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LINEAR ISOMETRIES OF FINITE CODIMENSIONS ON BANACH ALGEBRAS OF HOLOMORPHIC FUNCTIONS

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Abstract. Let K be a compact subset of the complex n-space and $A(K)$ the algebra of all continuous functions on K which are holomorphic on the interior of K. In this paper we show that under some hypotheses on K, there exists no linear isometry of finite codimension on $A(K)$. Several compact subsets including the closure of strictly pseudoconvex domain and the product of the closure of plane domains which are bounded by a finite number of disjoint smooth curves satisfy the hypotheses.

1. Introduction and preliminaries

In this paper we study non-existence theorems for finite codimension linear isometries on certain algebras of holomorphic functions of several complex variables, especially on the ball algebras and the polydisk algebras.

1.1. **Linear isometries of finite codimensions on function algebras.** A linear operator T on a Banach space B is said to be a shift operator (cf. [2]) if (1) T is an isometry; (2) the codimension of $T(B)$ in B is 1; (3) $\cap_{n=1}^{\infty} T^n(B) = \{0\}$. A unilateral shift operator on a Hilbert space is a shift operator in the sense of Crownover [2]. On the other hand there is a Banach space on which no shift operators are admitted. A linear operator which satisfies the above conditions (1) and (2) is called a codimension 1 linear isometry. If T is an linear isometry on a Banach space B and the codimension of $T(B)$ in B is a positive integer l for an linear isometry T on a Banach space B, then T is said to be

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a codimension l linear isometry, or simply a finite codimension linear isometry. Araujo and Font [1] studied and gave a structure theorem for codimension 1 linear isometries on *function algebras*. Here a function algebra on a compact Hausdorff space X is a uniformly closed subalgebra of the algebra $C(X)$ of all complex valued continuous functions on X which separates the points of X and contains constant functions. Izuchi [5] gave a condition for Douglas algebras which admit codimension 1 linear isometries. Takayama and Wada [11] characterized codimension 1 linear isometries on the disk algebra. They gave a sufficient and necessary condition for a codimension 1 linear isometry to be a shift operator. For the case of algebras of holomorphic functions of several complex variables, one of the authors showed that there is no codimension 1 linear isometry on the ball algebra and the polydisk algebra [6].

Font [3] studied finite codimension linear isometries on function algebras. For every positive integer l, there exists a function algebra on which codimension k linear isometries are admitted for every $k \geq l+1$, but are not admitted for every $1 \leq k \leq l$. We will give such an example in section 2. Thus it is interesting to study linear isometries on function algebras not only in the case of codimension 1 but also in the case of a finite codimension.

2. Propositions

In this section we give notations, definitions and some propositions.

Let S be a subset of $\mathbb{C}^n$, $\bar{S}$ the closure, ∂S the topological boundary and $\mathrm{int}S$ the interior. Let $B(p,\varepsilon) = \{z \in \mathbb{C}^n : |z-p| < \varepsilon\}$, where $p \in \mathbb{C}^n$ and $\varepsilon > 0$. The space of all holomorphic functions on an open subset D of $\mathbb{C}^n$ is denoted by $\mathcal{O}(D)$. Let K be a compact subset of $\mathbb{C}^n$. Let $A(K) = C(K) \cap \mathcal{O}(\,\mathrm{int}K)$. Let $H(K)$ be the closure in $C(K)$ of the functions that are holomorphic in a neighborhood of K. Let Δ be the open unit disc in the complex plane. Then $A(\bar{\Delta})$ is called the disk algebra on the disk. Note that $A(\bar{\Delta}) = H(\bar{\Delta})$.

Let X be a compact Hausdorff space. Let E be a linear subspace of $C(X)$. A subset Y of X is called a boundary for E if the absolute value of each function in E assumes its maximum on Y. If there exists a unique minimal closed boundary for E, it is called the Shilov boundary for E and it is denoted by ∂E. Note that function algebras admit the Shilov boundaries.

Let A be a function algebra on X and K a non-empty closed subset of X. We say that K is a peak set if there is a function $f \in A$ such that $f(x) = 1$ for $x \in K$ and $|f(y)| < 1$ for $y \in X \setminus K$. We also say that K is a p-set if it is the intersection of peak sets. A point $x \in X$ is a p-point if the singleton $\{x\}$ is a p-set.

For a positive integer l, put

$$A_l = \{f \in A(\bar{\Delta}) : f^{(k)}(0) = 0 \text{ for every } 1 \leq k \leq l\},$$

where $f^{(k)}(0)$ is the k-th derivative of f at the origin 0.

Then A_l is a function algebra on $\bar{\Delta}$ such that the unit circle $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ is the Choquet boundary. In fact, for $w \in \Gamma$, put $f(z) = \frac{1+\bar{w}^{l+1}z^{l+1}}{2}$. Then $f \in A_l$. Since $f(w) = 1$ and $|f(z)| < 1$ for $z \in \bar{\Delta}$ with $z \neq w$, a representing measure on A_l is only the Dirac measure δ_w. Then $\Gamma \subset Ch(A_l)$, where $Ch(A_l)$

is the Choquet boundary for A_l. On the other hand $Ch(A_l) \subset Ch(A(\bar{\Delta})) = \Gamma$. Hence $Ch(A_l) = \Gamma$. In general for a function algebra A on X the set which consists of all the p-points coincides with the Choquet boundary for A. Note that the closure of the Choquet boundary is the Shilov boundary.

Proposition 2.1. *Let l be a positive integer. Then for every integer m with $m \geq l+1$ there exists a codimension m linear isometry on A_l. On the other hand, for every integer k with $1 \leq k \leq l$, there is no codimension k linear isometry on A_l.*

Proof. Let $m \geq l+1$. Then the operator T defined by $T(f) = z^m f$ for $f \in A_l$ is obviously a codimension m linear isometry on A_l.

On the other hand, suppose that T is a codimension k linear isometry on A_l for some positive integer k. We will show that $k \geq l+1$. Since the Choquet boundary for A_l is the unit circle Γ and Γ has no isolated point, we see by the similar way to the used one in the proof of Theorem 1.1 in [11] that there exist a continuous map τ from Γ onto itself and a function $u \in A(\bar{\Delta})$ with $|u| = 1$ on Γ such that

$$Tf = u(f \circ \tau)$$

on Γ for every $f \in A_l$. Since $u = T1|\Gamma$ is unimodular on Γ and $u \in A_l$, we see that u is a constant of absolute value 1 or $u = z^{l+1}g$, where g is a finite-Blaschke product or a constant of absolute value 1. We will show that τ is a Möbius transformation. For a function $v \in A(\bar{\Delta})|\Gamma$, we denote by $\tilde{v}$ the function in $A(\bar{\Delta})$ with $\tilde{v}|\Gamma = v$.

First we consider the case where u is a constant and will show that the case does not occur. For each positive integer j, $\tau^{l+j} \in A_l|\Gamma$ since $Tz^{l+j} = u\tau^{l+j}$ and u is a constant. Then we see that

$$(\widetilde{\tau^{l+1}})^{k-1}\widetilde{\tau^{l+k+1}} = (\widetilde{\tau^{l+2}})^k$$

holds for every positive integer k since $(\tau^{l+1})^{k-1}\tau^{l+k+1} = (\tau^{l+2})^k$ on Γ. Thus zeros of $\widetilde{\tau^{l+1}}$ are zeros of $\widetilde{\tau^{l+2}}$. Let a be a zero of $\widetilde{\tau^{l+1}}$ with the order n_1 and n_2 the order of a as a zero of $\widetilde{\tau^{l+2}}$. Then by the above equation we have that $(k-1)n_1 \leq kn_2$ holds for every k, so we see that $n_1 \leq n_2$. It follows that $\widetilde{\tau^{l+2}}/\widetilde{\tau^{l+1}} \in A(\bar{\Delta})$. Hence $\tau \in A(\bar{\Delta})|\Gamma$ since $\tau = \widetilde{\tau^{l+2}}/\widetilde{\tau^{l+1}}$ on Γ. Clearly $|\tau| = 1$ on Γ and $\tau(\Gamma) = \Gamma$, τ is a finite-Blaschke product. If the number of the factor of τ is greater than 1, say m_1, then $\tau^{-1}(z_0)$ consists of m_1 points for every $z_0 \in \Gamma$. It follows that the codimension of $\{u(f \circ \tau) : f \in A_l|\Gamma\}$ in $A_l|\Gamma$ is infinite, which is a contradiction. Hence we see that τ is a Möbius transformation. Since $\tilde{\tau}^{l+1} = \widetilde{\tau^{l+1}} \in A_l$, we have

$$0 = (\tilde{\tau}^{l+1})^{(1)}(0) = (l+1)\tilde{\tau}^l(0)\tilde{\tau}^{(1)}(0),$$

so $\tilde{\tau}(0) = 0$, that is, $\tau(z) = cz$ with a unimodular constant c. Thus we have that T is invertible, which is a contradiction since T is of finite codimension.

