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ON THE STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION IN TOPOLOGICAL SPACES

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by T. Reidel

Abstract. In this paper we investigate the problem of the Hyers–Ulam stability of the generalized quadratic functional equation

$$f(x+y)+f(x-y)=g(x)+g(y),$$

where f, g are functions defined on a group with values in a linear topological space.

1. Introduction and preliminaries

S. M. Ulam [16] in 1940 presented the following question concerning the stability of group homomorphisms.
Let G be a group, G_1 a group with a metric d and $\varepsilon > 0$ a given number. Does there exist a $\delta > 0$ such that if a mapping $h\colon G \to G_1$ satisfies the inequality

$$d\big[h(xy), h(x)h(y)\big] < \delta \quad \text{for } x, y \in G,$$

then there exists a homomorphism $H\colon G \to G_1$ with

$$d\big[h(x), H(x)\big] < \varepsilon \quad \text{for } x \in G?$$

The first affirmative answer for the Cauchy additive equation under the assumption that G, G_1 are Banach spaces, has been done by D.H. Hyers [11].

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The reader can find a lot of references concerning the stability results of functional equations in the books [3], [4], [13], [14] and papers, e.g. [5], [10], [12], [15].
The problem of the stability of the quadratic functional equation has been investigated in the papers [1], [2], [7], [8], [9].
Let G be an abelian group and throughout this paper let X be a sequentially complete locally convex linear topological Hausdorff space. A mapping $f\colon G \to X$ is said to be quadratic if and only iff it satisfies the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in G. \tag{1.1}$$

Moreover, the above equation is called the quadratic functional equation.
Standard symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of natural, integer, rational and real numbers, respectively.
Given sets $A, B \subset X$ and a number $k \in \mathbb{R}$, we define the well known operations

$$A + B := \{x \in X\colon\ x = a + b,\ a \in A,\ b \in B\},$$
$$kA := \{x \in X\colon\ x = ka,\ a \in A\}.$$

By *conv* U we denote the convex hull of a set $U \subset X$ and by *cl* U the sequential closure of U.

We start with the following lemma [3].

Lemma 1.1. *Let Y_1 and Y_2 be linear spaces over $\mathbb{R}$. If $f\colon Y_1 \to Y_2$ is a quadratic function, then*

$$f(rx) = r^2 f(x), \quad r \in \mathbb{Q},\ x \in Y_1.$$

One can prove (see also [6]) the following lemmas.

Lemma 1.2. *If $A, B \subset X$ and $0 \le \alpha \le \beta$, then*

$$\alpha A \subset \beta\ conv\big[A \cup \{0\}\big],$$
$$conv\ A + conv\ B = conv(A + B).$$

Lemma 1.3. *For any sets $A, B \subset X$ and numbers $\alpha, \beta \in \mathbb{R}$ we have*

$$\alpha(A + B) = \alpha A + \alpha B,$$
$$(\alpha + \beta)A \subset \alpha A + \beta A.$$

Moreover, if A is a convex set and $\alpha, \beta \ge 0$, then

$$\alpha A + \beta A = (\alpha + \beta)A.$$

Let us recall that a set $A \subset X$ is said to be bounded iff for every neighbourhood U of zero there exists a number $\alpha > 0$ such that $\alpha A \subset U$.

Lemma 1.4. *If $A, B \subset X$ are bounded sets, then*

$$A \cup B, \quad A + B, \quad conv\ A$$

are also bounded subsets of X.

Now we shall prove

Lemma 1.5. *Let G be an abelian group and let $B \subset X$ be a nonempty set. If functions $f, g \colon G \to X$ satisfy*

$$f(x+y) + f(x-y) - g(x) - g(y) \in B, \quad x, y \in G, \tag{1.2}$$

then

$$f(x+y) + f(x-y) + 2f(0) - 2f(x) - 2f(y) \in 2 \; conv \; (B - B), \tag{1.3}$$

$$g(x+y) + g(x-y) + 2g(0) - 2g(x) - 2g(y) \in 2 \; conv \; (B - B) \tag{1.4}$$

for all $x, y \in G$.

