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ON POSITIVE DEFINITE DISTRIBUTIONS WITH COMPACT SUPPORT

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Abstract. We propose necessary and sufficient conditions for a distribution (generalized function) f of several variables to be positive definite. For this purpose, certain analytic extensions of f to tubular domains in complex space $\mathbb{C}^n$ are studied. The main result is given in terms of completely monotonic functions on convex cones in $\mathbb{R}^n$.

1. Introduction

Let $\mathbb{R}^n$ be the real n-dimensional Euclidean space imbedded in $\mathbb{C}^n$ so that $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. For $u \in \mathbb{R}^n$, we set $|u| = |u_1| + \cdots + |u_n|$. Let $|x|_2$ denote the standard Euclidean norm on $\mathbb{R}^n$. If, in addition, the entries of u are nonnegative integers, then we call u, throughout the following, a multi-index.

The space of test functions $\mathcal{E}(\mathbb{R}^n)$ is the set of $\varphi : \mathbb{R}^n \to \mathbb{C}$ such that $D_x^u \varphi$ is continuous for all multi-indices u. Here $D_x^u = D_{x_1}^{u_1} \cdot \cdots \cdot D_{x_n}^{u_n}$ and $D_{x_k} = \partial/\partial x_k$. Let $D(\mathbb{R}^n)$ denote the subspace of $\mathcal{E}(\mathbb{R}^n)$ consisting of functions with compact support. We assume that the topologies on $\mathcal{E}(\mathbb{R}^n)$ and on $D(\mathbb{R}^n)$ are introduced as usual (see, e.g., [7] or [16]). The elements of the conjugate spaces $\mathcal{E}'(\mathbb{R}^n)$ and $D'(\mathbb{R}^n)$ are called distributions or generalized functions. Since $D(\mathbb{R}^n)$ is continuously imbedded in $\mathcal{E}(\mathbb{R}^n)$, it follows that each $f \in D'(\mathbb{R}^n)$ gives an element of $\mathcal{E}'(\mathbb{R}^n)$ by restriction. Moreover, $\mathcal{E}'(\mathbb{R}^n)$ coincides with a subspace of compactly supported distributions of $D'(\mathbb{R}^n)$.

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If $f \in D'(\mathbb{R}^n)$, then the action of f on $\omega \in D(\mathbb{R}^n)$ is written as (f, ω). A distribution $f \in D'(\mathbb{R}^n)$ is said to be positive definite if for all $\varphi \in D(\mathbb{R}^n)$,

$$(f, \varphi * \widetilde{\varphi}) \geq 0, \tag{1.1}$$

were $\widetilde{\varphi}(x) := \overline{\varphi(-x)}$, and $*$ denotes the usual convolution operator. The Bochner-Schwartz theorem gives a representation of a positive definite distribution in terms of the Fourier transform. Let us recall some notion.

The Schwartz class $S(\mathbb{R}^n)$ can be defined as the set of $\omega \in \mathcal{E}(\mathbb{R}^n)$ satisfying

$$\|\omega\|_m := \sup_{x \in \mathbb{R}^n,\ |u| \leq m} \Big| (1 + |x|_2)^m D_x^u \omega(x) \Big| < \infty \tag{1.2}$$

for all nonnegative integers m. Semi-norms (1.2) turns $S(\mathbb{R}^n)$ into a Fréchet space. The elements of $S'(\mathbb{R}^n)$ are called tempered distributions. For $\varphi \in S(\mathbb{R}^n)$, we define the Fourier transform as

$$\widehat{\varphi}(x) = \int_{\mathbb{R}^n} e^{i(x,t)} \varphi(t)\, dt, \qquad x \in \mathbb{R}^n,$$

where $(x, t) = x_1 t_1 + \cdots + x_n t_n$. If $f \in S'(\mathbb{R}^n)$, then the Fourier transform $\mathcal{F}[f]$ can be defined by

$$\Big(\mathcal{F}[f], \varphi \Big) = (f, \widehat{\varphi}), \qquad \varphi \in S(\mathbb{R}^n). \tag{1.3}$$

The Bochner-Schwartz theorem states (see, e.g., [16, p. 125]) that $f \in D'(\mathbb{R}^n)$ is positive definite if and only if there exists nonnegative tempered measure $\eta \in S'(\mathbb{R}^n)$ such that $f = \mathcal{F}[\eta]$. We recall that a nonnegative measure η is said to be tempered if there exists $\alpha \geq 0$ such that

$$\int_{\mathbb{R}^n} \big(1 + |x|_2\big)^{-\alpha} d\eta < \infty.$$

Note that this theorem implies that any positive definite distribution belongs to $S'(\mathbb{R}^n)$.