Next we consider the case where $u = z^{l+1}g$, g is a finite-Blaschke product or a constant. Since $\tilde{u}^{k-1}\widetilde{u\tau^{kl+k}} = (\widetilde{u\tau^{l+1}})^k$ holds for every positive integer k, we see that $\widetilde{u\tau^{l+1}}/\tilde{u} \in A(\bar{\Delta})$ in the similar way to the above. Thus we see that $\tau^{l+1} \in A(\bar{\Delta})|\Gamma$ and in the same way we see that $\tau^{l+j} \in A(\bar{\Delta})|\Gamma$ for every positive integer j. So we see that τ is a Möbius tansformation as in the same way as before. Then we have that

$$u(A_l|\Gamma) \circ \tau \subset u(A(\bar{\Delta})|\Gamma) \circ \tau = uA(\bar{\Delta})|\Gamma \subset z^{l+1}A(\bar{\Delta})|\Gamma \subset A_l|\Gamma.$$

Since $\dim A_l|\Gamma/z^{l+1}A(\bar{\Delta})|\Gamma = 1$ and $\dim u(A(\bar{\Delta})|\Gamma) \circ \tau/u(A_l|\Gamma) \circ \tau = l$, we see that $\dim A_l|\Gamma/u(A_l|\Gamma) \circ \tau \geq l+1$. We conclude that $\dim A_l/T(A_l) \geq l+1$ since A_l is isometrically isomorphic to $A_l|\Gamma$ and $(T(A_l))|\Gamma = u(A_l|\Gamma) \circ \tau$. □

Proposition 2.2. *Let D be a bounded domain in $\mathbb{C}^n$. Let $A = A(\bar{D})$. Let $T : A \to A$ be a codimension l linear isometry for a positive integer l. Then $(\partial A)_0 = \partial A$, where $(\partial A)_0$ is the closed boundary for $T(A)$ described in Theorem 1 in* [3].

Proof. Suppose that $(\partial A)_0 \neq \partial A$. By Proposition 1 in [3], $\partial A \setminus (\partial A)_0$ has at most l points. Clearly $(\partial A)_0$ is closed and $\partial A \setminus (\partial A)_0$ is open, each point of $\partial A \setminus (\partial A)_0$ is an isolated point of ∂A. Since the set of p-points is dense in ∂A, each point x of $\partial A \setminus (\partial A)_0$ is a p-point. Since $\bar{D}$ is metrizable, the singleton $\{x\}$ is a peak set, that is, there is a function $f \in A$ such that $f(x) = 1$, and $|f(y)| < 1$ for $y \in \partial A \setminus \{x\}$. Therefore $f^j \to \chi_{\{x\}}$ uniformly on ∂A as $j \to \infty$, where $\chi_{\{x\}}$ denotes the characteristic function of $\{x\}$. Now $\chi_{\{x\}} \in A$ and $\chi_{\{x\}}^2 = \chi_{\{x\}}$. Therefore the Gelfand transform $\widehat{\chi}_{\{x\}}$ attains 1 or 0 on the maximal ideal space M_A. Since $D \subset M_A$ and D is connected, $\widehat{\chi}_{\{x\}} = 1$ on D or $\widehat{\chi}_{\{x\}} = 0$ on D. In either case we arrive at a contradiction. □

Proposition 2.3. *Let K be a nonempty compact subset of $\mathbb{C}^n$ which satisfies $K = \cap_{j=1}^{\infty} D_j$ where D_j is a bounded and holomorphically convex open subset of $\mathbb{C}^n$ and $D_j \supset \overline{D_{j+1}}$. Then, for any $z_0 \in \partial K$, any $\epsilon > 0$, there exists an integer j_ϵ such that $S_j \setminus K_j \neq \emptyset$ for any $j > j_\epsilon$, where S_j is a connected component of $B(z_0, \epsilon) \cap D_j$ which contains z_0 and $K_j = \{z \in D_j : |f(z)| \leq ||f||_{\infty(K)}$ for any $f \in \mathcal{O}(D_j)\}$.*

Proof. Let $d_j = d(z_0, D_j^c)$, where $d(z_0, D_j^c) = \inf\{|z_0 - w| : w \in D_j^c\}$. Then there exists a point $w_j \in \partial D_j$ such that $|z_0 - w_j| = d_j$. In fact, there exists a sequence $\{a_k\} \subset D_j^c$ such that $\lim_{k\to\infty} |z_0 - a_k| = d_j$. Since $\{a_k\}$ is a bounded set, there exists a subsequence $\{a_{k_j}\} \subset \{a_k\}$ and a point a_0 such that $\lim_{j\to\infty} a_{k_j} = a_0$. Then $|z_0 - a_0| = d_j$. Since D_j^c is closed, $a_0 \in D_j^c$. Therefore $a_0 \in \partial D_j$. In fact, if $a_0 \notin \partial D_j$, then $a_0 \in$int(D_j^c). Then there exists an $\epsilon_0 > 0$ such that $B(a_0, \epsilon_0) \subset D_j^c$. On the other hand $|a_0 - z_0|\frac{\epsilon_0}{2d_j} = \frac{\epsilon_0}{2} < \epsilon_0$. Then $|a_0 - (a_0 - (a_0 - z_0)\frac{\epsilon_0}{2d_j})| < \epsilon_0$. Hence $a_0 - (a_0 - z_0)\frac{\epsilon_0}{2d_j} \in B(a_0, \epsilon_0) \subset D_j^c$. Then $|z_0 - (a_0 - (a_0 - z_0)\frac{\epsilon_0}{2d_j})| = |z_0 - a_0||1 - \frac{\epsilon_0}{2d_j}| = d_j - \frac{\epsilon_0}{2}$. This is a contradiction.

Clearly $D_j \supset D_{j+1}, D_j^c \subset D_{j+1}^c$. Then $d_j \geq d_{j+1}$. Therefore $\lim_{j\to\infty} d_j = 0$. In fact, since the sequence $\{d_j\}$ is positive and monotone decreasing, there exists a $d \geq 0, \lim_{j\to\infty} d_j = d$. Suppose $d > 0$. Then $B(z_0, \frac{d}{2}) \subset D_j$ for any j. Therefore $B(z_0, \frac{d}{2}) \subset \cap_{j=1}^{\infty} D_j = K$. Since $z_0 \in \partial K$, this is a contradiction. Hence $\lim_{j\to\infty} d_j = 0$

There exists an integer $j_\epsilon > 0$ such that $d_j < \epsilon$ for any $j > j_\epsilon$. Fix $j > j_\epsilon$. Put $\inf\{t > 0 : z_0 + t(w_j - z_0) \notin S_j\} = t_0$. Now $w_j \in \partial D_j$ and D_j is open. Therefore

$w_j \notin S_j$. Hence $t_0 \leq 1$. Since $B(z_0, \epsilon) \cap D_j$ is open, S_j is open. Hence $t_0 > 0$. Put $a_0 = z_0 + t_0(w_j - z_0)$. Suppose $a_0 \in S_j$. Since S_j is open, there exists $\delta > 0$ such that $z_0 + t(w_j - z_0) \in S_j$ for any t with $t_0 < t < t_0 + \delta$. This is a contradiction. Hence $a_0 \notin S_j$.

Suppose $a_0 \in \mathrm{int}(S_j^c)$. Then there exists a $\delta > 0$ such that $z_0 + t(w_j - z_0) \in \mathrm{int}(S_j^c) \subset S_j^c$ for any t with $t_0 - \delta < t < t_0$. Therefore $z_0 + t(w_j - z_0) \notin S_j$. This is a contradiction. Hence $a_0 \notin \mathrm{int}(S_j^c)$.

Then $a_0 \in \partial S_j$. Now $|a_0 - z_0| = |t_0(w_j - z_0)| \leq |w_j - z_0| = d_j < \epsilon$. Therefore $a_0 \in B(z_0, \epsilon)$. Hence $a_0 \in (\partial S_j) \cap B(z_0, \epsilon)$. Note that $a_0 \notin K_j$. In fact, if $a_0 \in K_j$, $a_0 \in D_j$. Then $a_0 \in B(z_0, \epsilon) \cap D_j$. Therefore there exists a $\delta > 0$ such that $B(a_0, \delta) \subset B(z_0, \epsilon) \cap D_j$. On the other hand, for any t with $0 < t < t_0$, $z_0 + t(w_j - z_0) \in S_j$. Then $B(a_0, \delta) \cup S_j$ is connected and open. And $B(a_0, \delta) \cup S_j \subset B(z_0, \epsilon) \cap D_j$. Since S_j is a connected component, $B(a_0, \delta) \subset S_j$. Since $a_0 \in \partial S_j$, this is a contradiction. Hence $a_0 \notin K_j$.