Proof. Put $x = y = 0$ in (1.2). We get

$$2f(0) - 2g(0) \in B. \tag{1.5}$$

For $y = 0$ in (1.2), we obtain

$$2f(x) - g(x) - g(0) \in B, \quad x \in G. \tag{1.6}$$

Setting $x = y$ in (1.6), we have

$$2f(y) - g(y) - g(0) \in B, \quad y \in G. \tag{1.7}$$

To prove (1.3) we will use (1.2), (1.5), (1.6) and (1.7). Therefore by Lemma 1.2 and Lemma 1.3 we get

$$\begin{aligned} &f(x+y) + f(x-y) + 2f(0) - 2f(x) - 2f(y) \\ &= \big[f(x+y) + f(x-y) - g(x) - g(y)\big] + \big[2f(0) - 2g(0)\big] \\ &\quad - \big[2f(x) - g(x) - g(0)\big] - \big[2f(y) - g(y) - g(0)\big] \\ &\in \big[B + B + (-B) + (-B)\big] \subset \big[conv \; B + conv \; B + conv \; (-B) + conv \; (-B)\big] \\ &= 2 \; conv \; B + 2 \; conv \; (-B) = 2 \; conv \; (B - B), \quad x, y \in G. \end{aligned}$$

If we replace x by $x + y$ and x by $x - y$ in (1.6), respectively, then we obtain

$$2f(x+y) - g(x+y) - g(0) \in B, \quad x, y \in G, \tag{1.8}$$

$$2f(x-y) - g(x-y) - g(0) \in B, \quad x, y \in G. \tag{1.9}$$

To prove (1.4) we will use (1.2), (1.8) and (1.9). Therefore

$$\begin{aligned} &g(x+y) + g(x-y) + 2g(0) - 2g(x) - 2g(y) \\ &= \big[2f(x+y) + 2f(x-y) - 2g(x) - 2g(y)\big] - \big[2f(x+y) - g(x+y) - g(0)\big] \\ &\quad - \big[2f(x-y) - g(x-y) - g(0)\big] \\ &\in \big[2B + (-B) + (-B)\big] \subset \big[conv \; 2B + conv \; (-B) + conv \; (-B)\big] \\ &= 2 \; conv \; B + 2 \; conv \; (-B) = 2 \; conv \; (B - B), \quad x, y \in G. \end{aligned}$$

The prove is complete. □

Remark 1.6. A trivial observation is that $0 \in conv \; (B - B)$, which will play an essential role in the further considerations.

2. Main result

Now we shall prove the main result of the paper.

Theorem 2.1. *Let G be an abelian 2-divisible group and let $B \subset X$ be a nonempty bounded set. If functions $f, g\colon G \to X$ satisfy*

$$f(x+y) + f(x-y) - g(x) - g(y) \in B, \quad x, y \in G,$$

then there exists exactly one quadratic function $Q\colon G \to X$ such that

$$Q(x) + f(0) - f(x) \in \frac{2}{3}\ cl\ conv\ (B - B), \quad x \in G, \tag{2.1}$$

$$2Q(x) + g(0) - g(x) \in \frac{2}{3}\ cl\ conv\ (B - B), \quad x \in G. \tag{2.2}$$

Moreover, the function Q is given by the formulae

$$Q(x) = \lim_{n\to\infty} f_n(x) = \frac{1}{2} \lim_{n\to\infty} g_n(x), \quad x \in G, \tag{2.3}$$

where

$$f_n(x) = \frac{1}{2^{2n}} f(2^n x), \quad g_n(x) = \frac{1}{2^{2n}} g(2^n x), \quad n \in \mathbb{N},\ x \in G$$

and the convergence is uniform on G.

