definitions and state a list of basic properties of superquadratic functions.

Definition 1.1. [4, Definition 2.1] A function $f : [0, b) \rightarrow \mathbb{R}$ is superquadratic provided that for all $0 \leq x < b$, there exists a constant $C(x) \in \mathbb{R}$ such that

$$f(y) - f(x) - f(|y - x|) \geq C(x)(y - x)$$

for all $y \in [0, b)$. We say that f is subquadratic if $-f$ is a superquadratic function.

Lemma 1.2. [5, Lemma 2.3] Suppose that f is a superquadratic function on $[0, b)$, $x_i \in [0, b)$, $i = 1, \dots, n$, and $a_i \geq 0$, $i = 1, \dots, n$, are such that $A_n = \sum_{i=1}^n a_i > 0$. Then

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(\bar{x}) \geq \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - \bar{x}|), \quad (1.3)$$

where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$.

Lemma 1.3. [4, Lemma 2.1] Let f be a superquadratic function with $C(x)$ as in Definition 1.1.

- (i) Then $f(0) \leq 0$.
- (ii) If $f(0) = f'(0) = 0$, then $C(x) = f'(x)$ whenever f is differentiable at $0 < x < b$.
- (iii) If $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

Lemma 1.4. [4, Lemma 3.1] Suppose that $f : [0, b) \rightarrow \mathbb{R}$ is a continuously differentiable function and $f(0) \leq 0$. If f' is superadditive or $\frac{f'(x)}{x}$ is nondecreasing, then f is superquadratic.

We quote now some theorems that we refine in the sequel.

Theorem 1.5. [1, Theorem 1] *Let $f : [0, b) \rightarrow \mathbb{R}$ be a differentiable superquadratic and nonnegative function. Let $x_i \in [0, b)$, $i = 1, \dots, n$, be such that*

$$x_1 \leq x_2 \leq \dots \leq x_n \quad \text{or} \quad x_1 \geq x_2 \geq \dots \geq x_n.$$

Let \mathbf{a} be a real n -tuple satisfying (1.1) and $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$. Then

$$\begin{aligned} \sum_{i=1}^n a_i f(x_i) - A_n f(\bar{x}) &\geq \sum_{i=1}^{k-1} A_i f(x_{i+1} - x_i) + A_k f(\bar{x} - x_k) \\ &\quad + \bar{A}_{k+1} f(x_{k+1} - \bar{x}) + \sum_{i=k+2}^n \bar{A}_i f(x_i - x_{i-1}) \\ &\geq \left(\sum_{i=1}^k A_i + \sum_{i=k+1}^n \bar{A}_i \right) f \left(\frac{\sum_{i=1}^n a_i |x_i - \bar{x}|}{\sum_{i=1}^k A_i + \sum_{i=k+1}^n \bar{A}_i} \right) \\ &\geq (n-1) A_n f \left(\frac{\sum_{i=1}^n a_i (|x_i - \bar{x}|)}{(n-1) A_n} \right), \end{aligned}$$

where $k \in \{1, \dots, n-1\}$ satisfies

$$x_k \leq \bar{x} \leq x_{k+1}.$$

Lemma 1.6. [6, Lemma 1a] *Let $f : (a, b) \rightarrow \mathbb{R}$, where $-\infty \leq a < b \leq \infty$, be a convex function. Let $z \in (a, b)$ be fixed. Then the function $\Delta : (a, b) \rightarrow \mathbb{R}$ defined by*

$$\Delta(y) = f(y) - f(z) - f'(z)(y - z)$$

is nonnegative on (a, b) , nonincreasing on $(a, z]$ and nondecreasing on $[z, b)$.

Theorem 1.7. [6, Theorem 1] *Let $f : (a, b) \rightarrow \mathbb{R}$ be a convex function and $a_i \in \mathbb{R}$, $i = 1, \dots, n$, be such that (1.1) holds. Then for any $x_i \in (a, b)$, $i = 1, \dots, n$, such that*

$$x_1 \leq x_2 \leq \dots \leq x_n \quad \text{or} \quad x_1 \geq x_2 \geq \dots \geq x_n,$$

the inequalities

$$f(c) + f'(c)(\bar{x} - c) \leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \leq f(d) + \frac{1}{A_n} \sum_{i=1}^n a_i f'(x_i)(x_i - d)$$

hold for all $c, d \in (a, b)$, where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$.