There are many other than the Bochner-Schwartz theorem characterizations of positive definite functions (see, e.g., [8, p.p. 70-83]). As far as we known, it is perhaps surprising that there are almost no such results for distributions. We mention only [13], where attention has been paid to positive definite distributions of order zero on $\mathbb{R}$, with applications to a Volterra equation. See also survey article [11]. Note also that in [4] the Bochner-Schwartz theorem was generalized for the spaces of Fourier hyperfunctions and hyperfunctions.

In this paper, we wish to explore the idea of how to describe positive definite $f \in S'(\mathbb{R}^n)$ by means of its analytic representations in $\mathbb{C}^n$. Let us start with the case of one variable.

For $f \in D'(\mathbb{R})$, Tillman [15] has proved that there exists a pair of functions $f^{(+)}$ and $f^{(-)}$, analytic in the open upper $\mathbb{C}^{(+)}$ and in the open lower half-plane $\mathbb{C}^{(-)}$, respectively, such that

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \Big(f^{(+)}(t + i\varepsilon) - f^{(-)}(t - i\varepsilon) \Big) \varphi(t)\, dt = (f, \varphi) \tag{1.4}$$

for all $\varphi \in D(\mathbb{R})$. This pair $\left(f^{(+)}, f^{(-)}\right)$ (or sectionally analytic function on $\mathbb{C}^{(+)} \cup \mathbb{C}^{(-)} = \mathbb{C} \setminus \mathbb{R}$) is called an analytic representation of f.

If a distribution f has a compact support, then an analytic representation of f can be obtained using an explicit construction. Indeed, if $f \in \mathcal{E}'(\mathbb{R})$, then

$$K(f)(z) = \frac{1}{2\pi i}\Big(f(\cdot),\, (\cdot - z)^{-1}\Big) = \frac{1}{2\pi i}\Big(f_t,\, (t-z)^{-1}\Big) \tag{1.5}$$

is well defined for all $z \in \mathbb{C}\setminus\mathbb{R}$. The function $K(f)$ is called the Cauchy transform of f. If we take $\left(f^{(+)}, f^{(-)}\right) = K(f)$, then we obtain an analytic representation of f (see, e.g., [1, p. 155]). Note that analytic representations of the same distribution differ by at most an entire function [1].

For any fixed $z \in \mathbb{C}\setminus\mathbb{R}$, the Cauchy kernel $k(t) = (t-z)^{-1}$ belongs to $\mathcal{E}(\mathbb{R})$ but not to $S(\mathbb{R})$. Hence, for all $f \in S'(\mathbb{R})$, the analytic representation (1.5), in general, does not exists (see details in [1, p. 156]). On the other hand, if $f \in S'(\mathbb{R})$, then there exists a nonnegative integer m_0 such that f is continuous in the semi-norm $\|\cdot\|_{m_0}$ defined in (1.2), i.e., there exist $A > 0$ such that $|(f, \varphi)| \le A\|\varphi\|_{m_0}$ for all $\varphi \in S(\mathbb{R})$ (see [16, p. 74]). We call the smallest m_0 that satisfies the above inequality the S-*order* of $f \in S'(\mathbb{R})$. Let us write $\varrho_S(f)$ for this order. Note that $\varrho_S(f)$ is different from the usual order $\varrho_D(f)$ of f as distribution in $D'(\mathbb{R})$. Let us define the generalized Cauchy kernel $\widetilde{k}_m(t)$ to be $(t-z)^{-(m+1)}$. Now, if $f \in S'(\mathbb{R})$ and m is a nonnegative integer such that $m \ge \varrho_S(f)$, then $\left(f_t, \widetilde{k}_m(t)\right)$ is well defined. We derived in [9] necessary and sufficient conditions for $f \in S'(\mathbb{R})$ to be a positive definite distribution in terms of this transform. Let us recall that a function $\theta : (a, b) \to \mathbb{R}$, $-\infty \le a < b \le \infty$, is said to be completely monotonic if it is infinitely differentiable and $(-1)^n\theta^{(n)}(y) \ge 0$ for each $y \in (a, b)$ and all $n = 0, 1, 2, \ldots$. Further, $\theta(y)$ is said to be absolutely monotonic on (a, b) if a $\theta(-y)$ is completely monotonic on $(-b, -a)$.

