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ON THE STABILITY OF DRYGAS FUNCTIONAL EQUATION ON GROUPS

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by S. Saitoh

Abstract. In this paper, we study the stability of the system of functional equations $f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1})$ and $f(yx) + f(y^{-1}x) = 2f(x) + f(y) + f(y^{-1})$ on groups. Here f is a real-valued function that takes values on a group. Among others we proved the following results: 1) the system, in general, is not stable on an arbitrary group; 2) the system is stable on Heisenberg group $UT(3, K)$, where K is a commutative field with characteristic different from two; 3) the system is stable on certain class of n-Abelian groups; 4) any group can be embedded into a group where this system is stable.

1. Introduction

Given an operator T and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $\|T(v)\| \leq \delta$ for a $\delta > 0$ imply that $\|u - v\| \leq \varepsilon(\delta)$ for some u and for some $\varepsilon > 0$? This problem is called the stability of the functional transformation (ref. [16]). A great deal of work has been done in connection with the ordinary and partial differential equations. If f is a function from a normed vector space into a Banach space, and $\|f(x + y) - f(x) - f(y)\| \leq \delta$, Hyers [8] proved that there exists an additive function A such that $\|f(x) - A(x)\| \leq \delta$ (cf. [15]). If $f(x)$ is a real continuous function of x over $\mathbb{R}$, and $|f(x + y) - f(x) - f(y)| \leq \delta$,

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it was shown by Hyers and Ulam [10] that there exists a constant k such that $|f(x) - kx| \leq 2\delta$. Taking these results into account, we say that the additive Cauchy equation $f(x+y) = f(x)+f(y)$ is stable in the sense of Hyers and Ulam. The interested reader should refer to the book by Hyers, Isac and Rassias [9] for an in depth account on the subject of stability of functional equations.

Drygas [2] obtained a Jordan and von Neumann type characterization theorem for quasi-inner product spaces. In Drygas's characterization of quasi-inner product spaces the functional equation

$$f(x) + f(y) = f(x - y) + 2\left\{f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right)\right\}$$

played an important role. If we replace y by $-y$ in the above functional equation and add the resulting equation to the above equation, then we obtain

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y). \tag{1.1}$$

The Drygas functional equation (1.1) on an arbitrary group G takes the form

$$f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0 \tag{1.2}$$

for all $x, y \in G$. The stability of the equation (1.2) was studied in [11] and [17]. In [3] the system of equations

$$f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0, \qquad f(zyx) = f(zxy) \tag{1.3}$$

for all $x, y, z \in G$ was solved without any regularity assumption on f. It was shown there that the general solution $f : G \to K$ (a commutative field with characteristic different from two) of the system of functional equations (1.3) is given by

$$f(x) = H(x, x) + A(x)$$

where $H : G \times G \to K$ is a symmetric bihomomorphism and $A : G \to K$ is a homomorphism.

In this sequel, we will write the arbitrary group G in multiplicative notation so that 1 will denote the identity element of G. In this paper we consider the stability of the system of functional equations

$$\begin{cases} f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) = 0, \\ f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1}) = 0 \end{cases} \tag{1.4}$$

for all $x, y \in G$. Here $f : G \to \mathbb{R}$ (the set of real numbers) is the unknown function to be determined. In [6], it was shown that on an arbitrary group G, the system (1.4) is a generalization of the system (1.3).

This paper is organized as follows: In Section 2, we prove some preliminary results that will be used for proving stability of the Drygas system of functional equations (1.4). In Section 3, we investigate the stability of the system (1.4) in Heisenberg groups, and in a class of n-Abelian groups. In Section 4, we prove an embedding theorem for the stability of the system (1.4).