Theorem 1.8. [6, Theorem 4] *Suppose that all the conditions of Theorem 1.7 are satisfied and additionally assume that f is differentiable on $I = (a, b)$ and M satisfies (1.2). If f is nondecreasing and f' is concave on I , or if f is nonincreasing and f' is convex on I , then*

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \leq f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i f'(x_i)(x_i - \bar{x}) \leq f(M). \tag{1.4}$$

In this paper we obtain Lemma 1.2 and related results for superquadratic functions, but this time the coefficients a_i ($i = 1, \dots, n$) are not only nonnegative. There, in Lemma 1.2, the coefficients satisfy

$$a_i \geq 0 \quad (i = 1, \dots, n) \quad \text{and} \quad A_n = \sum_{i=1}^n a_i > 0.$$

Here we deal with coefficients that satisfy (1.1) and $0 \leq x_1 \leq x_2 \leq \dots \leq x_n < b$.

We generalize also part of the theorems proved in [3]. In particular we extend the following version of [3, Theorem 2.1a], as most of the theorems there, in [3], result from this theorem.

Theorem 1.9. [3, Theorem 2.1a] *Let $0 \leq x_k < b$, $a_k > 0$, $k = 1, \dots, n$. If f is superquadratic on $[0, b)$, then*

$$\begin{aligned} & \sum_{i=1}^n a_i f(x_i) - A_n f\left(\frac{\sum_{i=1}^n a_i x_i}{A_n}\right) - \left(\sum_{i=m+1}^n a_i f(x_i) - \bar{A}_{m+1} f\left(\frac{\sum_{i=m+1}^n a_i x_i}{\bar{A}_{m+1}}\right) \right) \\ & \geq \sum_{i=1}^m a_i f\left(\left|x_i - \frac{\sum_{j=1}^n a_j x_j}{A_n}\right|\right) + \bar{A}_{m+1} f\left(\left|\frac{\sum_{i=m+1}^n a_i x_i}{\bar{A}_{m+1}} - \frac{\sum_{i=1}^n a_i x_i}{A_n}\right|\right). \end{aligned} \quad (1.5)$$

This type of inequality, but for convex functions, was dealt with in [7].

In the sequel, all the results dealing with $0 \leq x_1 \leq x_2 \leq \dots \leq x_n < b$ hold also for $b > x_1 \geq x_2 \geq \dots \geq x_n \geq 0$.

2. MAIN RESULTS

Analogous to Lemma 1.6 and Theorem 1.7 we prove the following Lemma 2.1 and Theorem 2.2. We use a technique similar to the one used in [6].

Lemma 2.1. *Let f be a continuously differentiable function on $[0, b)$ and f' be superadditive on $[0, b)$. Then the function $D : [0, b) \rightarrow \mathbb{R}$, defined by*

$$D(y) = f(y) - f(z) - f'(z)(y - z) - f(|y - z|) + f(0)$$

is nonnegative on $[0, b)$, nonincreasing on $[0, z)$ and nondecreasing on $[z, b)$, for $0 \leq z < b$.

Proof. Since f' is superadditive, then for $0 \leq z \leq y < b$ we have

$$\begin{aligned} 0 & \leq \int_z^y (f'(t) - f'(z) - f'(t - z)) dt \\ & = f(y) - f(z) - f'(z)(y - z) - f(y - z) + f(0) \end{aligned}$$

and for $0 \leq y \leq z < b$,

$$\begin{aligned} 0 & \leq \int_y^z (f'(z) - f'(t) - f'(z - t)) dt \\ & = f'(z)(z - y) - f(z) + f(y) + f(0) - f(z - y). \end{aligned}$$

Together these show that for any $y, z \in [0, b)$ the inequality

$$f(y) - f(z) - f'(z)(y - z) - f(|y - z|) + f(0) \geq 0$$

holds. So we conclude that the function D is nonnegative on $[0, b)$.