Theorem 1.1. ([9], Theorem 1.3) *Let $f \in S'(\mathbb{R})$ and let n be an integer such that $2n \ge \varrho_f$. Suppose that $a_1, a_2 \in \mathbb{R}$, $a_1 \ne a_2$. Let*

$$\widetilde{K}(f, j)(z) = (-1)^n \frac{i}{\pi}\Big(e^{ia_j t} f_t, (z-t)^{-(2n+1)}\Big),$$

$z \in \mathbb{C}\setminus\mathbb{R}$, $j = 1, 2$. Then f is positive definite if and only if:

*(**i**) $y \to \widetilde{K}(f, j)(iy)$, $j = 1, 2$ are completely monotonic functions for $y \in (0, \infty)$;*

*(**ii**) $y \to -\widetilde{K}(f, j(iy)$, $j = 1, 2$ are absolutely monotonic functions for $y \in (-\infty, 0)$.*

It is quite possible that similar results are also valid for $f \in S'(\mathbb{R}^n)$. Of course, for $n > 1$, it is natural to use the Cauchy kernel K_Γ with respect to a cone Γ in $\mathbb{R}^n$ (see its definition (1.6)). Then the Cauchy transform $K_\Gamma(f)$ of f is defined as the convolution of K_Γ with f. Note that the case of several variables is much more difficult than the one-dimensional case. At first, we do not fully understand how to define the generalized Cauchy kernel $\widetilde{K_\Gamma}$ to get well defined transform $f * \widetilde{K_\Gamma}$ on $S'(\mathbb{R}^n)$. Second, the process of taking boundary values as in (1.4) are investigated only for some proper subclasses of $S'(\mathbb{R}^n)$ (see, e.g., [3]). Finally, we note that the main purpose of this paper is to provide a criterion for a distribution to be positive definite. Therefore, to simplify the technical details, we obtain here a criterion only for compactly supported distributions.

A set $\Gamma \subset \mathbb{R}^n$ is said to be a cone if $x \in \Gamma$ implies $\alpha x \in \Gamma$ for all $\alpha > 0$. The dual cone of Γ is defined by

$$\Gamma^* = \{t \in \mathbb{R}^n : (x, t) \geq 0 \text{ for all } x \in \Gamma\}.$$

The cone Γ^* is always closed convex, and $(\Gamma^*)^* = \overline{\operatorname{ch}\Gamma}$, where $\operatorname{ch}\Gamma$ denotes the convex hull of Γ. Next, Γ is called salient (acute) if $\overline{\operatorname{ch}\Gamma}$ does not contain any straight line in $\mathbb{R}^n$. This is equivalent to $\operatorname{int}(\Gamma^*) \neq \emptyset$. A cone Γ is said to be regular if Γ is an open convex salient cone.

Let $\{\Lambda_j\}_1^m$ be a family of regular cones. We say that $\{\Lambda_j\}_1^m$ covers exactly $\mathbb{R}^n$ if

$$\overline{\cup_{k=1}^m \Lambda_j} = \mathbb{R}^n$$

and the Lebesgue measure of $\overline{\Lambda_i} \cap \overline{\Lambda_j}$ is equal to zero whenever $i \neq j$. Any $\omega = (\omega_1, \dots, \omega_n)$ whose entries ω_k are -1 or 1 defines the cone $Q_\omega = \{x \in \mathbb{R}^n : x_k\omega_k > 0 \text{ for } k = 1, \dots, n\}$. The cone Q_ω is called a quadrant. The collection of all 2^n regular cones $\{Q_\omega\}_\omega$ covers exactly $\mathbb{R}^n$.