2. Preliminary results

The system (1.4) is said to be stable if for any f satisfying the system of inequalities

$$\begin{cases} |f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1})| \le \delta, \\ |f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1})| \le \delta \end{cases} \tag{2.1}$$

for some positive number δ there is a φ, a solution of (1.4), and a positive number ε such that

$$|f(x) - \varphi(x)| \le \varepsilon \qquad (x \in G).$$

Definition 2.1. The function $f : G \to \mathbb{R}$ is said to be a quasidrygas function if the sets

$$\Delta D_1 = \big\{ f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) \,\big|\, \quad \forall\, x, y \in G \big\},$$

$$\Delta D_2 = \Big\{ f(yx) + f(y^{-1}x) - 2f(x) - f(y) - f(y^{-1}) \,\big|\, \quad \forall\, x, y \in G \Big\}$$

are bounded. The set of quasidrygas functions will be denoted by $KD(G)$.

Definition 2.2. The function $f : G \to \mathbb{R}$ is said to be a quasijensen function if the set

$$\Delta J = \big\{ f(xy) + f(xy^{-1}) - 2f(x) \,\big|\, \quad \forall\, x, y \in G \big\}$$

is bounded. A quasijensen function $f : G \to \mathbb{R}$ is said to be a pseudojensen function if it satisfies the condition $f(x^n) = nf(x)$ for all $n \in \mathbb{Z}$ (the set of integers). The set of quasijensen functions will be denoted by $KJ(G)$. The set of pseudojensen functions will be denoted by $PJ(G)$.

Definition 2.3. The function $f : G \to \mathbb{R}$ is said to be a quasiquadratic function if the set

$$\Delta Q = \big\{ f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \,\big|\, \quad \forall\, x, y \in G \big\}$$

is bounded. The quasiquadratic function $f : G \to \mathbb{R}$ that satisfies the condition $f(x^n) = n^2 f(x)$ for all $n \in \mathbb{Z}$ will be called a pseudoquadratic function. The set of quasiquadratic functions will be denoted by $KQ(G)$. The set of pseudoquadratic functions will be denoted by $PQ(G)$.

Lemma 2.4. *If f satisfies* (2.1)*, then $\varphi(x) := f(x) + f(x^{-1})$ belongs to $KQ(G)$ and $\psi(x) := f(x) - f(x^{-1})$ belongs to $KJ(G)$.*

Proof. Since

$$\begin{aligned} &\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(x) \\ &\qquad = f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \\ &\qquad\qquad + f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1}) - 2f(y^{-1}) \\ &\qquad = f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) \\ &\qquad\qquad + f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1}) - f(y) - f(y^{-1}), \end{aligned}$$

the set

$$\big\{ \varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(x) \mid \quad \forall\, x, y \in G \big\}$$

is bounded. Thus $\varphi \in KQ(G)$. Similarly, since

$$\begin{aligned}
&\psi(xy) + \psi(xy^{-1}) - 2\psi(x) \\
&\qquad = f(xy) + f(xy^{-1}) - 2f(x) \\
&\qquad\qquad -[f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1})] \\
&\qquad = f(xy) + f(xy^{-1}) - 2f(x) - f(y) - f(y^{-1}) \\
&\qquad\qquad -[f(y^{-1}x^{-1}) + f(yx^{-1}) - 2f(x^{-1}) - f(y) - f(y^{-1})],
\end{aligned}$$

the set

$$\big\{\psi(xy) + \psi(xy^{-1}) - 2\psi(x) \mid \quad \forall x, y \in G\big\}$$

is bounded. Hence $\psi \in KJ(G)$, and the proof of the lemma is now complete. $\square$

Lemma 2.5. *Every quasidrygas function $f : G \to \mathbb{R}$ is representable in the form*

$$f(x) = \varphi(x) + \psi(x), \tag{2.2}$$

where $\varphi \in KQ(G)$ and $\psi \in KJ(G)$.