Since

$$D'(y) = f'(y) - f'(z) - f'(|y - z|) \operatorname{sgn}(y - z),$$

and as f' is superadditive, then for $0 \leq y \leq z$ we have

$$\begin{aligned} D'(y) &= f'(y) - f'(z) + f'(z - y) \\ &\leq f'(y) - f'(z) + f'(z) - f'(y) \\ &= 0 \end{aligned}$$

and for $z \leq y < b$,

$$\begin{aligned} D'(y) &= f'(y) - f'(z) - f'(y - z) \\ &\geq f'(y) - f'(z) - f'(y) + f'(z) \\ &= 0. \end{aligned}$$

This completes the proof. \square

Now we can present the main results of this section where we show that inequality (1.3) is satisfied not only for nonnegative coefficients but also when (1.1) is satisfied and $x_1 \leq x_2 \leq \dots \leq x_n$.

Theorem 2.2. *Let f be a continuously differentiable function on $[0, b)$ and f' be superadditive on $[0, b)$. Let \mathbf{a} be a real n -tuple satisfying (1.1). Let $x_i \in [0, b)$, $i = 1, \dots, n$, be such that $x_1 \leq x_2 \leq \dots \leq x_n$ and $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$. Then*

(a) *the inequality*

$$f(c) - f(0) + f'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|) \leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \quad (2.1)$$

holds for all $c \in [0, b)$.

In particular,

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(\bar{x}) + f(0) \geq \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - \bar{x}|). \quad (2.2)$$

(b) *If in addition $f(0) \leq 0$, then f is superquadratic and*

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \geq f(c) + f'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|). \quad (2.3)$$

In particular,

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \geq f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - \bar{x}|). \quad (2.4)$$

(c) *If in addition $f \geq 0$ and $f(0) = f'(0) = 0$, then f is superquadratic and convex increasing and*

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(c) - f'(c)(\bar{x} - c) \geq \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|) \geq 0. \quad (2.5)$$

Proof. It was proved in [1] and in [2] that $x_1 \leq \bar{x} \leq x_n$.

(a) Let $D(x_i) = f(x_i) - f(c) - f'(c)(x_i - c) - f(|x_i - c|) + f(0)$, $i = 1, \dots, n$.

From Lemma 2.1 we know that $D(x_i) \geq 0$ for all $i = 1, \dots, n$.

Comparing c with x_1, \dots, x_n we must consider three cases.

Case 1. $x_n < c < b$: In this case $x_i \in [0, c)$ for all $i = 1, \dots, n$. Hence, according to Lemma 2.1 we have

$$D(x_1) \geq D(x_2) \geq \dots \geq D(x_n) \geq 0.$$

Denoting $A_0 = 0$ it follows

$$a_i = A_i - A_{i-1}, \quad i = 1, \dots, n,$$

and therefore

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^n (A_i - A_{i-1}) D(x_i) \\ &= A_1 D(x_1) + (A_2 - A_1) D(x_2) + \dots + (A_n - A_{n-1}) D(x_n) \\ &= \sum_{i=1}^{n-1} A_i (D(x_i) - D(x_{i+1})) + A_n D(x_n) \\ &\geq 0. \end{aligned}$$

Case 2. $0 \leq c < x_1$: In this case $x_i \in (c, b)$ for all $i = 1, \dots, n$. Hence, according to Lemma 2.1 we have

$$0 \leq D(x_1) \leq D(x_2) \leq \dots \leq D(x_n).$$

Denoting $\bar{A}_{n+1} = 0$ it follows

$$\begin{aligned} \bar{A}_k &= \sum_{i=k}^n a_i = A_n - A_{k-1}, \quad k = 1, \dots, n, \\ a_i &= \bar{A}_i - \bar{A}_{i+1}, \quad i = 1, \dots, n, \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^n (\bar{A}_i - \bar{A}_{i+1}) D(x_i) \\ &= \bar{A}_1 D(x_1) + \sum_{i=2}^n \bar{A}_i (D(x_i) - D(x_{i-1})) \\ &\geq 0. \end{aligned}$$

Case 3. $x_1 \leq c \leq x_n$: In this case there exists $k \in \{1, \dots, n-1\}$ such that $x_k \leq c \leq x_{k+1}$.