Since K_j is compact, there exists $\delta > 0$ such that $B(a_0, \delta) \cap K_j = \emptyset$. Now $a_0 \in \partial S_j$. Therefore there exists a point $b_0 \in B(a_0, \delta) \cap S_j$. Hence $b_0 \in S_j \setminus K_j$. □

3. Main result

In this section we show that there is no linear isometry of finite codimensions on $A(K)$ for certain compact subsets K of $\mathbb{C}^n$, especially in the case where K is a closed ball or a closed polydisk. In particular, we show that for such compact sets K a linear isometry with the codimension at most finite on $A(K)$ is surjective and represented by constant times of the composition operator induced by a holomorphic automorphism of $\mathrm{int}K$.

Theorem 3.1. *Let $n > 1$ be a positive integer and K a non-empty compact subset of $\mathbb{C}^n$ which satisfies the following five conditions.*

(i) $\mathrm{int}K$ *is connected and* $\overline{\mathrm{int}K} = K$.

(ii) $K = \cap_{j=1}^{\infty} D_j$ *where* D_j *is a bounded and holomorphically convex open subset of* $\mathbb{C}^n$ *and* $D_j \supset \overline{D_{j+1}}$.

(iii) *For every point p in K there exists an $\varepsilon_p > 0$ such that $B(p, \varepsilon) \cap \mathrm{int}K$ is connected for every ε with $0 < \varepsilon < \varepsilon_p$.*

(iv) $A(K) = H(K)$.

(v) *If a function u is in $A(K)$ and $|u| = 1$ on $\partial A(K)$, then u is constant or u has a zero in* $\mathrm{int}K$.

Suppose that T is a linear isometry on $A(K)$ such that the codimension of $T(A(K))$ in $A(K)$ is at most finite. Then T is surjective. In particular, there exist a complex number a of absolute value 1 and a homeomorphism φ of K onto itself which is a biholomorphic map of $\mathrm{int}K$ *onto itself such that $Tf = a(f \circ \varphi)$ for every $f \in A(K)$.*

Proof. We will simply write A instead of $A(K)$.

First we consider the case where T is surjective. In this case we see by a theorem of de Leeuw, Rudin and Wermer [7] that T has the form

$$Tf = aT_1 f,$$

where $a \in A$, $a^{-1} \in A$, $|a(z)| = 1$ for all $x \in K$, and T_1 is an automorphism of A. Since $\mathrm{int}K \neq \emptyset$, a is a constant function. By (ii), (iv) and Theorem 2.12 in [9], the maximal ideal space of A is K. Since T_1 is an automorphism of A, by a routine argument, there is a homeomorphism φ from K onto itself such that

$$T_1 f = f \circ \varphi$$

holds for every $f \in A$. It follows that φ is a biholomorphic map from $\mathrm{int}K$ onto itself. We see that $Tf = a(f \circ \varphi)$ holds for every $f \in A$.

Next we will show that there is no codimension l linear isometry on A for any positive integer l. Suppose that for some positive integer l there exists a codimension l linear isometry, say $T : A \to A$. We will show a contradiction. By Theorem $\mathcal{A}$ in [1] and Proposition 2.2, there exist a continuous map h from ∂A onto ∂A and a continuous map $a : \partial A \to \mathbb{C}$, such that $|a(x)| = 1$ for all $x \in \partial A$, and

$$(Tf)(x) = a(x)f(h(x))$$

for all $x \in \partial A$ and all $f \in A$. So by (v), $T1$ is constant or $T1$ has a zero in $\mathrm{int}K$.

Claim 1. $T1$ is constant.

To prove this, suppose that $T1$ has a zero in $\mathrm{int}K$. Let $F = \{x \in \mathrm{intK}\colon (T1)(x) = 0\}$. We show that

$$T(A) \subset \{f \in A : f = 0 \ \text{ on } F\}. \tag{3.1}$$

Fix $f \in A$. Set

$$\tilde{f} = \frac{Tf}{T1} \ \text{ on } \ \mathrm{int}K \setminus F.$$

We note that for any $\alpha \in \mathbb{C}$

$$\widetilde{\alpha f} = \frac{T(\alpha f)}{T1} = \frac{\alpha T(f)}{T1} = \alpha \tilde{f}. \tag{3.2}$$

Put $||f||_{\infty(K)} = \sup_{z\in K} |f(z)|$, the supremum norm of f. Then we see that the inequality $|\tilde{f}(x)| \leq ||f||_{\infty(K)}$ holds for every $x \in \mathrm{int}K \setminus F$. To prove this, we may asssume that $||f||_{\infty(K)} = 1$ by (3.2). Suppose that there exists a point x_0 in $\mathrm{int}K \setminus F$ such that $|\tilde{f}(x_0)| > 1$. Let k be a positive integer. Then $T(f^k) = af^k \circ h$ on ∂A. and $(Tf)^k = (af \circ h)^k = a^k(f \circ h)^k = a^k f^k \circ h$ on ∂A. Therefore $(Tf)^k = a^{k-1}T(f^k)$ on ∂A. Then $(Tf)^k = (T1)^{k-1}T(f^k)$ on ∂A. Hence we have

$$\frac{(Tf)^k}{(T1)^k} = \frac{T(f^k)}{T1}$$

on $\mathrm{int}K \setminus F$. Therefore $(\tilde{f})^k = \widetilde{f^k}$ on $\mathrm{int}K \setminus F$, and so we have $T1(\tilde{f})^k = (T1)\widetilde{f^k} = T(f^k)$ on $\mathrm{int}K \setminus F$. Recall that T is a linear isometry. Hence we have $||T(f^k)||_{\infty(K)} = ||f^k||_{\infty(K)} = ||f||^k_{\infty(K)} = 1$. Since $T1(\tilde{f})^k(x_0) = T1(x_0)(\tilde{f})^k(x_0)$, where $T1(x_0) \neq 0$ and $|\tilde{f}(x_0)| > 1$, we see that

$$|T1(\tilde{f})^k(x_0)| \to \infty$$

as $k \to \infty$. Hence $|T(f^k)(x_0)| \to \infty$ as $k \to \infty$. This is a contradiction. It follows that $\tilde{f}$ is bounded and holomorphic on $\mathrm{int}K \setminus F$. There is $g \in \mathcal{O}(\mathrm{int}K)$ such that $g = \tilde{f}$ on $\mathrm{int}K \setminus F$ by Theorem I.3.4 in [8] since F is a thin set. Clearly Tf and

$(T1)g$ are holomorphic on $\mathrm{int}K$, $Tf = (T1)\tilde{f} = (T1)g$ on $\mathrm{int}K \setminus F$ and $\mathrm{int}K \setminus F$ is dense in $\mathrm{int}K$, it holds that $Tf = (T1)g$ on $\mathrm{int}K$.

If $x \in F$, then $Tf(x) = (T1)(x)g(x) = 0$. Hence (3.1) holds. Since $n > 1$, F is an infinite set. Then $\dim A/TA \geq \dim A/\{f \in A : f = 0 \text{ on } F\} = \infty$. This is a contradiction. We conclude that $T1$ has no zero in $\mathrm{int}K$. Thus, by (v), we see that $T1$ is constant, that is, a is a constant of absolute value 1. So we see that Claim 1 holds.

Define the operator $\tilde{T} : A \to A$ by

$$\tilde{T}f = \bar{a}Tf$$

for $f \in A$. Then $\tilde{T}f = f \circ h$ on ∂A. By Remark 3.2 in [1], $\tilde{T}$ is a homomorphism. Let π_j be the jth coordinate function; $\pi_j(z) = z_j$, for $z = (z_1, \dots, z_n)$, $1 \leq j \leq n$. Since K is closed and bounded, $\pi_j \in A$. Put $\varphi_j = \tilde{T}\pi_j$, $\varphi = (\varphi_1, \dots, \varphi_n)$. Since $\varphi_j \in A$, $\varphi : K \to \mathbb{C}^n$ is continuous on K and holomorphic on $\mathrm{int}K$. We claim that $\varphi(K) \subset K$. To prove this, let $z_0 \in K$. Define the map $\tilde{T}_{z_0} : A \to \mathbb{C}$ by

$$\tilde{T}_{z_0}f = (\tilde{T}f)(z_0), \quad (f \in A).$$

Since $\tilde{T}1 = \bar{a}T1 = 1$, $\tilde{T}_{z_0}$ is a non-zero complex homomorphism. Since $M_A = M_{H(K)} = K$, there is a point $w_0 \in K$ such that $(\tilde{T}f)(z_0) = \tilde{T}_{z_0}f = f(w_0)$ for every $f \in A$. In particular, $\pi_j(w_0) = (\tilde{T}\pi_j)(z_0) = \varphi_j(z_0)$. Then $\varphi(z_0) = (\varphi_1(z_0), \dots, \varphi_n(z_0)) = (\pi_1(w_0), \dots, \pi_n(w_0)) = w_0 \in K$. Hence we see that $\varphi(K) \subset K$.