Proof. Indeed, if $\varphi(x) = f(x) + f(x^{-1})$ and $\psi = f(x) - f(x^{-1})$ are functions then from the previous lemma, we have

$$f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2}\psi(x). \tag{2.3}$$

Clearly if $\varphi \in KQ(G)$ and $\psi \in KJ(G)$, then $2\varphi \in KQ(G)$ and $2\psi \in KJ(G)$. Replacing φ by 2φ and ψ by 2ψ in (2.3), we have the asserted representation for f. The proof of the lemma is now complete. $\square$

Definition 2.6. A quasidrygas function $f : G \to \mathbb{R}$ is said to be a pseudodrygas function if there is a decomposition (2.2), that is $f(x) = \varphi(x) + \psi(x)$, such that $\varphi \in PQ(G)$ and $\psi \in PJ(G)$. The set of pseudodrygas functions on G will be denoted by $PD(G)$.

We denote the set of bounded functions $f : G \to \mathbb{R}$ on G by $B(G)$. It is clear that $PQ(G) \cap PJ(G) = \{0\}$. Thus it follows that the decomposition (2.2) is uniquely defined.

Lemma 2.7. *The linear space $PD(G)$ is a direct sum of linear spaces $PJ(G)$ and $PQ(G)$, that is*

$$PD(G) = PJ(G) \oplus PQ(G).$$

Similarly, the linear space $KD(G)$ is a direct sum of linear spaces $PD(G)$ and $B(G)$, that is

$$KD(G) = PD(G) \oplus B(G). \tag{2.4}$$

Proof. The first part of the lemma is easy to show. Thus we prove only the second part of the lemma. Let $f \in KD(G)$. By Lemma 2.5, we have

$$f(x) = \varphi(x) + \psi(x), \tag{2.5}$$

where $\varphi \in KJ(G)$ and $\psi \in KQ(G)$. It was shown in [4] that

$$KJ(G) = PJ(G) \oplus B(G).$$

Similarly from [5], we have

$$KQ(G) = PQ(G) \oplus B(G). \tag{2.6}$$

Therefore $\varphi(x) = \varphi_1(x) + \sigma_1$ and $\psi(x) = \psi_1(x) + \sigma_2$, where $\varphi_1 \in PJ(G)$, $\psi_1 \in PQ(G)$, and σ_1, σ_2 bounded functions. Thus from (2.5), we have

$$f(x) = \varphi_1(x) + \psi_1(x) + \sigma,$$

where $\sigma = \sigma_1 + \sigma_2$. Hence it is clear that

$$KD(G) = PJ(G) \oplus PQ(G) \oplus B(G) = PD(G) \oplus B(G)$$

and the proof of the lemma is now complete. □

We denote, by $D(G)$, the set of all functions satisfying the system (1.4). By $J(G)$, we denote the set of all functions satisfying the Jensen equation

$$f(xy) + f(xy^{-1}) - 2f(x) = 0 \tag{2.7}$$

for all $x \in G$. The subset of $J(G)$ consisting of all functions f satisfying the condition $f(1) = 0$ will be denoted by $J_0(G)$. By $Q(G)$, we denote the set of all functions satisfying the quadratic equation

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0 \qquad (x \in G).$$

The following theorem was established in [6].

Theorem 2.8. *The linear space $D(G)$ is a direct sum of $J_0(G)$ and $Q(G)$, that is $D(G) = J_0(G) \oplus Q(G)$.*

Proposition 2.9. *The system* (1.4) *is stable over a group G if and only if $PD(G) = D(G)$. In other words the system* (1.4) *is stable if and only if $PQ(G) = Q(G)$ and $PJ(G) = J_0(G)$.*

Proof. Suppose that $PD(G) = D(G)$, that is $PQ(G) = Q(G)$ and $PJ(G) = J_0(G)$. Let the function f satisfy (2.1) for some positive number δ. Then $f \in KD(G)$ and by Lemma 2.7, $f = g + \sigma$, where $g \in PD(G) = D(G)$ and $\sigma \in B(G)$. Thus we see that $f - g = \sigma$ is a bounded function on G. So the system (1.4) is stable.