Let Γ be an open cone in $\mathbb{R}^n$. Then $T_\Gamma = \mathbb{R}^n + i\Gamma = \{z = x+iy : x \in \mathbb{R}^n,\ y \in \Gamma\}$ is called a tube domain in $\mathbb{C}^n$. If Γ is regular, then the Cauchy kernel of Γ (or with respect to Γ) is defined as

$$K_\Gamma(z) = \int_{\Gamma^*} e^{i(z,\ t)}\, dt, \qquad z \in T_\Gamma. \tag{1.6}$$

The kernel K_Γ is an analytic function on T_Γ [16, p. 143]. If f is a distribution on $\mathbb{R}^n$, then

$$K_\Gamma(f)(z) = \frac{1}{(2\pi)^n}\Big(f(\cdot), K_\Gamma(z - \cdot)\Big) = \frac{1}{(2\pi)^n}\Big(f_t, K_\Gamma(z - t)\Big) \tag{1.7}$$

is called the Cauchy transform of f. For example, in $\mathbb{R}$ there are only two regular cones $(-\infty, 0)$ and $(0, \infty)$. If $\Gamma = (0, \infty)$, then we see that (1.7) coincides with (1.5).

Suppose that Γ is a regular cone in $\mathbb{R}^n$. The directional derivation of a function $\theta : \Gamma \to \mathbb{C}$ along $a = (a_1, \dots, a_n) \in \Gamma$ is defined as usual: $D_a\theta(y) = (a_1 D_{y1} + \cdots + a_n D_{y_n})\theta(y)$. Then θ is called completely monotonic if

$$(-1)^k D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k}\theta(y) \geq 0, \qquad k = 0, 1, \dots,$$

for all $y \in \Gamma$ and all $\gamma_1, \dots, \gamma_k \in \Gamma$ (see [6, p. 172]).

Now we are able to describe positive definite distributions in $\mathcal{E}'(\mathbb{R}^n)$.

Theorem 1.2. *Assume that $\{\Gamma_k\}_{k=1}^m$ is a set of regular cones such that $\{\Gamma_k^*\}_1^m$ covers exactly $\mathbb{R}^n$. A distribution $f \in \mathcal{E}'(\mathbb{R}^n)$ is positive definite if and only if $y \to K_{\Gamma_k}(f)(iy)$, $y \in \Gamma_k$, is a completely monotonic function for each $k = 1, 2, \dots, m$.*

2. Preliminaries and proofs

We define the inverse Fourier transform of a bounded measure μ on $\mathbb{R}^n$ as

$$\check{\mu}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(\xi,t)}\, d\mu(t).$$

In the case if μ has a density $\varphi \in L^1(\mathbb{R}^n)$, then the inverse transform of φ is defined similarly. In addition, the following inversion formula $\widehat{(\check{\varphi})} = \varphi$ holds for suitable functions φ.

Suppose that Λ is a convex salient cone in $\mathbb{R}^n$, and let $S'(\Lambda)$ be the set of all $F \in S'(\mathbb{R}^n)$ supported on Λ. For any fixed $y \in \mathbb{R}^n$, the Laplace transform of $F \in S'(\Lambda)$ is defined by

$$L_y(F)(x) = \mathcal{F}\Big[F(\cdot)e^{-(y,\cdot)}\Big](x) = \mathcal{F}_\xi\Big[F(\xi)e^{-(y,\xi)}\Big](x), \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where $\mathcal{F} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is the Fourier transform defined in (1.3). If $y \in \operatorname{int} \Lambda^*$, then $F(\cdot)e^{-(y,\cdot)}$ belongs to $S'(\mathbb{R}^n)$ (see, e.g., [16, p. 127]). Hence, $L_y(F)(x)$ is well defined for all $x \in \mathbb{R}^n$. Further, $L_y(F)(x)$ is analytic on $T_{\operatorname{int}\Lambda^*}$ as a function of $z = x + iy$, and

$$\frac{\partial^{|u|}}{\partial z_1^{u_1} \dots \partial z_n^{u_n}} L_y(F)(x) = i^{|u|}\mathcal{F}_\xi\Big[(\xi_1^{u_1} \cdots \xi_n^{u_n})F(\xi)e^{-(y,\xi)}\Big](x) \tag{2.2}$$

for each multi-index $u = (u_1, \dots, u_n)$ [16, p. 128].