Claim 2. $\tilde{T}f = f \circ \varphi$ on K for every $f \in A$.

To prove this, fix $f \in A$ and $z_0 \in K$. By (iv), there are a sequence $\{\Omega_m\}$ of neighborhoods of K and a sequence $\{f_m\}$ of functions such that each f_m is in $\mathcal{O}(\Omega_m)$ and $\{f_m\}$ converges to f uniformly on K. Then, for each integer m, there is an integer N_m such that $\cap_{j=1}^{N_m} \bar{D}_j \subset \Omega_m$ by the condition (ii). Therefore $f_m \in \mathcal{O}(D_{N_m})$. By Theorem VII.4.1 in [8], there are $Q_1, \dots, Q_n \in \mathcal{O}(D_{N_m} \times D_{N_m})$ such that

$$f_m(z) - f_m(w) = \sum_{j=1}^{n} (z_j - w_j) Q_j(z, w)$$

for all $z, w \in D_{N_m}$. Then

$$f_m(z) - f_m(\varphi(z_0)) = \sum_{j=1}^{n} (\pi_j(z) - \varphi_j(z_0)) Q_j(z, \varphi(z_0))$$

holds for all $z \in D_{N_m}$. Since $\tilde{T}$ is a homomorphism, we see that

$$\begin{aligned} (\tilde{T}(f_m))(z_0) - f_m(\varphi(z_0)) &= \sum_{j=1}^{n} (\varphi_j(z_0) - \varphi_j(z_0))(\tilde{T}(Q_j(\cdot, \varphi(z_0))(z_0)) \\ &= 0. \end{aligned}$$

Hence we have $(\tilde{T}(f_m))(z_0) = f_m(\varphi(z_0))$. Recall that $\tilde{T}$ is an isometry. Hence we see that $||\tilde{T}(f_m) - \tilde{T}(f)||_{\infty(K)} \to 0$ as $m \to \infty$. Then $(\tilde{T}(f))(z_0) = f(\varphi(z_0))$. Since $z_0 \in K$ is arbitrary, we conclude that $\tilde{T}(f) = f \circ \varphi$ holds on K. So we see that Claim 2 holds.

Claim 3. φ is locally one-to-one in K, that is, each point $x \in K$ has a neighborhood on which φ is one-to-one.

To prove this, suppose that φ is not locally one-to-one in K. Then there are a point $x_0 \in K$, and two sequences $\{x_j\}$ and $\{y_j\}$ in K such that $\lim_{j\to\infty} x_j = x_0$, $\lim_{j\to\infty} y_j = x_0$, $\varphi(x_j) = \varphi(y_j)$, $|x_j - x_0| > |x_{j+1} - x_0| > 0$ and $|y_j - x_0| > |y_{j+1} - x_0| > 0$ for all j. We can find $l+1$ functions $\{g_1, g_2, \ldots, g_{l+1}\} \subset A$ such that

$$g_j(x_i) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

and $g_j(y_i) = 0$ for every i and j with $1 \leq i \leq l+1$, $1 \leq j \leq l+1$. Suppose that $g = \alpha_1 g_1 + \ldots + \alpha_{l+1} g_{l+1}$ is in $\tilde{T}A$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{l+1} \in \mathbb{C}$, that is, $g = h \circ \varphi$ for some $h \in A$. Then $g(x_j) = \alpha_j$ and $g(y_j) = 0$ for each $1 \leq j \leq l+1$. Since $\varphi(x_j) = \varphi(y_j)$ and $g = h \circ \varphi$, we have $g(x_j) = g(y_j)$. Hence $\alpha_j = 0$ for every $1 \leq j \leq l+1$. Thus $\{g_1 + \tilde{T}A, \ldots, g_{l+1} + \tilde{T}A\}$ are linearly independent. Therefore the codimension of $\tilde{T}A$ in A is greater than $l+1$. This is a contradiction. Hence we have shown that for every point $x \in K$ there is an open neighborhood G_x of x in K such that φ is one-to-one in G_x. So we see that Claim 3 holds.

If $x \in \mathrm{int}K$, we may assume that $G_x \subset \mathrm{int}K$. We see by Theorem I.2.14 in [8] that φ is biholomorphic from G_x onto $\varphi(G_x)$. Since $\varphi(G_x)$ is open in $\mathbb{C}^n$ and $\varphi(G_x) \subset \mathrm{int}K$, we see that $\varphi(\mathrm{int}\ K) \subset \mathrm{int}K$. Let $k \in \mathbb{N}$ and $\pi^k = (\pi_1^k, \ldots, \pi_n^k)$. Put $F = (F_1, \ldots, F_n) = \sum_{k=1}^N a_k \pi^k$. Then there is an open set U in $\mathrm{int}K$ such that F is univalent on U. In fact, put $J = [\frac{\partial F_i}{\partial z_j}]$. Since $J = \left[\sum_{k=1}^N a_k \frac{\partial \pi_i^k}{\partial z_j}\right]$,

$$\det J = \sum_{k=1}^N a_k k \pi_1^{k-1} \sum_{k=1}^N a_k k \pi_2^{k-1} \ldots \sum_{k=1}^N a_k k \pi_n^{k-1}.$$

For j, $\sum_{k=1}^N a_k k \pi_j^{k-1} \not\equiv 0$, since $\sum_{k=1}^N a_k \pi_j^k$ is not a constant. Therefore $\det J \not\equiv 0$. By Theorem I.1.19 in [8], $\{z = (z_1, \ldots, z_n) \in D : (\det J)(z) = 0\}$ is a closed set and does not contain an open subset of D. Then there exists a point $a \in D$ such that $(\det J)(a) \neq 0$. By Theorem I.2.5 in [8], there exists an open set U in $\mathrm{int}K$ such that F is univalent on U.

Claim 4. φ is univalent on $\mathrm{int}K$.

To pove this, suppose that φ is not univalent on $\mathrm{int}K$. Then there are two different points p, $p' \in \mathrm{int}K$ such that $\varphi(p) = \varphi(p')$. Put $b = \varphi(p)$. Recall that φ is locally univalent. Hence there are open neighborhoods U_p and $U_{p'}$ of p and p' respectively with $U_p \cap U_{p'} = \emptyset$ such that both $\varphi|_{U_p}$ and $\varphi|_{U_p'}$ are univalent. By Theorem I.2.14 in [8], $\varphi(U_p)$ is open. Since $\varphi(p) \in \varphi(U_p)$, $\varphi(U_p)$ is an open neighborhood of b. Since $\varphi(p) = \varphi(p')$, $\varphi(U_{p'})$ is also an open neighborhood of b. So there is a sequence $\{w_j\}$ of different points in $\varphi(U_p) \cap \varphi(U_{p'})$ with $w_j \to b$ as $j \to \infty$, so there are points $z_j \in U_p$ and $z_j' \in U_{p'}$ such that $\varphi(z_j) = \varphi(z_j') = w_j$. Then we can find $l+1$ functions $\{g_1, g_2, \ldots, g_{l+1}\} \subset A$ such that

$$g_j(z_i) = \begin{cases} 1, & j = i \\ 0, & j \neq i, \end{cases}$$

$$g_j(z_i') = 0 \ \text{ for } 1 \le j \le l+1, \ 1 \le i \le l+1.$$

In the same way as before, this implies that $\{g_1 + \tilde{T}A, \ldots, g_{l+1} + \tilde{T}A\}$ is linearly independent, which is a contradiction since the codimension of $\tilde{T}A$ in A is l. So we see that Claim 4 holds.

Let $A^n = \{(f_1, \ldots, f_n) : f_k \in A\}$ and $A^n \circ \varphi = \{(f_1 \circ \varphi, \ldots, f_n \circ \varphi) : f_k \in A\}$. Then $\dim A^n/A^n \circ \varphi = ln$. In fact, there exist $g_{j,1}, \ldots, g_{j,n} \in A \backslash A \circ \varphi$ $(1 \le j \le l)$ such that

$$A^n = A^n \circ \varphi + \mathbb{C}(g_{1,1}, g_{1,2}, \ldots, g_{1,n}) + \mathbb{C}(g_{2,1}, g_{2,2}, \ldots, g_{2,n}) + \cdots + \mathbb{C}(g_{l,1}, g_{l,2}, \ldots, g_{l,n}).$$

Suppose

$$c_{1,1}g_{1,1}e_1 + c_{1,2}g_{1,2}e_2 + \cdots + c_{1,n}g_{1,n}e_n + c_{2,1}g_{2,1}e_1 + \cdots + c_{2,n}g_{2,n}e_n + \cdots + c_{l,1}g_{l,1}e_1 + \cdots + c_{l,n}g_{l,n}e_n = 0,$$

where $e_1, \ldots, e_n$ are the standard orthonormal basis elements in $\mathbb{C}^n$ and $c_{i,j}$ $(1 \le i \le l, 1 \le j \le n)$ is constant. Since $\dim A/(A \circ \varphi) = l$, we have $c_{1,1} = c_{1,2} = \cdots = c_{l,n} = 0$. Hence $\dim A^n/A^n \circ \varphi = ln$.