Now suppose that the system of equations (1.4) is stable. Let us show in this case that $PJ(G) = J_0(G)$ and $PQ(G) = Q(G)$. Let us verify that if $PQ(G) \neq Q(G)$, then the system (1.4) is not stable. Let $g \in PQ(G) \setminus Q(G)$ and suppose that there is a $f \in D(G)$ and a positive number δ such that $|g(x) - f(x)| \leq \delta$ for any $x \in G$. By Theorem 2.8 there are $j \in J_0(G)$ and $q \in Q(G)$ such that $f(x) = j(x) + q(x)$. Therefore, we have

$$|g(x) - f(x)| = |g(x) - j(x) - q(x)| \leq \delta.$$

Hence $g'(x) := g(x) - q(x) \in PQ(G)$ and for some bounded function ξ we have $g'(x) = j(x) + \xi(x)$. It follows that for any $n \in \mathbb{N}$ we have

$$\begin{aligned} g'(x^n) &= j(x^n) + \xi(x^n), \\ n^2 g'(x) &= nj(x) + \xi(x^n), \\ g'(x) &= \frac{1}{n} j(x) + \frac{1}{n^2}\xi(x^n). \end{aligned}$$

So we see that $g'(x) \equiv 0$. Hence $g(x) = q(x)$ for all $x \in G$, and we came to a contradiction with the assumption about g. So $PQ(G) = Q(G)$.

Now let $g \in PJ(G) \setminus J_0(G)$. Suppose there is a $f \in D(G)$ and a positive number δ such that $|g(x) - f(x)| \le \delta$ for any $x \in G$. Then by Theorem 2.8 there are $j \in J_0(G)$ and $q \in Q(G)$ such that $f(x) = j(x) + q(x)$. Therefore we have

$$|g(x) - f(x)| = |g(x) - j(x) - q(x)| \le \delta. \tag{2.8}$$

Let $g_1(x) = g(x) - j(x)$ then $g_1 \in PJ(G) \setminus J_0(G)$ and from (2.8) we see that there is a bounded function σ such that $g_1(x) = q(x) + \sigma(x)$. Thus, for any $n \in \mathbb{N}$, we get

$$\begin{aligned} g_1(x^n) &= q(x^n) + \sigma(x^n), \\ n g_1(x) &= n^2 q(x) + \sigma(x^n), \\ q(x) &= \frac{1}{n} g_1(x) - \frac{1}{n^2}\sigma(x^n). \end{aligned}$$

Hence, $q(x) \equiv 0$, and we get

$$|g(x) - f(x)| = |g(x) - j(x)| \le \delta.$$

It follows that $f(x) \equiv j(x)$, and thus $PJ(G) = J_0(G)$. Hence we came to a contradiction about the assumption on g. Therefore if the system (1.4) is stable then $PQ(G) = Q(G)$ and $PJ(G) = J_0(G)$, and $PD(G) = D(G)$. This completes the proof of the theorem. $\square$

Remark 2.10. In the paper [4], we have shown that if a group G has nontrivial pseudocharacter (for example G is a nonabelian free group), then $PJ(G) \ne J_0(G)$. Hence by Proposition 2.9 the system 1.4 is not stable over such a group G. So, in general, the system (1.4) is not stable.

Proposition 2.11. *Let G be an arbitrary group and $f \in PQ(G)$. Then*
1. $f(1) = 0$;
2. $f(x^{-1}) = f(x)$, *and*
3. *f is invariant relative to inner automorphisms of G.*

Proof. Let $f \in PQ(G)$, then we have

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \le \delta \qquad (x, y \in G) \tag{2.9}$$

for some $\delta > 0$.

1. If we let $x = y = 1$, then from (2.9), we obtain $|f(1)| \le \frac{1}{2}\delta$. Since $f \in PQ(G)$, for any $n \in \mathbb{N}$ we have $n^2|f(1)| = |f(1^n)| \le \frac{1}{2}\delta$. Hence it follows that $f(1) = 0$.

2. If $x = 1$, then from (2.9) we have

$$|f(y) + f(y^{-1}) - 2f(1) - 2f(y)| \le \delta$$

and

$$|f(y^{-1}) - f(y)| \le \delta \qquad (y \in G).$$

From the last relation it follows that for any $n \in \mathbb{N}$

$$|f(y^{-n}) - f(y^n)| \le \delta.$$

So

$$n^2|f(y^{-1}) - f(y)| = |f(y^{-n}) - f(y^n)| \leq \delta,$$

and thus $f(y^{-1}) = f(y)$ for all $y \in G$.