By Lemma 2.1 we have

$$D(x_1) \geq D(x_2) \geq \dots \geq D(x_k) \geq 0 \quad \text{and} \quad 0 \leq D(x_{k+1}) \leq D(x_{k+2}) \leq \dots \leq D(x_n).$$

Then

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^k a_i D(x_i) + \sum_{i=k+1}^n a_i D(x_i) \\ &= \sum_{i=1}^{k-1} A_i (D(x_i) - D(x_{i+1})) + A_k D(x_k) \\ &\quad + \bar{A}_{k+1} D(x_{k+1}) + \sum_{i=k+2}^n \bar{A}_i (D(x_i) - D(x_{i-1})) \\ &\geq 0. \end{aligned}$$

In all three cases we get

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^n a_i [f(x_i) - f(c) - f'(c)(x_i - c) - f(|x_i - c|) + f(0)] \\ &\geq 0, \end{aligned}$$

and therefore, the inequality (2.1) holds. Inserting $c = \bar{x}$ in (2.1) we get (2.2).

(b) From Lemma 1.4 we get that as $f(0) \leq 0$, f is superquadratic. The inequality (2.3) follows from (2.1). Inserting $c = \bar{x}$ in (2.3) we get (2.4).

(c) First, from Lemma 1.4 we get that as $f(0) = 0$, f is superquadratic. Then from Lemma 1.3 it follows that f is convex increasing. We only need to show that under our conditions the inequality

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|) \geq 0 \tag{2.6}$$

holds.

We will show that (2.6) holds in the case that $x_k \leq c \leq x_{k+1}$, $k = 1, \dots, n - 1$. The other cases we can prove similarly.

We use the identity

$$\begin{aligned} \sum_{i=1}^n a_i f(|x_i - c|) &= \sum_{i=1}^{k-1} A_i (f(c - x_i) - f(c - x_{i+1})) \\ &\quad + A_k f(c - x_k) + \bar{A}_{k+1} f(x_{k+1} - c) \\ &\quad + \sum_{i=k+2}^n \bar{A}_i (f(x_i - c) - f(x_{i-1} - c)). \end{aligned}$$

Since the function f is nonnegative and convex, and $f(0) = 0$, then for $0 \leq x_i \leq x_{i+1} \leq c$, $i = 1, \dots, k - 1$, we have

$$\begin{aligned} f(c - x_i) - f(c - x_{i+1}) &\geq f(x_{i+1} - x_i) - f(0) \\ &= f(x_{i+1} - x_i) \\ &\geq 0 \end{aligned}$$

and for $c \leq x_{i-1} \leq x_i < b$, $i = k + 2, \dots, n$,

$$\begin{aligned} f(x_i - c) - f(x_{i-1} - c) &\geq f(x_i - x_{i-1}) - f(0) \\ &= f(x_i - x_{i-1}) \\ &\geq 0. \end{aligned}$$

Therefore, as (1.1) holds, (2.6) is satisfied and together with (2.1) it gives (2.5). This completes the proof. \square

So, we see that the inequality (2.4) in Part (b) of Theorem 2.2 extends Lemma 1.2 to coefficients satisfying (1.1) and $x_1 \leq x_2 \leq \dots \leq x_n$. In (2.3) we get refinement of Lemma 1.2. Also, the inequality (2.5) for $c = \bar{x}$ improves Theorem 1.5.

Remark 2.3. Suppose that f is continuously differentiable on $[0, b]$, $f(0) = 0$ and f' is subadditive on $[0, b]$. Then f is subquadratic and the reverse of (2.3) holds.

Theorem 2.4. *Let f be continuously differentiable and convex increasing on $[0, b]$, $f(0) = f'(0) = 0$ and f' be concave on $[0, b]$. Let \mathbf{a} be a real n -tuple satisfying (1.1) and $x_i \in [0, b]$, $i = 1, \dots, n$, be such that $x_1 \leq x_2 \leq \dots \leq x_n$. Then*

$$\begin{aligned} &\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \\ &\leq \min \left\{ f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i f'(x_i)(x_i - \bar{x}), \quad f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - \bar{x}|) \right\} \quad (2.7) \end{aligned}$$

holds, where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$.