Put $l' = ln$. Since $\{\pi^1 + A^n \circ \varphi, \ldots, \pi^{l'+1} + A^n \circ \varphi\}$ are linearly dependent, there exist $l'+1$ constants (not all zero) $\alpha_1, \ldots, \alpha_{l'+1}$ such that $\sum_{j=1}^{l'+1} \alpha_j \pi^j \in A^n \circ \varphi$. Then there exists $u = (u_1, \ldots, u_n) \in A^n$ such that $\sum_{j=1}^{l'+1} \alpha_j \pi^j = u \circ \varphi$. Put $f = \sum_{j=1}^{l'+1} \alpha_j \pi^j$. Put $D = \mathrm{int} K$ and

$$M = \left\{ z \in D : \left(\det \left[\frac{\partial u_i}{\partial z_j} \right] \right)(z) = 0 \right\}.$$

By Theorem I.3.8 in [8], $M = D$ or M is a thin subset of D. In the same way as before, there is an open subset U of D such that f is univalent on U. Since φ is univalent on D, $f \circ \varphi^{-1} = u$ is univalent on $\varphi(U)$. By Theorem I.2.14 in [8], $\det \left[\frac{\partial u_i}{\partial z_j} \right] \neq 0$ on $\varphi(U)$. Then $M \neq D$. Hence M is a thin subset of D. For a subset X of D we denote by ∂X the topological boundary of X in D. We consider two cases: (a) $\partial \varphi(D) \subset M$; (b) $\partial \varphi(D) \not\subset M$. We will show that there is a thin subset E of D such that $\varphi(D) = D \setminus E$ in the case (a). We will also show that (b) never happens.

First we consider the case (a). Recall that M is a thin subset of D. Hence $\partial \varphi(D)$ is a thin subset of D. By Corollary I.3.6 in [8], $D \setminus \partial \varphi(D)$ is connected. Since $\varphi(D)$ is an open subset of D, it holds that

$$D \setminus \partial \varphi(D) = \varphi(D) \cup \{(D \setminus \varphi(D)) \setminus \partial \varphi(D)\}.$$

Clearly the closure $\overline{\varphi(D)}$ of $\varphi(D)$ in D equals to $\varphi(D) \cup \partial \varphi(D)$, we see that $D \setminus \partial \varphi(D) = \varphi(D) \cup (D \setminus \overline{\varphi(D)})$. Note that $\varphi(D)$ and $D \setminus \overline{\varphi(D)}$ are disjoint open subsets of D. Since $D \setminus \partial \varphi(D)$ is connected and $\varphi(D) \neq \emptyset$, we see that $D \setminus \overline{\varphi(D)} = \emptyset$. Therefore $D \setminus \partial \varphi(D) = \varphi(D)$. Let $E = \partial \varphi(D)$. Recall that $\partial \varphi(D) \subset M$. Hence E is a thin subset of D.

Claim 5. Case (b) never happens.

To prove this, we consider Case (b). Since $\partial\varphi(D) \not\subset M$, there is a point $w_0 \in \partial\varphi(D) \cap M^c$. Since $\det\left[\frac{\partial u_i}{\partial z_j}(w_0)\right] \neq 0$, by Theorem I.2.5 in [8], there is an open neighborhood U_{w_0} of w_0 in D such that $u : U_{w_0} \to u(U_{w_0})$ is biholomorphic. Recall that $w_0 \in \partial\varphi(D)$. Hence there is a sequence $\{w_j\}$ of different points in $\varphi(D) \cap U_{w_0}$ such that $w_j \to w_0$ as $j \to \infty$. Put $z_j = \varphi^{-1}(w_j)$ for every positive integer j. By passing to a subsequence, we may suppose that there is a point $z_0 \in K$ such that $z_j \to z_0$ as $j \to \infty$. Since $\varphi(z_j) = w_j$, we have $\varphi(z_0) = w_0$. If $z_0 \notin \partial K$, then $w_0 = \varphi(z_0) \in \varphi(D)$. This contradicts that $w_0 \in \partial\varphi(D)$. Hence $z_0 \in \partial K$.

Since $f = u \circ \varphi$ on K, $f(z_0) = u \circ \varphi(z_0) = u(w_0)$. Put $V_{z_0} = f^{-1}(u(U_{w_0}))$. Since u is univalent on U_{w_0}, then $u(U_{w_0})$ is open, so V_{z_0} is an open neighborhood of z_0 in $\mathbb{C}^n$. By the hypothesis (iii), there exists an $\varepsilon > 0$ such that $\overline{V'_{z_0}} \subset V_{z_0}$ and $V'_{z_0} \cap D$ is connected, where we denote $V'_{z_0} = \{z \in \mathbb{C}^n : |z - z_0| < \varepsilon\}$. Define the function $\varphi_0 : V'_{z_0} \to D$ by $\varphi_0(z) = (u|U_{w_0})^{-1} \circ f(z)$ for $z \in V'_{z_0}$. Clearly $f(V'_{z_0}) \subset f(V_{z_0}) \subset u(U_{w_0})$ and u is biholomorphic on U_{w_0}, φ_0 is well-defined and holomorphic on V'_{z_0} into U_{w_0}. Now $D \ni z_j \to z_0$ as $j \to \infty$ and V'_{z_0} is an open neighborhood of z_0. Therefore there is an integer j_0 such that $z_{j_0} \in V'_{z_0} \cap D$. Recall that $\varphi(z_{j_0}) = w_{j_0} \in U_{w_0}$ and φ is a biholomorphic map from D onto $\varphi(D)$. Hence there is an open neighborhood $V_{z_{j_0}}$ of z_{j_0} in D such that $V_{z_{j_0}} \subset V'_{z_0}$ and $\varphi(V_{z_{j_0}}) \subset U_{w_0}$. Let $z \in V_{z_{j_0}}$. Since $V_{z_{j_0}} \subset D$, $f(z) = u \circ \varphi(z)$. On the other hand, since $\varphi_0(z) = (u|U_{w_0})^{-1} \circ f(z)$, we have $f(z) = u \circ \varphi_0(z)$. Therefore $u(\varphi(z)) = u(\varphi_0(z))$. Clearly $\varphi_0(V'_{z_0}) \subset U_{w_0}$ and $V_{z_{j_0}} \subset V'_{z_0}$, we see that $\varphi_0(z) \in U_{w_0}$. Since $\varphi(V_{z_{j_0}}) \subset U_{w_0}$, we see that $\varphi(z) \in U_{w_0}$. Recall that u is univalent on U_{w_0}. Hence we have $\varphi(z) = \varphi_0(z)$ for $z \in V_{z_{j_0}}$. Since $V_{z_{j_0}}$ is an open subset of $V'_{z_0} \cap D$, we see that $\varphi = \varphi_0$ on $V'_{z_0} \cap D$ by Theorem I.1.19. in [8], so $\varphi = \varphi_0$ on $V'_{z_0} \cap K$. Define the function $\tilde{\varphi} : K \cup V'_{z_0} \to K$ by

$$\tilde{\varphi}(z) = \begin{cases} \varphi(z), & z \in K \\ \varphi_0(z), & z \in \overline{V'_{z_0}}. \end{cases}$$

Recall that $\varphi_0(V'_{z_0}) \subset U_{w_0} \subset D$. Hence $\tilde{\varphi}$ is a holomorphic map from $D \cup V'_{z_0}$ into D. We will show a contradiction. Put

$$B_0 = \{g \in A : \exists \tilde{g} : \text{holomorphic on } V'_{z_0} \cup D \text{ such that } \tilde{g}|_D = g \text{ on } D\}.$$

Then $\dim A/B_0 = \infty$. In fact, fix $\epsilon > 0$ such that $B(z_0, \epsilon) \subset V'_{z_0}$. For any j, we may assume that $d_j < \epsilon$. By Proposition 2.3, $S_1 \setminus K_1 \neq \emptyset$. Fix $z_1 \in S_1 \setminus K_1$. Then there exists a $f_1 \in \mathcal{O}(D_1)$ such that $|f_1(z_1)| > ||f_1||_{\infty(K)}$. By Proposition 2.3, $S_2 \setminus K_2 \neq \emptyset$. Since $S_2 \setminus K_2$ is open and $f_1^{-1}(f_1(z_1))$ is a thin set, $S_2 \setminus K_2 \setminus f_1^{-1}(f_1(z_1)) \neq \emptyset$. Fix $z_2 \in S_2 \setminus K_2 \setminus f_1^{-1}(f_1(z_1))$. Then there exists a $f_2 \in \mathcal{O}(D_2)$ such that $|f_2(z_2)| > ||f_2||_{\infty(K)}$. By induction, fix

$$z_k \in S_k \setminus K_k \setminus f_1^{-1}(f_1(z_1)) \cup \ldots \cup f_{k-1}^{-1}(f_{k-1}(z_{k-1})).$$