3. Taking into account that $f(yx^{-1}) = f(xy^{-1})$ the following relations

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \leq \delta,$$
$$|f(yx) + f(yx^{-1}) - 2f(x) - 2f(y)| \leq \delta$$

imply that

$$|f(xy) - f(yx)| \leq 2\delta.$$

Substituting yx^{-1} for y in the last inequality, we get

$$|f(xyx^{-1}) - f(y)| \leq 2\delta.$$

Since $f \in PQ(G)$, now for any $n \in \mathbb{N}$ we have

$$n^2|f(xyx^{-1}) - f(y))| = |f(xy^n x^{-1}) - f(y^n))| \leq 2\delta.$$

The last relation is possible only if $f(xyx^{-1}) = f(y)$. Thus f is invariant relative to inner automorphisms of G. $\square$

3. Stability

In this section, we establish the stability of the system (1.4) on noncommutative group $UT(3, K)$, and on a subclass $\mathcal{K}_n$ of n-Abelian groups. Here K is a commutative field and n is a positive integer greater than or equal to 2.

Let n be an integer. A group G is said to be an n-Abelian group if $(xy)^n = x^n y^n$ for every x and y in G. For more on n-Abelian groups the interested reader should refer to [12], [1], [7], and [13]. For any $n \in \mathbb{N}$ (the set of natural numbers), let $\mathcal{K}_n$ denote the class of groups G satisfying the relation

$$(xy)^n = x^n y^n = y^n x^n \tag{3.1}$$

for any $x, y \in G$. Obviously, the class $\mathcal{K}_n$ is a subclass of the class of n-Abelian groups. For $n \in \mathbb{N}$, let G^n be the subgroup of G generated by the set $\{ x^n \mid x \in G\}$. The subspace of $J_0(G)$ consisting of real additive characters will be denoted by $X(G)$. The following theorem was proved in [5].

Theorem 3.1. *If G is n-Abelian group then $PQ(G) = Q(G)$, and any element f in $Q(G)$ can be represented in the form $f(x) = \frac{1}{n^2}\varphi(x^n)$, where $\varphi \in Q(G^n)$.*

Theorem 3.2. *If $G \in \mathcal{K}_n$ then the system* (1.4) *is stable on G, and any element f in $PD(G)$ can be represented in the form $f = j + q$, where $j \in X(G)$ and $q \in Q(G)$.*

Proof. From Proposition 2.9 and Theorem 3.1 it follows that we need only to establish the relation $PJ(G) = J(G)$. Let us verify it. The subgroup G^n is abelian. Hence by Theorem 3.11 from [4] we have $PJ(G^n) = J_0(G^n) = X(G^n)$. Now let $f \in PJ(G)$, then for any x the element x^n belongs to G^n. Then we have

$$\begin{aligned} f(xy) &= \tfrac{1}{n} \cdot nf(xy) \\ &= \tfrac{1}{n} f((xy)^n) \\ &= \tfrac{1}{n} f(x^n y^n) \\ &= \tfrac{1}{n} f(x^n) + \tfrac{1}{n} f(y^n) \\ &= f(x) + f(y). \end{aligned}$$

So

$$PJ(G) = X(G) = J(G).$$

□

Next, we examine stability of the system (1.4) on the noncommutative groups $UT(3, K)$ of 3-by-3 upper triangular matrices. Let K be an arbitrary commutative field. Let K^* be the set nonzero elements of K with operation of multiplication. Let $UT(3, K)$ be a group consisting of upper triangular matrices of the form

$$\begin{bmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

for all $x, y, t \in K$. The group $UT(3, K)$ is called the Heisenberg group. We proceed to examine the stability of the system (1.4) on the group $UT(3, K)$.