Proof. Define a set $C := 2\ conv\ (B - B)$. Then from (1.3) we have

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) \in \big(C - 2f(0)\big), \quad x, y \in G. \tag{2.4}$$

Setting $y = x$ in (2.4), we obtain

$$f(2x) - 4f(x) \in \big(C - 3f(0)\big), \quad x \in G.$$

Define a set $\widetilde{C} := C - 3f(0)$. Then we have

$$\frac{1}{2^2} f(2x) - f(x) \in \frac{1}{4}\widetilde{C} \subset \frac{1}{4}\ conv\ \widetilde{C}, \quad x \in G. \tag{2.5}$$

By the induction we can prove that

$$\frac{1}{2^{2n}} f(2^n x) - f(x) \in \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right) conv\ \widetilde{C}, \quad n \in \mathbb{N},\ x \in G. \tag{2.6}$$

For $n = 1$ we get (2.5), obviously. Now, let us assume that (2.6) is satisfied for some $n \in \mathbb{N}$. Then for $n+1$ on account of Lemma 1.3 we have

$$\begin{aligned}
\frac{1}{2^{2(n+1)}} f(2^{(n+1)}x) - f(x) &= \left[\frac{1}{2^{2(n+1)}} f(2^{(n+1)}x) - \frac{1}{2^2} f(2x)\right] \\
+ \left[\frac{1}{2^2} f(2x) - f(x)\right] &= \frac{1}{2^2}\left[\frac{1}{2^{2n}} f(2^n \cdot 2x) - f(2x)\right] + \left[\frac{1}{2^2} f(2x) - f(x)\right] \\
&\in \frac{1}{2^2} \cdot \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right) conv\ \widetilde{C} + \frac{1}{4}\ conv\ \widetilde{C} = \frac{1}{3}\left(1 - \frac{1}{2^{2(n+1)}}\right) conv\ \widetilde{C},
\end{aligned}$$

which proves (2.6) for all $n \in \mathbb{N}$ and $x \in G$.
Define

$$Q_n^1(x) := \frac{1}{2^{2n}} f(2^n x), \quad n \in \mathbb{N},\ x \in G. \tag{2.7}$$

For all $m, n \in \mathbb{N}$ and $x \in G$, we have by (2.6)

$$Q^1_{m+n}(x) - Q^1_n(x) = \frac{1}{2^{2(m+n)}} f(2^{(m+n)}x) - \frac{1}{2^{2n}} f(2^n x)$$

$$= \frac{1}{2^{2n}} \left[\frac{1}{2^{2m}} f(2^m \cdot 2^n x) - f(2^n x) \right] \in \frac{1}{2^{2n}} \cdot \frac{1}{3} \left(1 - \frac{1}{2^{2m}} \right) conv\ \widetilde{C}.$$

From boundedness of the set $conv\ \widetilde{C}$ (see the Lemma 1.4) we have that $\left\{Q^1_n\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence of elements of X. Since we have assumed the sequential completeness of X, the sequence (2.7) is convergent for all $x \in G$ and the convergence is uniform on G.

Define

$$Q^1(x) := \lim_{n\to\infty} Q^1_n(x), \quad x \in G.$$

Thus from (2.6) and the definition of the set $\widetilde{C}$, we have for $n \to \infty$

$$Q^1(x) + f(0) - f(x) \in \frac{2}{3}\ cl\ conv\ (B - B), \quad x \in G. \tag{2.8}$$

We shall check that Q^1 is a quadratic function. Substituting $2^n x$, $2^n y$ instead of x and y in (2.4), respectively, we get

$$\frac{1}{2^{2n}} f\big(2^n(x+y)\big) + \frac{1}{2^{2n}} f\big(2^n(x-y)\big) - 2\frac{1}{2^{2n}} f(2^n x) - 2\frac{1}{2^{2n}} f(2^n y)$$

$$\in \frac{1}{2^{2n}}\big(C - 2f(0)\big), \quad x, y \in G.$$

Since by Lemma 1.4 the set $C - 2f(0)$ is bounded, letting $n \to \infty$ we obtain

$$Q^1(x+y) + Q^1(x-y) - 2Q^1(x) - 2Q^1(y) = 0, \quad x, y \in G,$$

i.e. Q^1 is a quadratic function satisfying (2.8).