Proof. Since $f'(0) = 0$ and f' is concave, it follows that f' is subadditive and therefore, f is subquadratic.

Then by Remark 2.3 the inequality

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \leq f(c) + f'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|)$$

holds. Choosing $\bar{x} = c$ we get

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \leq f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - \bar{x}|). \quad (2.8)$$

By combining (2.8) with (1.4) it follows (2.7). \square

In the following example we show that there are cases where the inequality (2.7), which is derived in Theorem 1.8 from the convexity of f , gives a better bound than the bound derived from (2.8) for subquadratic f and vice-versa.

Example 2.5. Consider the function $f(x) = x^{1+\alpha}$ for $0 \leq \alpha \leq 1$ and $x \geq 0$. It's clear that f is convex and f' is concave on $[0, b]$. Since $f(0) = 0$, then f is

also subquadratic.

If we choose $a_1 = a_2 = \frac{1}{2}$, $x_1 = 0$, $x_2 = 1$, from (2.7) it follows

$$\frac{1}{2} \leq \min \left\{ \frac{1}{2^{1+\alpha}} + \frac{1+\alpha}{4}, \frac{1}{2^{1+\alpha}} + \frac{1}{2^{1+\alpha}} \right\}.$$

Then for $\alpha = 1$ we have

$$\frac{1}{2} \leq \min \left\{ \frac{3}{4}, \frac{1}{4} \right\} = \frac{1}{4}$$

and for $\alpha = 0$

$$\frac{1}{2} \leq \min \left\{ \frac{3}{4}, 1 \right\} = \frac{3}{4}.$$

We see that in the first case we get better bound derived from the subquadracity of f and in the second case the better bound is derived from the convexity of f and the concavity of f' .

Now we prove an analog to Theorem 1.9 for coefficients that satisfy (1.1).

Theorem 2.6. *Let f be a continuously differentiable function on $[0, b)$, $f(0) = 0$ and f' be superadditive on $[0, b)$. Let \mathbf{a} be a real n -tuple satisfying (1.1) and $\bar{A}_j > 0$ for all $j = 1, \dots, n$. Then for any $x_i \in [0, b)$, $i = 1, \dots, n$, such that $x_1 \leq x_2 \leq \dots \leq x_n$, the inequality (1.5) holds.*

Proof. It is given that $0 \leq x_1 \leq x_2 \leq \dots \leq x_n < b$ and $\bar{A}_j > 0$ for all $j = 1, \dots, n$. Therefore, according to Theorem 399 in [8] we get

$$\sum_{i=m+1}^n a_i(x_i - x_m) \geq 0$$

which is equivalent to

$$\frac{1}{\bar{A}_{m+1}} \sum_{i=m+1}^n a_i x_i \geq x_m.$$

Denote

$$\begin{aligned} y_i &= x_i, \quad i = 1, \dots, m, & y_{m+1} &= \frac{1}{\bar{A}_{m+1}} \sum_{i=m+1}^n a_i x_i, \\ b_i &= a_i, \quad i = 1, \dots, m, & b_{m+1} &= \bar{A}_{m+1}, \\ B_i &= A_i, \quad i = 1, \dots, m, & B_{m+1} &= \sum_{i=1}^{m+1} b_i. \end{aligned} \tag{2.9}$$