Let χ_A denote the indicator function of $A \subset \mathbb{R}^n$. If we compare (1.6) and (2.1), then we have that

$$\begin{aligned} K_\Gamma(z) = \int_{\Gamma^*} e^{i(z,\xi)}\, d\xi = \int_{\mathbb{R}^n} \chi_{\Gamma^*}(\xi) e^{i(x,\xi)} e^{-(y,\xi)}\, d\xi = \mathcal{F}_\xi\Big[\chi_{\Gamma^*}(\xi)e^{-(y,\xi)}\Big](x) \\ = L_y\Big(\chi_{\Gamma^*}\Big)(x) \end{aligned}$$

for all $z = x + iy \in T_\Gamma$. This together with (1.7) and (2.2) implies the following lemma, where we collect certain facts about the Cauchy transform, which we need in this section. For a proof of this lemma we refer to [16, p.p. 144-145].

Lemma 2.1. *Let Γ be a regular cone in $\mathbb{R}^n$. The Cauchy kernel $K_\Gamma(z)$ is an analytic function for $z \in T_\Gamma = \mathbb{R}^n + i\Gamma$. If $f \in \mathcal{E}'(\mathbb{R}^n)$, then the Cauchy transform (1.7) is well defined on T_Γ. Moreover, $K_\Gamma(f)$ is analytic on T_Γ and*

$$\frac{\partial^{|u|}}{\partial z_1^{u_1} \dots \partial z_n^{u_n}} K_\Gamma(f)(z) = \frac{1}{(2\pi)^n}\Big(f_t, \frac{\partial^{|u|}}{\partial z_1^{u_1} \dots \partial z_n^{u_n}} K_\Gamma(z-t)\Big) \tag{2.3}$$

for all multi-index $u \in \mathbb{R}^n$.

Recall that a complex-valued function u on $\mathbb{R}^n$ is said to be positive definite if

$$\sum_{j,k=1}^{n} u(x_j - x_k) c_j \bar{c}_k \geq 0 \tag{2.4}$$

for any finite sets $x_1, \dots, x_n \in \mathbb{R}^n$ and for any $c_1, \dots, c_n \in \mathbb{C}$. The Bochner theorem (see, e.g., [2, p. 58], [5, p. 293] and [12, p.p. 41-47]) states that a continuous function $u : \mathbb{R}^n \to \mathbb{C}$ is positive definite if and only if it is the Fourier transform of a positive finite measure μ on $\mathbb{R}^n$. Note that if u is continuous on $\mathbb{R}^n$, then the definition (2.4) is equivalent to

$$\int_{\mathbb{R}^n} u(x)\big(\varphi * \widetilde{\varphi}\big)(x)\, dx \geq 0, \tag{2.5}$$

where φ ranges over $L^1(\mathbb{R}^n)$ (or over all continuous functions on $\mathbb{R}^n$ with compact support). As usual, we identify a locally integrable function u on $\mathbb{R}^n$ with a regular distribution by the formula

$$(u, \varphi) = \int_{\mathbb{R}^n} u(x)\varphi(x)\, dx \tag{2.6}$$

for suitable test functions. Of course, for regular distributions, both definitions (1.1) and (2.5) coincide. Note that any locally bounded measure μ on $\mathbb{R}^n$ also defines in a similar way as in (2.6) an integrable distribution.

We need a few simple facts about positive definite functions. The next lemmas are not new, but their proofs are added here for completeness.

Lemma 2.2. *A distribution $f \in S'(\mathbb{R}^n)$ is positive definite if and only if*

$$(f, \omega) \geq 0 \tag{2.7}$$

for all positive definite $\omega \in D(\mathbb{R}^n)$.

Proof. Assume that both $f \in S'(\mathbb{R}^n)$ and $\omega \in D(\mathbb{R}^n)$ are positive definite. Using the Bochner theorem in $S(\mathbb{R}^n)$ for ω, and the Bochner-Schwartz theorem in $S'(\mathbb{R}^n)$ for f, we have that $\check{\omega}$ is a nonnegative function in $S(\mathbb{R}^n)$ and $\mathcal{F}[f]$ is a nonnegative tempered measure on $\mathbb{R}^n$. Then $(\mathcal{F}[f], \check{\omega})$ may be derived as the usual integral

$$(\mathcal{F}[f], \check{\omega}) = \int_{\mathbb{R}^n} \check{\omega}(x)\, d\big(\mathcal{F}[f](x)\big).$$

Moreover, $(\mathcal{F}[f], \check{\omega}) \geq 0$. Therefore, (1.3) implies that $(f, \omega) = (\mathcal{F}[f], \check{\omega}) \geq 0$.