Define

$$Q^2_n(x) := \frac{1}{2^{2n}} g(2^n x), \quad n \in \mathbb{N},\ x \in G.$$

Similarly as before applying (1.4) we shall check that

$$\frac{1}{2^{2n}} g(2^n x) - g(x) \in \frac{1}{3}\left(1 - \frac{1}{2^{2n}}\right) conv\ D, \quad n \in \mathbb{N},\ x \in G, \tag{2.9}$$

where $D := C - 3g(0)$. Then $\left\{Q^2_n\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence of elements of X uniformly convergent on G. Denote

$$Q^2(x) := \lim_{n\to\infty} Q^2_n(x), \quad x \in G.$$

Letting $n \to \infty$ in (2.9), one gets

$$Q^2(x) + g(0) - g(x) \in \frac{2}{3}\ cl\ conv\ (B - B), \quad x \in G. \tag{2.10}$$

Similarly as in a previous case we can check that Q^2 is a quadratic function satisfying (2.10).

Now we prove the equality $2Q^1 = Q^2$. Applying (1.6), (2.8) and (2.10) we consider the following difference

$$2Q^1(x) - Q^2(x) = \big[2Q^1(x) - 2f(x)\big] - \big[Q^2(x) - g(x)\big] + \big[2f(x) - g(x)\big]$$

$$\in \big[2\ cl\ conv\ (B - B) + B - 2f(0) + 2g(0)\big] =: M,$$

i.e.

$$2Q^1(x) - Q^2(x) \in M, \quad x \in G.$$

In view of Lemma 1.4 the set M is bounded. Then

$$2\frac{1}{2^{2n}}Q^1(x) - \frac{1}{2^{2n}}Q^2(x) \in \frac{1}{2^{2n}}M, \quad x \in G.$$

Replacing x by $2^n x$ in the above condition we get

$$2\frac{1}{2^{2n}}Q^1(2^n x) - \frac{1}{2^{2n}}Q^2(2^n x) \in \frac{1}{2^{2n}}M, \quad x \in G.$$

We have by Lemma 1.1

$$2Q^1(x) - Q^2(x) \in \frac{1}{2^{2n}}M, \quad x \in G.$$

Letting $n \to \infty$ we obtain

$$2Q^1(x) - Q^2(x) = 0, \quad x \in G,$$

i.e. $2Q^1 = Q^2$. Assuming $Q := Q^1$, we can see that the conditions (2.1), (2.2) and (2.3) are satisfied.

To prove the uniqueness assume that there exists another quadratic function $\widetilde{Q}\colon G \to X$ satisfying the condition

$$\widetilde{Q}(x) + f(0) - f(x) \in \frac{2}{3}\ cl\ conv\ (B - B), \quad x \in G$$

and for the contrary suppose that there exists a $y \in G$ such that $c := \widetilde{Q}(y) - Q(y) \neq 0$. Then we have

$$\widetilde{Q}(x) - Q(x) = \widetilde{Q}(x) - f(x) - \big[Q(x) - f(x)\big] \in \frac{4}{3}\ cl\ conv\ (B - B) =: \widetilde{M},$$

i.e.

$$\widetilde{Q}(x) - Q(x) \in \widetilde{M}, \quad x \in G.$$

Applying the same method as before we get

$$\widetilde{Q}(x) - Q(x) = 0, \quad x \in G,$$

i.e. $\widetilde{Q} = Q$, which contradicts $c \neq 0$. The contradiction implies that $c = 0$, which completes the proof. $\square$

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