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ON DIFFERENT PRODUCTS OF CLOSED OPERATORS

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Submitted by H. Dedania

Abstract. The purpose of this work is to give different products of closed operators together with their faults and assets.

1. Introduction

Let H be a complex Hilbert space. All operators are assumed to be linear and defined from H into H. The domain of an operator A is denoted by $D(A)$ which we assume to be dense. The operator A is said to be closed if its graph $G(A) = \{(x, Ax) | x \in D(A)\}$ is closed in $H \times H$. The adjoint of A is denoted by A^*. The identity operator is denoted by I. The set of bounded operators on H is denoted by $B(H)$ while $C(H)$ denotes the set of closed operators (by the closed graph theorem $B(H) \subset C(H)$). $C_0(H)$ is the set of all contractions A on H (i.e., $\|A\| \leq 1$) such that $\ker(I - A^*A) = \{0\}$.
If A and B are two closed operators, then denote the projections onto $G(A)$ (respectively $G(B)$) by $P_{G(A)}$ (respectively $P_{G(B)}$). If

$$\delta(A, B) = \|(1 - P_{G(B)})P_{G(A)}\|,$$

then

$$g(A, B) = \max(\delta(A, B), \delta(B, A))$$

is a metric on $C(H)$.

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Any other result or notion about unbounded operators that has not been mentioned and which will be used is assumed to be known by the reader. The literature on this subject is vague. We cite [6], [13] and [15] among others.

The product of closed operators plays a major role in the theory of partial differential equations since by factorization theorems for real valued analytic functions, it was shown (cf. [5, 8, 9]) that the problems of local solvability of partial differential equations reduce to the existence and the uniqueness of local solutions of ordinary differential equations in a Hilbert space. So we give different definitions of a product of closed operators. The first one is the usual product which we will recall shortly together with its major deficiencies. Then, we give a definition due to J. Dixmier [4], which we believe to the best of our knowledge, it is not known to many authors. Then we give a new definition of a product in the last section which has properties not shared by the usual product.

For a start the standard and the known definition of the product of two operators A and B with domains respectively $D(A)$ and $D(B)$ is just

Definition 1.1. Let A and B be two operators with domains respectively $D(A)$ and $D(B)$. Then $(AB)f := A(Bf)$ for $f \in D(AB) = B^{-1}(D(A))$.

This definition, although being the most natural, has some faults and we will show that the product of two closed operators can behave cantankerously even if stronger conditions are imposed on them.

First, the most notable deficiency perhaps, is that the set $\mathcal{C}(\mathcal{H})$ cannot be given a structure of a group since, as a matter of fact, the product of two closed operators is not always closed. In fact, it is seldom closed unless we impose strong conditions. One of which is that the "left" operator in the product has bounded inverse. Another possibility is to assume that both closed operators are also Fredholm (for this particular result, the interested reader might consult [12]). It is worth noticing that B. Messirdi and G. Djellouli are working on a condition that makes the product (of two closed operators) closed. They have found so far that if the distance $g(A, B)$ is sensible then AB is closed.

Now, if both operators are even self-adjoint, then their product need not be closed, see e.g. [10] and assuming that the "left" operator is bounded does not make the result true as will be shown in the next section.

If we talk about adjoints, then the results are not better. For instance, the adjoint of AB if both operators are unbounded (and closed as well) does not equal to B^*A^* (we have equality though if A is bounded in this order).

The product of closed operators can behave in such an awkward way so that the product of a closed (and symmetric) operator with itself can have a domain that reduces to $\{0\}$. This was first done by Naimark [11] who gave a non-explicit way of constructing such operators. Then Chernoff [1] gave simpler and more explicit operators A satisfying $D(A^2) = \{0\}$ (they are also semibounded in his case). About this precise question, some credit should also be given to J. Dixmier [4] as he gave in the end of his paper a way of constructing symmetric operators whose squares (and even their adjoint's squares) have trivial domain.

Getting back to P.R. Chernoff, his idea was based on the Cayley transform (see [15]) and the operator A was given by multiplication by $F = i(\Omega + 1)(\Omega - 1)^{-1}$

(where Ω is some complex function, see [1]) on the domain $D(A) = (\Omega - 1)\mathcal{H}^2(S)$ where $\mathcal{H}^2(S)$ denotes the Hardy space on the unit circle S (see [16]).

This example can further be exploited to show another deficiency of this product as regards to the distributive law. For example, the inclusion $A(B+C) \supset AB + AC$ may be strict. For if it were true then one would have on the one hand $D(A(A-A)) = D(A)$ and on the other hand $D(A^2 - A^2) = D(A^2) = \{0\}$.

Last but not least, the operator AB can just not make any sense if $D(A) \cap Ran(B) = \{0\}$ ("Chernoff" again is an example of that).