Let G be an arbitrary group and $a, b \in G$. Then the commutator $[a, b]$ of a and b is defined as $[a, b] = a^{-1}b^{-1}ab$.

Definition 3.3. A group G is said to be a metabelian group if $[[x, y], z] = 1$ for any $x, y, z \in G$.

It is clear that if $[x, y] = 1$, then $[[x, y], z] = 1$, that is, every Abelian group is metabelian. Let $G := UT(3, K)$. Then it is easy to check that G is a metabelian group. It was shown in [4] that the Jensen functional equation is stable on any metabelian group. So to establish stability of the system (1.4) we need to show that $PQ(G) = Q(G)$. Denote by A, B, C subgroups of G consisting matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (a,\, b,\, c \in K)\,,$$

respectively. Let H be the subgroup of G generated by B and C.

Proposition 3.4. *If $\varphi \in PQ(G)$, then φ can be represented in the form $\varphi(x) = q(\tau(x))$, where $\tau : G \to K \times K$ is a homomorphism defined by the rule*

$$\tau : \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \to (a, b)$$

and $q \in Q(K \times K)$. Therefore $PQ(G) = Q(G)$.

Proof. Let $\varphi \in PQ(G)$. By Proposition 2.11, the map φ is invariant relative to inner automorphisms of G. From the relation

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b & ba_1 - b_1 a + c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$

it follows that

$$\varphi\left(\begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 & b & ba_1 - b_1 a + c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}\right). \tag{3.2}$$

Let us verify that $\varphi\big|_C \equiv 0$. Let δ be a positive number such that

$$|\,\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y)\,| \le \delta \qquad (x, y \in H).$$

Then for any $\beta \in B$ and $\gamma \in C$ we have

$$|\varphi(\gamma\beta^2) + \varphi(\gamma) - 2\varphi(\gamma\beta) - 2\varphi(\beta)| \le \delta. \tag{3.3}$$

Let

$$\beta = \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $b \neq 0$. Then from (3.2) it follows that

$$\varphi\left(\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \tag{3.4}$$

for any $c \in K$. So, $\varphi(\gamma\beta^2) = \varphi(\beta^2)$ and from (3.3) we get

$$\begin{aligned} &|\varphi(\beta^2) + \varphi(\gamma) - 2\varphi(\beta) - 2\varphi(\beta)| \le \delta, \\ &|4\varphi(\beta) + \varphi(\gamma) - 2\varphi(\beta) - 2\varphi(\beta)| \le \delta, \\ &|\varphi(\gamma)| \le \delta. \end{aligned}$$

So we see that φ is bounded on C. Hence

$$\varphi\big|_C \equiv 0. \tag{3.5}$$

Now from the relations (3.2) and (3.5) it follows that φ is constant on any coset C of G. Therefore there is a $q \in Q(K \times K)$ such that $\varphi(x) = q(\tau(x))$, and we see that $\varphi \in Q(G)$. The proof of the proposition is now complete. □

From Proposition 3.4 and the discussion immediately prior to Proposition 3.4, we see that the Drygas system (1.4) is stable on Heisenberg groups.

4. An embedding theorem for stability

Denote by G^* a group with two generators a and b and one defining relation $a^{-1}ba = b^m$, where $m \ge 2$ is an integer.

Lemma 4.1. *If $\psi \in PJ(G^*)$, then $\psi(b) = 0$, and $PJ(G^*) = J(G^*) = X(G^*)$.*

Proof. Denote by A and B the subgroups of G^* generated by b and b^m, respectively. Then we see that G^* is an HNN–extension, where an isomorphism $\varphi : A \to B$ is given by the rule $\varphi(b) = b^m$. (For more information about HNN–extensions see [14].) We have $A = 1 \cdot A$, and $A = B \cup bB \cup b^2 B \cdots \cup b^{m-1}B$ cosets A by A, and A by B respectively. Let $x_0 = b,\ x_1 = ax_0a^{-1}, \cdots, x_{k+1} = ax_ka^{-1}$. Then we have

$$x_1^m = x_0, \cdots, x_{k+1}^m = x_k \tag{4.1}$$

for any $k \in \mathbb{N}$. It is clear that the normal closure H of A in G^* is generated by the set $\{x_k \mid k \in \mathbb{N}\}$. Let X_k be a subgroup of G^* generated by x_k. From the relations (4.1) we see that