Then there exists a $f_k \in \mathcal{O}(D_k)$ such that $|f_k(z_k)| > ||f_k||_{\infty(K)}$. Put $h_k = \frac{1}{f_k - f_k(z_k)}$ on $D_k \setminus f_k^{-1}(f_k(z_k))$. By the hypothesis (ii), $M_{A(K)} = K$. Since $K_k \cap f_k^{-1}(f_k(z_k)) = \emptyset$, $h_k \in A(K)$. Then $\{h_1 + B_0, \ldots, h_k + B_0\}$ are linearly independent. In fact, suppose $\{h_1 + B_0, \ldots, h_k + B_0\}$ are not linearly independent. Then there exist

$k_1, \ldots, k_j$ with $k_1 < \ldots < k_j$ and $\alpha_1, \ldots, \alpha_j \in \mathbb{C}$ with $\alpha_1 \cdots \alpha_j \neq 0$ such that $\alpha_1 h_{k_1} + \cdots + \alpha_j h_{k_j} \in B_0$. Therefore there exists a holomorphic function H on $V'_{z_0} \cup D$ such that $H = \alpha_1 h_{k_1} + \cdots + \alpha_j h_{k_j}$ on D. On the other hand, S_{k_j} is connected component of $B(z_0, \epsilon) \cap D_{k_j}$ which contains z_0. Furthermore, $S_{k_j} \subset B(z_0, \epsilon)$ and $S_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j}))$ is a nonempty connected open set. In fact, $f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j}))$ is a thin set. Note that S_{k_j} is the connected open set. By Corollary I.3.6 in [8], $S_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \cdots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j}))$ is connected. Since

$$K \cap f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \cdots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j})) = \emptyset$$

and

$$S_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \cdots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j})) \ni z_0,$$

H and $\alpha_1 h_{k_1} + \cdots + \alpha_j h_{k_j}$ is holomorphic on $S_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j}))$. Furthermore

$$z_0 \in S_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j}))$$

and

$$D \cap (S_{k_j}) \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j})) \neq \emptyset.$$

Then

$$H = \alpha_1 h_{k_1} + \cdots + \alpha_j h_{k_j}$$

on

$$S_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j}^{-1}(f_{k_j}(z_{k_j})).$$

Since

$$z_{k_j} \in S_{k_j} \setminus K_{k_j} \setminus f_{k_1}^{-1}(f_{k_1}(z_1)) \cup \ldots \cup f_{k_j-1}^{-1}(f_{k_j}(z_{k_j-1}))$$

and $k_{j-1} \leq k_j - 1$,

$$\lim_{z \to z_{k_j}} \alpha_1 h_{k_1}(z) + \cdots + \alpha_{j-1} h_{k_{j-1}}(z) = \alpha_1 h_{k_1}(z_{k_j}) + \cdots + \alpha_{j-1} h_{k_{j-1}}(z_{k_j}).$$

And $|\alpha_j h_{k_j}(z)| \to \infty$ as $z \to z_{k_j}$. Since $\lim_{z \to z_{k_j}} H(z) = H(z_{k_j})$, this is a contradiction.

We also see that $A \circ \varphi \subset B_0$. Suppose that $h \in A \circ \varphi$. Then there is $g \in A$ with $g \circ \varphi = h$. Put $\tilde{g} = g \circ \tilde{\varphi}$. Then $\tilde{g}$ is well-defined and holomorphic on $V'_{z_0} \cup D$. Since $\tilde{\varphi}|_D = \varphi$, $\tilde{g}|_D = g \circ \varphi = h$. It follows that $h \in B_0$. We have proved that $A \circ \varphi \subset B_0$. Thus we see that $\dim(A/B_0) < \infty$ since $\dim A/(A \circ \varphi) < \infty$. This is a contradiction. We conclude that the case (b) never occurs. So we see that Claim 5 holds. It follows that only the case (a) occurs.

We consider the case (a) from now on. As we already showed, $\varphi(D) = D \setminus E$ for a thin set E. Therefore φ is a continuous map from K onto K.

Claim 6. φ is one-to-one on K.

To prove this, suppose that φ is not one-to-one. Then there exists a point $b_1 \in K$ such that $\varphi^{-1}(b_1)$ contains at least two points. We will show that $\varphi^{-1}(b_1)$ contains at most $l+1$ points. To prove this, suppose that $\varphi^{-1}(b_1)$ contains at least $l+2$ points. Take $l+2$ points $\{a_1, \ldots, a_{l+2}\}$ in $\varphi^{-1}(b_1)$. We can find $l+1$ functions $\{g_1, g_2, \ldots, g_{l+1}\} \subset A$ such that

$$g_j(a_i) = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

for $1 \leq j \leq l+1, 1 \leq i \leq l+2$. We will show that $\{g_1 + \tilde{T}A, \ldots, g_{l+1} + \tilde{T}A\}$ is linearly independent. Let $\alpha_1 g_1 + \ldots + \alpha_{l+1} g_{l+1} \in \tilde{T}A$ with $\alpha_1,\ \alpha_2,\ \ldots, \alpha_{l+1} \in \mathbb{C}$. Then there is a function $g \in A$ such that $g \circ \varphi = \alpha_1 g_1 + \ldots + \alpha_{l+1} g_{l+1}$. Then $g \circ \varphi(a_j) = g(b_1)$ for $1 \leq j \leq l+2$. Since $g_j(a_{l+2}) = 0$ for $1 \leq j \leq l+2$, we have that $g \circ \varphi(a_{l+2}) = 0$, so $\alpha_j = \alpha_1 g_1(a_j) + \ldots + \alpha_{l+1} g_{l+1}(a_j) = 0$ for $1 \leq j \leq l+1$. Then $\{g_1 + \tilde{T}A,\ \ldots,\ g_{l+1} + \tilde{T}A\}$ are linearly independent. Therefore the codimension of $\tilde{T}A$ in A is greater than $l+1$. This is a contradiction. Hence we have $\varphi^{-1}(b_1) = \{a_1, \ldots, a_j\}$ for some j with $2 \leq j \leq l+1$. In a similar way to the above we see that the set $\{x \in K : \varphi^{-1}(x)$ contains at least two points$\}$ consists of at most l points. Therefore there exists an $\varepsilon > 0$ such that $B_K(a_k,\ \varepsilon)'$s are disjoint, where $B_K(a_k,\ \varepsilon) = \{x \in K : |x - a_k| < \varepsilon\}$, and $\varphi^{-1}(\varphi(x)) = \{x\}$ for every $x \in B_K(a_k,\ \varepsilon) \setminus \{a_k\}$ and $1 \leq k \leq j$. We can easily see that

$$K \setminus \varphi(K \setminus (B_K(a_k,\ \varepsilon) \setminus \{a_k\})) = \varphi(B_K(a_k,\ \varepsilon) \setminus \{a_k\}). \tag{3.3}$$

It follows that $\varphi((B_K(a_k,\ \varepsilon) \setminus \{a_k\}))$ is open in K. By the way of the choice of the number $\epsilon > 0$, we can also easily show that for a sufficiently small number $\delta > 0$,

$$\varphi^{-1}(B_K(b_1,\ \delta)) \subset \bigcup_{k=1}^{j} B_K(a_k,\ \varepsilon),$$