From this notation it follows that

$$\begin{aligned} 0 &\leq B_i \leq B_{m+1}, \quad i = 1, \dots, m, & B_{m+1} &> 0, \\ B_{m+1} &= \sum_{i=1}^{m+1} b_i = A_n, \end{aligned}$$

and

$$0 \leq y_1 \leq y_2 \leq \dots \leq y_{m+1}.$$

Applying (2.4) to $y_i, b_i, B_i, i = 1, \dots, m + 1$, we get

$$\frac{1}{B_{m+1}} \sum_{i=1}^{m+1} b_i f(y_i) \geq f(\bar{y}) + \frac{1}{B_{m+1}} \sum_{i=1}^{m+1} b_i f(|y_i - \bar{y}|), \quad (2.10)$$

where

$$\bar{y} = \frac{1}{B_{m+1}} \sum_{i=1}^{m+1} b_i y_i.$$

Multiplying inequality (2.10) with B_{m+1} we get

$$\begin{aligned} & \sum_{i=1}^m b_i f(y_i) + b_{m+1} f(y_{m+1}) \\ & \geq B_{m+1} f(\bar{y}) + \sum_{i=1}^m b_i f(|y_i - \bar{y}|) + b_{m+1} f(|y_{m+1} - \bar{y}|). \end{aligned} \quad (2.11)$$

From (2.9) it follows that

$$\bar{y} = \frac{1}{A_n} \left(\sum_{i=1}^m a_i x_i + \sum_{i=m+1}^n a_i x_i \right) = \frac{1}{A_n} \sum_{i=1}^n a_i x_i \quad (2.12)$$

and therefore, using (2.9) and (2.12), the inequality (2.11) becomes

$$\begin{aligned} & \sum_{i=1}^m a_i f(x_i) + \bar{A}_{m+1} f\left(\frac{1}{\bar{A}_{m+1}} \sum_{i=m+1}^n a_i x_i\right) \\ & \geq A_n f\left(\frac{1}{A_n} \sum_{i=1}^n a_i x_i\right) + \sum_{i=1}^m a_i f\left(\left|x_i - \frac{1}{A_n} \sum_{i=1}^n a_i x_i\right|\right) \\ & \quad + \bar{A}_{m+1} f\left(\left|\frac{1}{\bar{A}_{m+1}} \sum_{i=m+1}^n a_i x_i - \frac{1}{A_n} \sum_{i=1}^n a_i x_i\right|\right), \end{aligned}$$

which is equivalent to (1.5). □

REFERENCES

- [1] S. Abramovich, S. Banić, M. Matić and J.E. Pečarić, *Jensen–Steffensen’s and related inequalities for superquadratic functions*, Math. Inequal. Appl. **11** (2007), no. 1, 23–41.
- [2] S. Abramovich, M. Klaričić Bakula, M. Matić and J.E. Pečarić, *A variant of Jensen–Steffensen’s inequality and quasi-arithmetic means*, J. Math. Anal. Appl. **307** (2005), 370–386.
- [3] S. Abramovich, B. Ivanković and J.E. Pečarić, *Improvement of Jensen–Steffensen’s inequality*, Australian J. Math. Anal. Appl., to appear.
- [4] S. Abramovich, G. Jameson and G. Sinnamon, *Refining Jensen’s inequality*, Bull. Math. Sc. Math. Roum. **47** (2004), 3–14.
- [5] S. Abramovich, G. Jameson and G. Sinnamon, *Inequalities for averages of convex and superquadratic functions*, J. Inequal. Pure and Appl. Math. **5** (2004), no. 4, article 91.
- [6] M. Klaričić Bakula, M. Matić and J.E. Pečarić, *Generalization of the Jensen–Steffensen and related inequalities* Cent. Eur. J. Math. **7** (2009), no. 4, 787–803.
- [7] S.S. Dragomir, *A new refinement of Jensen’s inequality in linear spaces with applications*, RGMIA, Vol. 12 2009, online: [www.staff.vu.edu.au/rgmia/v12\(E\).asp](http://www.staff.vu.edu.au/rgmia/v12(E).asp).

- [8] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, New York, 1952.
- [9] J.A. Oguntuase, L.E. Perssons, E.K. Essel and B.A. Popoola, *Refined multidimensional Hardy-type inequalities via superquadracity*, Banach J. Math. Anal. **2** (2008), no. 2, 129–139.
- [10] J.E. Pečarić, *A companion inequality to Jensen–Steffensen’s inequality*, J. Approx.Theory **44** (1985), no. 3, 289–291.
- [11] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, 1992.
- [12] M.L. Slater, *A companion inequality to Jensen’s inequality*, J. Approx. Theory **32** (1981), 106–166.

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