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_k \subset X_{k+1} \subset \cdots$$

All the groups $X_1, X_2, \cdots, X_k, X_{k+1}, \cdots$ are cyclic. Hence H is a locally cyclic group. And we have a semidirect product $G^* = A \cdot H$, where A acts on H as follows: $x_{k+1}^a = x_k$, $k \in \mathbb{N}$ and $x_o^a = x_o^m$.

Let us note that the map $\tau : A \cdot H \to A$ defined by the rule $\tau(au) = \tau(a)$ is an epimorphism and the corresponding mapping $\tau^* : X(A) \to X(G^*)$ such that $\tau^*(\xi) \to \xi^*$, where $\xi^*(au) = \xi(a)$ is an embedding of $X(A)$ into $X(G^*)$. So, if $\psi \in PJ(G^*)$ and $\psi(a) = \alpha$ we can construct a character ξ of G^* such that $\xi(a) = \alpha$. Now if we consider $f = \psi - \xi$ we have $f\big|_A \equiv 0$.

Now let us show that $f(b) = 0$. Let δ be a nonnegative number such that

$$|f(xy) + f(xy^{-1}) - 2f(x)| \le \delta, \qquad (x, y \in G^*).$$

Let $x = cu,\ y = dv$, where $c, d \in A$ and $u, v \in H$. Then we have $xy = cdu^dv,\ xy^{-1} = cd^{-1}u^{d^{-1}}(v^{-1})^{d^{-1}}$. Hence

$$\begin{aligned}&|f(xy) + f(xy^{-1}) - 2f(x)|\\ &\quad = |f(cdu^dv) + f(cd^{-1}u^{d^{-1}}(v^{-1})^{d^{-1}}) - 2f(cu)| \le \delta.\end{aligned}$$

If $u = 1$ and $c = d$, we get

$$|f(c^2v) + f((v^{-1})^{c^{-1}})| \le \delta,$$

that is

$$|f(c^2v) - f(v^{c^{-1}})| \le \delta. \tag{4.2}$$

For any $n \in \mathbb{N}$ we have $(c^2v)^n = c^{2n}v^{c^{2(n-1)}}v^{c^{2(n-2)}} \cdots v^{c^2}v$. From (4.2) it follows

$$|f(c^{2n}v^{c^{2(n-1)}}v^{c^{2(n-2)}} \cdots v^{c^2}v) - f(v^{c^{2(n-1)-n}}v^{c^{2(n-2)-n}} \cdots v^{c^{2-n}}v^{c^{-n}})| \le \delta.$$

Hence

$$|nf(c^2v) - f(v^{c^{2(n-1)-n}}v^{c^{2(n-2)-n}} \cdots v^{c^{2-n}}v^{c^{-n}})| \le \delta.$$

If $v = b$, $c = a$ then the latter inequality implies

$$|nf(a^2b) - f(b^{a^{2(n-1)-n}}b^{a^{2(n-2)-n}} \cdots b^{a^{2-n}}b^{a^{-n}})| \le \delta,$$

or

$$|nf(a^2b) - f(b^{a^{n-2}}b^{a^{n-4}} \cdots b^{a^{n-2(n-1)}}b^{a^{n-2n}})| \le \delta,$$

or

$$\left|nf(a^2b) - \sum_{i=1}^{n} f(b^{a^{n-2i}})\right| \leq \delta.$$

If $f(b) = \lambda$, then

$$\begin{array}{l} f(b^a) = f(b^m) = mf(b) = m\lambda, \\ f(b^{a^2}) = f(b^{m^2}) = m^2 f(b) = m^2\lambda, \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ f(b^{a^n}) = f(b^{m^n}) = m^n f(b) = m^n\lambda. \end{array}$$