Now if we talk about adjoints, then the results are not better. For instance, the adjoint of AB if both operators are unbounded does not equal to B^*A^* (one only has $A^*B^* \subset (BA)^*$ and equality if B is bounded in this order).

2. The Dixmier Product

J. Dixmier [4] gave another definition of a product. For the sake of convenience of the reader we recall it and the noteworthy results he obtained.

Definition 2.1. The product $A \cdot B$ of two operators A and B is defined in the following way. We say that $f \in D(A \cdot B)$ and $g := A \cdot Bf$ if there exist two sequences, $(f_n)_n$ in $D(B)$ and $(g_n)_n$ in $RanA$, $f_n \to f$ and $g_n \to g$, such that $A^{-1}g_n - Bf_n \longrightarrow 0$ (for some well-chosen $A^{-1}g_n$ and Bf_n).

By adopting this definition he obtained

Theorem I. The operator $A \cdot B$ is closed.

Theorem II. (1) The two products AB and $A \cdot B$ coincide if either of the following occurs:
(a) A closed and B bounded;
(b) A^{-1} bounded and B closed.
(2) $A \cdot B$ coincide with the closure of AB if the roles of A and B in a) or b) are inverted.

In the previous theorem, Property 1) is no longer true if A is bounded and B is closed since in this case AB may not be closed. There is a nice example due to A.M. Davie [3] which is as follows: on $L^2(U)$, where U is the strip $0 < x < 1$ in the (x, y) plane, let $Bf(x, y) = yf(x, y)$ and $Af(x, y) = xf(x, y)$. Then if f is such that $xyf(x, y)$ is in L^2 but $yf(x, y)$ is not then f is in the closure of $D(AB)$ (in the graph norm) but f is not in $D(B)$, and whence not in $D(AB)$.

J. Dixmier also got a formula for adjoints which is

Theorem III. If A and B are closed , then $(AB)^* = B^* \cdot A^*$; and in general, $(A \cdot B)^* = B^*A^*$.

3. The MM-Product

As alluded to in the introduction we now give a new definition of a product of two closed operators on a Hilbert space. If A is a densely defined closed operator in a Hilbert space with domain $D(A)$. Then, it is known (cf. [2, 15]) that the operator $B = (1+A^*A)$ is closed (it is even self-adjoint) and has bounded inverse.

Its inverse, which we denote by R_A, is also positive. Hence by the square root lemma (see [15]), we know that there exists a unique positive operator C such that $C^2 = R_A$. Hence, it is legitimate to consider the operator

$$S_A = (1 + A^*A)^{-\frac{1}{2}}$$

(as denoted by Labrousse–Mercier [7]).

It is also known (cf. [2, 15]) that AR_A is bounded as long as A is closed. Besides $\mathrm{ran}R_A$ is equal to $D(A^*A)$ and $A^*AR_A = I - R_A$. And for $x \in D(A)$ one has $R_{A^*}Ax = AR_Ax$ so that $(AR_A)^* = A^*R_{A^*}$. We also have

$$\forall x \in H \ , \ \left\| \left(\frac{1}{2} - R_A \right) x \right\|^2 + \|AR_Ax\|^2 = \frac{1}{4}\|x\|^2.$$

Hence $\|R_A\| \leq 1$ and $\|AR_A\| \leq \frac{1}{2}$. Another property is $\ker(AR_A) = \ker A$. Now we give some properties of S_A that will be needed.

Proposition 3.1. *For all $x \in H$ one has*

$$\|S_Ax\|^2 + \|AS_Ax\|^2 = \|x\|^2. \tag{3.1}$$

The range of S_A is $\mathrm{ran}S_A = D(A)$.

Proof. From

$$\|R_Ax\|^2 + \|AR_Ax\|^2 =< x, R_Ax > \ , \forall x \in H,$$

we obtain

$$\|S_AS_Ax\|^2 + \|AS_AS_Ax\|^2 = \|S_Ax\|^2 \ , \forall x \in H. \tag{3.2}$$

Hence AS_A is bounded on $\mathrm{ran}(R_A)$ with a norm smaller than 1. Now since $\mathrm{ran}(R_A) \subset \mathrm{ran}(S_A) \subset D(A)$ is dense in $D(A)$ with respect to the graph norm and since Equation 3.2 shows that $\mathrm{ran}(S_A)$ is dense in $D(A)$ then it follows that $\mathrm{ran}(S_A) = D(A)$. □

Remark 3.2. From the last proposition one obtains $\|S_A\|_{B(H)} \leq 1$ and $\|AS_A\|_{B(H)} \leq 1$.

Proposition 3.3. *The following properties hold*
1) If $x \in D(A)$, $S_{A^*}Ax = AS_Ax$,
2) $(AS_A)^* = A^*S_{A^*}$,
3) $\ker(AS_A) = \ker(A)$.