where $B_K(b_1, \delta) = \{x \in K : |x - b_1| < \delta\}$. By the hypothesis (iii), we see that there exists a connected open neighborhood V of b_1 in K such that $V \subset B_K(b_1, \delta)$. Then $V \setminus \{b_1\}$ is also connected. In fact, for any $p \in K$, any connected open neighborhood G in K of p, it is enough to prove that $G \setminus \{p\}$ is connected. Suppose this is false. Then there exists a point $p \in K$ and a connected open neighborhood G_p in K of p such that $G_p \setminus \{p\}$ is not connected. Note that $p \in \partial K$. Then there are two nonempty open sets V_1, V_2 such that $V_1 \cup V_2 = G_p \setminus \{p\}, V_1 \cap V_2 = \emptyset$. Then there exists $0 < \epsilon < \epsilon_p$ such that $B(p, \epsilon) \cap K \subset G_p$. Therefore $B(p, \epsilon) \cap V_1 \neq \emptyset$ and $B(p, \epsilon) \cap V_2 \neq \emptyset$. In fact, we suppose that $B(p, \epsilon) \cap V_1 = \emptyset$. Let $\tilde{V_2} = (B(p, \epsilon) \cap K) \cup V_2$. Then $\tilde{V_2}$ is an open set in K and $\tilde{V_2} \cup V_1 = G_p$. Therefore $\tilde{V_2} \cap V_1 = ((B(p, \epsilon) \cap K) \cup V_2) \cap V_1 = (B(p, \epsilon) \cap K \cap V_1) \cup (V_2 \cap V_1) = \emptyset$. Since $V_1 \neq \emptyset$ and $\tilde{V_2} \neq \emptyset$, we have G_p is connected, which is a contradiction. Also $B(p, \epsilon) \cap V_2 \neq \emptyset$. Then $B(p, \epsilon) \cap \mathrm{int}K$ is not connected. In fact, $(B(p, \epsilon) \cap \mathrm{int}K) \cap V_1 \neq \emptyset$, $(B(p, \epsilon) \cap \mathrm{int}K) \cap V_2 \neq \emptyset$. Then

$$\begin{aligned} &((B(p, \epsilon) \cap \mathrm{int}K) \cap V_1) \cap ((B(p, \epsilon) \cap \mathrm{int}K) \cap V_2) \\ &\qquad = ((B(p, \epsilon) \cap \mathrm{int}K) \cap (V_1 \cap V_2)) \subset V_1 \cap V_2 = \emptyset \end{aligned}$$

and

$$B(p, \epsilon) \cap \mathrm{int}K \subset G_p = V_1 \cup V_2.$$

Therefore $B(p,\epsilon)\cap \mathrm{int}K$ is not connected, which is a contradiction. Then we have

$$V \subset B_K(b_1,\ \delta) \subset \cup_{k=1}^{j}\varphi(B_K(a_k,\varepsilon)).$$

Since

$$V\setminus\{b_1\} \subset \cup_{k=1}^{j}(\varphi(B_K(a_k,\varepsilon))\setminus\{b_1\}) = \cup_{k=1}^{j}(\varphi(B_K(a_k,\varepsilon)\setminus\{a_k\})),$$

we have

$$V\setminus\{b_1\} = \cup_{k=1}^{j}((V\setminus\{b_1\})\cap\varphi(B_K(a_k,\varepsilon)\setminus\{a_k\}).$$

Let $\{x_\nu\}$ a sequence in $B_K(a_k,\varepsilon)\setminus\{a_k\}$ such that $x_\nu \to a_k$ as $\nu\to\infty$. Then $\varphi(x_\nu)\to\varphi(a_k) = b_1$ as $\nu\to\infty$. For a sufficiently large ν, $\varphi(x_\nu)\in V$. Since $x_\nu \neq a_k$, $\varphi(x_\nu)\neq b_1$. Hence $\varphi(x_\nu)\in V\setminus\{b_1\}$. Therefore $\varphi(x_\nu)\in (V\setminus\{b_1\})\cap \varphi(B_K(a_k,\varepsilon))\setminus\{a_k\})\neq\emptyset$. On the other hand, it is easy to see that

$$\varphi(B_K(a_k,\varepsilon))\setminus\{a_k\})\cap\varphi(B_K(a'_k,\varepsilon))\setminus\{a'_k\}) = \emptyset$$

holds if $k\neq k'$. Thus we have

$$((V\setminus\{b_1\})\cap\varphi(B_K(a_k,\varepsilon))\setminus\{a_k\}))\cap((V\setminus\{b_1\})\cap\varphi(B_K(a'_k,\varepsilon))\setminus\{a'_k\})) = \emptyset$$

if $k\neq k'$. Therefore $V\setminus\{b_1\}$ is not connected. This is a contradiction. We conclude that φ is one-to-one. So we see that Claim 6 holds.

It follows that φ is a homeomorphism of K onto K and φ is holomorphic on $\mathrm{int}K$. Thus φ^{-1} is a homeomorphism of K onto K and φ^{-1} is holomorphic on $\mathrm{int}K\setminus E$. Define the operator $\tilde{S} : A\to A$ by

$$\tilde{S}f = f\circ\varphi^{-1}\ \ (f\in A).$$

Then $\tilde{T}\tilde{S} = \tilde{S}\tilde{T}$ is the identity operator. Hence the codimension of $\tilde{T}A$ in A is 0. This is a contradiction. □

4. Examples

In this section we give examples of domains which satisfy the five hypotheses in Theorem in the previous section.

Let K be a compact subset of $\mathbb{C}^n$. Recall that a point $p\in\partial K$ is called a peak point for $A(K)$ if there is an $h\in A(K)$ with $h(p)=1$ and $|h(z)|<1$ for $z\in K\setminus\{p\}$. The family of invertible elements of $A(K)$ is denoted by $A(K)^{-1}$.

Example 4.1. Let n be a positive integer greater than 1. Let D be a bounded strictly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. Let $K=\bar{D}$. Then K satisfies the five hypotheses (i), (ii), (iii), (iv) and (v) in Theorem.

Proof. By the definition of K, (i) holds. Since D is strictly pseudoconvex, there are a neighborhood U of ∂D and a strictly plurisubharmonic function $r\in C^2(U)$ such that $D\cap U = \{z\in U : r(z)<0\}$. Put $D_j = \{z\in U : r(z)<\frac{1}{j}\}$. Then D_j is strictly pseudoconvex. By Theorem VI.1.17 in [8], D_j is holomorphically convex. Thus (ii) holds.

Suppose (iii) does not hold, that is, there exists p in K, for every $\epsilon_p>0$, there exists ϵ such that $0<\epsilon<\epsilon_p$ and $B(p,\epsilon)\cap\mathrm{int}K$ is not connected. Without loss of generality we may assume p in ∂K. Since ∂D is of class C^2, there are an open neighborhood U of p and a real valued function $r\in C^2(U)$ such that

$U \cap D = \{x \in U : r(x) < 0\}$ and $dr(x) \neq 0$ for $x \in U$. We may assume that $U \cap D$ is not connected. Then there are two nonempty sets O_1, O_2 such that $O_1 \cup O_2 = U \cap D$ and $O_1 \cap O_2 = \emptyset$. Note that O_1 and O_2 are closed and bounded. Then r has a local maximum or a local minimum on O_1. Since $dr(x) \neq 0$ for $x \in U$, this is a contradiction. Hence (iii) holds.

By Theorem VII.2.1 in [8], (iv) holds.

By Theorem VI.1.13 in [8], every $p \in \partial K$ is a peak point, so $\partial A(K)$ coincides with the topological boundary ∂K of K in $\mathbb{C}^n$. Let $u \in A(K)$. Suppose that $|u| = 1$ on $\partial A(K)$ and u has no zero in $\mathrm{int}K$. Then we have $|u| \leq 1$ on K and $u \in A(K)^{-1}$, so $|u^{-1}| \leq 1$ on K since $|u^{-1}| = 1$ on $\partial A(K)$. It follows that $|u| = 1$ on K. Since $|u| = 1$ on K and $\mathrm{int}K = D \neq \emptyset$, u is constant. $\square$

Example 4.2. Let n be a positive integer greater than 1. Let K be a compact and convex subset of $\mathbb{C}^n$ such that $\overline{\mathrm{int}K} = K$. Then K satisfies the five hypotheses (i), (ii), (iii), (iv) and (v) in Theorem.

Proof. (i) is obvious. Without loss of generality we may assume that the origin is in $\mathrm{int}K$. Since K is convex, $\mathrm{int}K$ is convex, and so it is holomophically convex by lemma II.3.6 in [8]. Let $D_m = \left(1 + \frac{1}{m}\right)\mathrm{int}K$ for every positive integer m. Then D_m is bounded and holomorphically convex since K is compact and D_m is convex. We also see that $K = \cap_{m=1}^{\infty} D_m$ and $D_m \supset \overline{D_{m+1}}$.

And (iii) clearly holds, because $B(p, \epsilon) \cap \mathrm{int}K$ is convex for all $p \in K$ and all $\epsilon > 0$. Without loss of generality we may assume that $0 \in \mathrm{int}\, K$. By the definitions we have that $H(K) \subset A(K)$. We will show that $A(K) \subset H(K)$. Suppose that $f \in A(K)$. Put $f_j(z) = f\left(\frac{1}{1+\frac{1}{j}} z\right)$. Then f_j is holomorphic on $D_j = (1 + \frac{1}{j})\mathrm{int}K$ and converges to f uniformly on K. Therefore we have that $f \in H(K)$, so (iv) holds.