Hence

$$\begin{aligned} \left|nf(a^2b) - \sum_{i=1}^{n} f(b^{a^{n-2i}})\right| &= \left|nf(a^2b) - \sum_{i=1}^{n} m^{n-2i}\lambda\right| \\ &= \left|nf(a^2b) - \lambda\sum_{i=1}^{n} m^{n-2i}\right| \\ &\leq \delta. \end{aligned}$$

Therefore

$$\left|f(a^2b) - \lambda\tfrac{1}{n}\sum_{i=1}^{n} m^{n-2i}\right| \leq \frac{\delta}{n} \quad \rightarrow 0 \quad as \quad n \rightarrow \infty.$$

Since $\frac{1}{n}\sum_{i=1}^{n} m^{n-2i} \rightarrow \infty$ as $n \rightarrow \infty$ we see that the previous relation is possible only if $\lambda = 0$. So we have the relation $f(b) = 0$. It follows that $f\big|_H \equiv 0$.

Now we have

$$\begin{aligned} &\mid f(a^k a^k u) + f(a^k (a^k u)^{-1}) - 2f(a^k) \mid \\ &\qquad = \mid f(a^k a^k u) + f((u^{-1})^{a^{-k}}) - 2f(a^k) \mid \\ &\qquad = \mid f(a^{2k} u) \mid \leq \delta. \end{aligned}$$

Therefore, f is bounded on subgroup G_2 generated by a^2 and H. Hence $f\big|_{G_2} \equiv 0$. Now if g is an arbitrary element of G^* then $g^2 \in G_2$ and we get $f(g) = \frac{1}{2}f(g^2) = 0$. Therefore $f \equiv 0$ on G^*. It follows that $\psi \in X(A)$ and $PJ(G^*) = X(G^*) = X(A)$. The proof of the lemma is now complete. □

In the following theorem we show that any group G can be embedded into a group $\tilde{G}$ where the Drygas system (1.4) is stable.

Theorem 4.2. *Any group G can be embedded into a group $\tilde{G}$ such that $PD(\tilde{G}) = D(G)$.*

Proof. Using the construction of HNN-extension [14], we can construct a group $\tilde{G}$ containing the group G such that for any $c \in \tilde{G}$ there is an element $d \in \tilde{G}$ such that $d^{-1}cd = c^m$, where $m \geq 2$. Let $\tilde{G}(c,d)$ denote a subgroup of $\tilde{G}$ generated by c and d. It is clear that the mapping $\pi : a \rightarrow d$, $\pi : b \rightarrow c$ can be extended as an epimorphism of the group $G^* =< a, b \mid a^{-1}ba = b^m >$ onto $\tilde{G}(c,d)$. Now let $\varphi \in PJ(\tilde{G})$ and $\tilde{\varphi} = \varphi\big|_{\tilde{G}(c,d)} \in PJ(\tilde{G}(c,d))$. Then the function φ^* defined by the rule $\varphi^*(x) = \tilde{\varphi}(\pi(x))$ is an element of $PJ(G^*)$. By Lemma 4.1 we have

$\varphi^*(b) = 0$. This relation implies $\tilde{\varphi}(c) = \tilde{\varphi}(\pi(b)) = \varphi^*(b) = 0$. So $\varphi(c) = 0$ for any $c \in \tilde{G}$. Therefore $PJ(\tilde{G}) = 0$.

Now let us verify that $PQ(\tilde{G}) = 0$. Let $\varphi \in PQ(\tilde{G})$. As we know any pseudo-quadratic function is constant on every class of conjugate elements. Using this fact and the relation $d^{-1}cd = c^m$ we get $\varphi(c) = \varphi(d^{-1}cd) = \varphi(c^m) = m^2\varphi(c)$. It follows that $\varphi(c) = 0$. So $PQ(\tilde{G}) = 0$. Now from Proposition 2.9 we have $PD(\tilde{G}) = 0$ and the system (1.4) is stable on $\tilde{G}$. The proof of the theorem is now finished. □

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