Proof. 1) Since R_A is a positive contraction, then there exists a sequence of polynomials $P_n(R_A)$ (see [14]) such that the degree of $P_n(R_A)$ is 2^{n-1} and

$$P_n(0) = 0 \ , \ P_n(R_{A^*})A = AP_n(R_A) \text{ and } \lim_{n \to +\infty} \|P_n(R_A) - S_A\|_{B(H)} = 0.$$

So if $x \in D(A)$ then

$$S_{A^*}Ax = \lim_{n \to +\infty} P_n(R_{A^*})Ax = \lim_{n \to +\infty} AP_n(R_A)x = AS_Ax.$$

2) Let $x \in D(A)$. Then

$$\forall y \in H, \ < AS_Ax, y >=< S_{A^*}Ax, y >=< x, A^*S_{A^*}y > .$$

Hence $AS_A = (A^*S_{A^*})^*$ on $D(A)$ and hence on all H since $D(A)$ is dense in H.
3) This property is trivial. □

Definition 3.4 (**Labrousse–Mercier** [7])**.** Let $A \in C(H)$. Then the bisecting of A is the operator $\tilde{A}$ defined as

$$\tilde{A} = AS_A(1+S_A)^{-1}.$$

Proposition 3.5. *Let $A \in C(H)$. Then*
1) $\tilde{A} \in C_0(H)$.
2) $(\tilde{A})^* = \widetilde{A^*}$.
3) $R_{\tilde{A}} = \frac{I+S_A}{2}$.
4) $\tilde{A}R_{\tilde{A}} = \frac{AS_A}{2}$.

Proof. Let $A \in C(H)$. Then

$$\|\tilde{A}\|_{B(H)} \leq \|AS_A\|_{B(H)}\|(I+S_A)^{-1}\|_{B(H)} \leq 1$$

which proves 1). Then by the previous proposition, propriety 2) we have $(\tilde{A})^* = (I+S_A)^{-1}A^*S_{A^*}$. But if $x \in D(A^*)$ then $A^*S_{A^*}x = S_AA^*x$ and hence

$$(A^* + A^*S_{A^*})x = (A^* + S_AA^*)x.$$

So

$$A^*(I+S_{A^*})x = (I+S_A)A^*x$$

or

$$(I+S_A)^{-1}A^*x = A^*(I+S_{A^*})^{-1}x.$$

Accordingly,

$$(\tilde{A})^* = A^*(I+S_{A^*})^{-1}S_{A^*} = A^*S_{A^*}(I+S_{A^*})^{-1} = \widetilde{A^*},$$

establishing 2).

To prove 3) we do the following

$$I + \tilde{A}^*\tilde{A} = I + (I+S_A)^{-1}A^*S_{A^*}AS_A(I+S_A)^{-1}.$$

Since

$$A^*S_{A^*}AS_A = S_AA^*AS_A = S_A(I+A^*A)S_A - R_A = S_AR_A^{-1}S_A - R_A = I - R_A,$$

then

$$I + \tilde{A}^*\tilde{A} = I + (I+S_A)^{-1}(I-R_A)(I+S_A)^{-1}.$$

Therefore

$$I + \tilde{A}^*\tilde{A} = (I+S_A)^{-1}[(I+S_A) + (I-S_A)(I+S_A)(I+S_A)^{-1}] = 2(I+S_A)^{-1}.$$

Thus

$$R_{\tilde{A}} = (I + \tilde{A}^*\tilde{A})^{-1} = \frac{I+S_A}{2}.$$

The last property is proved straightforwardly. $\square$

Our approach in defining a new product is based upon the following result (also due to Labrousse–Mercier [7])

Theorem 3.6. *1) If $A \in B(H)$ then*

$$\|\tilde{A}\|_{B(H)} = \frac{\|A\|_{B(H)}}{1 + \sqrt{1 + \|A\|^2_{B(H)}}}$$

(in particular $\|\tilde{I}\|_{B(H)} = \frac{1}{1+\sqrt{2}}$). Conversely, if $\|\tilde{A}\|_{B(H)} < 1$ then $A \in B(H)$ and

$$\|A\| = \frac{2\|\tilde{A}\|_{B(H)}}{1 - \|\tilde{A}\|^2_{B(H)}}.$$

2) The map $F : A \longrightarrow F(A) = \tilde{A}$ from $(C(H), g)$ onto $(C_0(H), \|\cdot\|_{B(H)})$ is bijective and open.

Remark 3.7. For any closed T in $C_0(H)$ one has

$$(F^{-1}(T))^* = F^{-1}(T^*).$$

So the definition we give in this paper is the following

Definition 3.8. Let $A, B \in C(H)$. Then we define the product $\bullet$ of A and B by

$$A \bullet B = F^{-1}(F(A)F(B))$$

where $F(A)F(B)$ is the usual product defined on H.