Suppose that $u \in A(K)$ and $|u| = 1$ on $\partial A(K)$. Suppose that u has no zero in K. Then $u^{-1} \in A(K)$. Therefore $|u^{-1}| \leq 1$ on K since $|u^{-1}| = 1$ on $\partial A(K)$. Hence $|u| = 1$ on K, so u is constant since $\mathrm{int}K \neq \emptyset$.

Suppose that u has a zero in K and no zero in $\mathrm{int}K$. Then u has a zero in ∂K. Let $p \in \partial K$ such that $u(p) = 0$. Put $u_j(z) = u\left(\frac{1}{1+\frac{1}{j}} z\right)$. Then u_j is holomorphic on D_j and has no zero on K. Therefore $\frac{1}{u_j} \in A(K)$. On the other hand $u_j(p) = u\left(\frac{1}{1+\frac{1}{j}} p\right) \to 0$ as $j \to \infty$. Then there is an integer N such that $|u_j(p)| < \frac{1}{2}$ if $j > N$. Since $|u|$ is uniformly continuous on K, there is an integer M such that $|u(z) - u_j(z)| < \frac{1}{3}$ if $j > M,\ z \in K$. In particular $|1 - |u_j(z)|| < \frac{1}{3}$ if $j > M,\ z \in \partial A(K)$. Therefore $\frac{2}{3} < |u_j|$ on $\partial A(K)$ if $j > M$. Thus $\left|\frac{1}{u_j}\right| < \frac{3}{2}$ on $\partial A(K)$ if $j > M$. Recall that $\frac{1}{u_j} \in A(K)$. Hence $\left|\frac{1}{u_j}\right| < \frac{3}{2}$ on K if $j > M$. If $j > \max\{N, M\}$, then $\left|\frac{1}{u_j}\right| < \frac{3}{2}$ on K. Since $|\frac{1}{u_j(p)}| > 2$, this is a contradiction. Hence u has a zero in $\mathrm{int}K$. $\square$

Example 4.3. Let n be a positive integer greater than 1 and K_j a compact subset of $\mathbb{C}$ such that ∂K_j consists of a finite number of disjoint smooth closed curves. Let $K = \prod_{j=1}^n K_j$. Then K satisfies the five hypotheses (i), (ii), (iii), (iv) and (v) in Theorem.

Proof. (i) is obvious. We will show that (iii) holds. We consider only for $n = 2$. A proof is similar for general n. It suffices to show the following : for every point $p = (p_1, p_2)$ in K there exists an $\epsilon_p > 0$ such that $(D(p_1, \epsilon) \times D(p_2, \epsilon) \cap \mathrm{int}K$ is connected for every ϵ with $0 < \epsilon < \epsilon_p$, where $D(p_1, \epsilon) = \{z \in \mathbb{C} : |z - p_1| < \epsilon\}$. Fix $p \in K$. Write int$K$=int$K_1 \times$intK_2. Since ∂K_j consists of a finite number of disjoint smooth closed curves, there exists an $\epsilon_{p_1} > 0$ such that $D(p_1, \epsilon) \cap \mathrm{int}K_1$ is connected for every ϵ with $0 < \epsilon < \epsilon_{p_1}$. And there exists an $\epsilon_{p_2} > 0$ such that $D(p_2, \epsilon) \cap \mathrm{int}K_2$ is connected for every ϵ with $0 < \epsilon < \epsilon_{p_2}$. Put $\epsilon_p = \min\{\epsilon_{p_1}, \epsilon_{p_2}\}$. Then

$$\begin{aligned}(D(p_1, \epsilon) \cap \mathrm{int}K_1) \times (D(p_2, \epsilon) \cap \mathrm{int}K_2)& \\ = (D(p_1, \epsilon) \times D(p_2, \epsilon)) \cap (\mathrm{int}K_1 \times \mathrm{int}K_2)& \\ = (D(p_1, \epsilon) \times D(p_2, \epsilon)) \cap \mathrm{int}K&\end{aligned}$$

is connected for every ϵ with $0 < \epsilon < \epsilon_p$. Hence (iii) holds.

Let $D(K_j, \frac{1}{k}) = \{z \in \mathbb{C} : d(z, K_j) < \frac{1}{k}\}$, where

$$d(z, K_j) = \inf\{|z - w| : w \in K_j\},\ 1 \leq j \leq n$$

and $D_k = \prod_{j=1}^n D(K_j, \frac{1}{k})$. Then we have that $K = \cap_{k=1}^{\infty} D_k$ and $D_k \supset \overline{D_{k+1}}$. By Proposition II.3.8 in [8], D_k is holomorphically convex. Hence (ii) holds.

Let $R(K_j)$ be the algebra of all continuous functions on K_j which can be approximated uniformly on K_j by rational functions with poles off K_j. By Theorem II.10.4 in [4], $R(K_j) = A(K_j)$. Since $R(K_j) \subset H(K_j) \subset A(K_j),\ A(K_j) = H(K_j)$. By Corollaire 8 in [10], $A(K) = H(K)$. Hence (iv) holds.

Suppose $u \in A(K)$ such that $|u| = 1$ on $\partial A(K)$. If u has no zero in K, then $\frac{1}{u} \in A(K)$. Hence $|u| = 1$ on K. Since int$K \neq \emptyset$, u is constant. Now consider the case that u has a zero in K. We claim that u has a zero in int$K$. Suppose not. Then there exists a $p = (p_1, \ldots, p_n) \in \partial K$ with $u(p) = 0$. We note that $\partial A(K) = \prod_{j=1}^n \partial K_j$. We consider only for $n = 2$. A proof is similar for general n. Let U be a neighborhood of $x \in \partial K_1$. Then there exists $f \in A(K_1)$ such that $||f||_\infty = 1$ and $|f| < 1$ on $K_1 \backslash U$. Let V be a neighborhood of $y \in \partial K_2$. Then there exists $g \in A(K_2)$ such that $||g||_\infty = 1$ and $|g| < 1$ on $K_2 \backslash V$. Then $fg \in A(K_1 \times K_2), ||fg||_\infty = 1$ and $|fg| < 1$ on $K_1 \times K_2 \backslash U \times V$. Hence $(x, y) \in \partial A(K_1 \times K_2)$. Conversely for $F \in A(K_1 \times K_2)$, there exists $(x, y) \in K_1 \times K_2$ such that $|F(x, y)| = ||F||_\infty$. Since $F(x, \cdot) \in A(K_2)$, there exists $y_0 \in \partial A(K_2)$ such that $|F(x, \cdot)| = |F(x, y_0)|$. Since $F(\cdot, y_0) \in A(K_1)$, there exists $x_0 \in \partial A(K_1)$ such that $|F(x, y)| = |F(x_0, y_0)|$. Therefore $\partial K_1 \times \partial K_2$ is a boundary for $A(K_1 \times K_2)$. Hence $\partial A(K_1 \times K_2) \subset \partial K_1 \times \partial K_2$.

Since $\partial A(K) = \prod_{j=1}^n \partial K_j$, we see that there exists a j with $1 \leq j \leq n$ such that $p_j \in \mathrm{int}K_j$. Without loss of generality we may suppose that there exists $1 \leq j_0 \leq n-1$ such that $p_j \in \partial K_j$ for $1 \leq j \leq j_0$ and $p_j \in \mathrm{int}K_j$ for $j_0+1 \leq j \leq n$. Suppose that $\{(z_{1,m}, \ldots, z_{j_0,m})\}$ is a sequence of $\prod_{j=1}^{j_0} \mathrm{int}K_j$ which converges to $(p_1, \ldots, p_{j_0})$. Put $u_m : K_{j_0+1} \to \mathbb{C}$ (resp. $u_\infty : K_{j_0+1} \to \mathbb{C}$) defined by $u_m(z) =$

$u(z_{1,m}, \ldots, z_{j_0,m}, z, p_{j_0+2}, \ldots, p_n)$ (resp. $u_\infty(z) = u(p_1, \ldots, p_{j_0}, z, p_{j_0+2}, \ldots, p_n)$). Then u_m, $u_\infty \in A(K_{j_0+1})$ and u_m has no zero in $\mathrm{int}K_{j_0+1}$ since we have assumed that u has no zero in $\mathrm{int}K$. We also see that u_m converges to u_∞ uniformly on K_{j_0+1}. Since $u_\infty(p_{j_0+1}) = 0$ and $p_{j_0+1} \in \mathrm{int}K_{j_0+1}$, we have that $u_\infty = 0$ on K_{j_0+1} by Rouché's theorem. It follows by induction we see that $u = 0$ on $\{(p_1, \ldots, p_{j_0}, z_{j_0+1}, \ldots, z_m, \ldots, z_n) \in \mathbb{C}^n : z_m \in K_m \text{ for every } j_0 + 1 \leq m \leq n\}$. This is a contradiction since $|u| = 1$ on $\prod_{j=1}^n \partial K_j$. We conclude that u has a zero in $\mathrm{int}K$. $\square$

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