The first result we have got is the following

Theorem 3.9. *Let A, $B \in C(H)$.*
1) if $\|F(A)F(B)\|_{B(H)} < 1$, then

$$A \bullet B = 2F(A)F(B)\left(1 - F(B^*)F(A^*)F(A)F(B)\right)^{-1},$$

i.e., $A \bullet B$ is bounded on $\mathcal{H}$ and

$$\|A \bullet B\| = \frac{2\|F(A)F(B)\|_{B(H)}}{1 - \|F(A)F(B)\|^2_{B(H)}}.$$

2) If $\|F(A)F(B)\|_{B(H)} = 1$, then $A \bullet B$ is an unbounded densely defined closed *operator with domain $D(A \bullet B) = \operatorname{ran}(I - F(B^*)F(A^*)F(A)F(B))$.*

And for $y = [I - F(B^*)F(A^*)F(A)F(B)]x \in D(A \bullet B)$ we have

$$(A \bullet B)y = 2F(A)F(B)x.$$

Proof. 1) If $\|F(A)F(B)\|_{B(H)} < 1$, then the operator $(I - F(B^*)F(A^*)F(A)F(B))$ is invertible with bounded inverse given by the Neumann series

$$\sum_{n=0}^{\infty}(F(B^*)F(A^*)F(A)F(B))^n.$$

Accordingly, $2F(A)F(B)(I - F(B^*)F(A^*)F(A)F(B))^{-1} \in B(H)$ and it is easy to check that

$$F[2F(A)F(B)(I - F(B^*)F(A^*)F(A)F(B))^{-1}] = F(A)F(B)$$

and the $B(H)$-norm of $A \bullet B$ is easily computed.

2) First, the operator $A\bullet B$ is densely defined since $\operatorname{ran}(I-F(B^*)F(A^*)F(A)F(B))$ is dense in H. It is also closed. For if $(y_n)_n$ is a sequence in $D(A\bullet B)$ which converges to y in H for which $A\bullet By_n$ converges to z in H then by setting

$$x_n=[I-F(B^*)F(A^*)F(A)F(B)]^{-1}y_n \text{ and } t_n=R^{-1}_{F(A)F(B)}x_n,$$

one gets

$$y_n=[I-F(B^*)F(A^*)F(A)F(B)]x_n=(2R_{F(A)F(B)}-I)R^{-1}_{F(A)F(B)}x_n=(2R_{F(A)F(B)}-I)t_n$$

which converges to y and

$$(A\bullet B)y_n=2F(A)F(B)x_n=2F(A)F(B)R_{F(A)F(B)}t_n\longrightarrow z.$$

By virtue of Equation (3.1), $(t_n)_n$ is a Cauchy sequence that converges to x in H with

$$R_{F(A)F(B)}x=\omega\ ,\ y=[I-F(B^*)F(A^*)F(A)F(B)]\omega\in D(A\bullet B)$$

and

$$z=2F(A)F(B)\omega=(A\bullet B)y.$$

Now one easily shows that $F(A\bullet B)=F(A)F(B)$ and finally $A\bullet B$ is unbounded since $D(A\bullet B)$ is strictly contained in H.

To show this assume $D(A\bullet B)=H$ then $(I-F(B^*)F(A^*)F(A)F(B))$ would then be invertible which would in turn lead to $\|F(A)F(B)\|_{B(H)}<1$ which is a contradiction. □

Remark 3.10. The law $\bullet$ is not commutative (unless $\tilde{A}$ and $\tilde{B}$ commute!). It is, however, associative but it does not have an identity element. Nevertheless, this law has a fundamental property about adjoints that is no shared by the usual product in the unbounded case.

We have

Proposition 3.11. *We have $(B\bullet A)^*=A^*\bullet B^*$.*

Proof. We have

$$(A\bullet B)^*=\left(F^{-1}(F(A)F(B))\right)^*.$$

Hence, by Remark 3.7, we get

$$(A\bullet B)^*=F^{-1}\left[(F(A)F(B))^*\right].$$

Then since $F(A)$ and $F(B)$ are bounded and by Proposition 3 we deduce that

$$(A\bullet B)^*=F^{-1}\left[F(B^*)F(A^*)\right]=B^*\bullet A^*.$$

□

The last result in this paper is again a property that is not always true for arbitrary closed operators in the usual product.

Corollary 3.12. *If either A or B is in $\mathcal{B}(\mathcal{H})$, then so is $A\bullet B$.*

Proof. We only prove the result for A. If $A\in B(H)$, then $\|F(A)\|_{B(H)}<1$ and hence $\|F(A)F(B)\|_{B(H)}<1$. Thus $A\bullet B\in B(H)$. □

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