Let $f \in S'(\mathbb{R}^n)$ and let $\varphi \in D(\mathbb{R}^n)$. The Fourier transform of $\varphi * \widetilde{\varphi}$ is equal to $\hat{\varphi}\overline{\hat{\varphi}} = |\widehat{\varphi}|^2 \geq 0$. Hence, $\varphi * \widetilde{\varphi}$ is positive definite on $\mathbb{R}^n$. If f satisfies (2.7) for all positive definite $\omega \in D(\mathbb{R}^n)$, then we can take $\omega = \varphi * \widetilde{\varphi}$. Thus, (1.1) holds.

Lemma 2.3. *Suppose that $\varphi \in \mathcal{E}(\mathbb{R}^n)$ is positive definite. Then there exists a sequence (ψ_k) of positive definite $\psi_k \in D(\mathbb{R}^n)$, $k = 1, 2, \dots,$ such that $\lim_{k\to\infty} \psi_k = \varphi$ in the topology of $\mathcal{E}(\mathbb{R}^n)$.*

Proof. Take any $\sigma_1 \in D(\mathbb{R}^n)$ such that $\|\sigma_1\|_{L^2(\mathbb{R}^n)} = 1$. Set $\sigma = \sigma_1 * \widetilde{\sigma_1}$. Then σ is positive definite. Hence,

$$|\sigma(x)| \leq \sigma(0) = \|\sigma_1\|_{L^2(\mathbb{R}^n)} = 1, \quad x \in \mathbb{R}^n. \tag{2.8}$$

Now we define the function $\psi_k(x)$ to be $\sigma(x/k)\varphi$ for $k = 1, 2, \dots$. Of course, $\psi_k(x) \in D(\mathbb{R}^n)$ and $\psi_k(x)$ is positive definite as a product of two positive definite functions. We recall that a sequence $\{\theta_j\}_j \in \mathcal{E}(\mathbb{R}^n)$ converges in $\mathcal{E}(\mathbb{R}^n)$ to $\theta \in \mathcal{E}(\mathbb{R}^n)$ if and only if for every multi-index $u \in \mathbb{R}^n$, the sequence $\{D_x^u\theta_j\}_j$ converges uniformly to $D_x^u\theta$ on every compact subset of $\mathbb{R}^n$. By (2.8), it is easy to see that for any fixed multi-index $u \in \mathbb{R}^n$, the sequence $D_x^u\big[\sigma(x/k) - 1\big]$, $k = 1, 2, \dots,$

converges to the zero function as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}^n$. Finally, since

$$\varphi(x) - \psi_k(x) = \varphi(x)\big[\sigma(x/k) - 1\big],$$

we finish the proof.

We are now in a position to prove the necessity of Theorem 1.2.

Proof of Theorem 1.2 (Necessity).

Suppose that Γ is a regular cone in $\mathbb{R}^n$ and let $y \in \Gamma$. If $t \in \mathbb{R}^n$, then using (2.1) and (2.2), we see that

$$D_y^u K_\Gamma(iy - t) = \frac{\partial^{|u|}}{\partial y_1^{u_1} \ldots \partial y_n^{u_n}} K_\Gamma(iy - t) = (-1)^{|u|} \int_{\Gamma^*} (\xi_1^{u_1} \cdots \xi_n^{u_n}) e^{-(y,\ \xi)} e^{-i(t,\xi)}\, d\xi$$

for each multi-index $u \in \mathbb{R}^n$. In particular, if $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma$, then for the directional derivative D_γ of K_Γ, we have

$$\begin{aligned} D_\gamma K_\Gamma(iy - t) &= (\gamma, D_y) K_\Gamma(iy - t) = \sum_{s=1}^n \gamma_s \frac{\partial}{\partial y_s} K_\Gamma(iy - t) \\ &= -\int_{\Gamma^*} \Big(\sum_{s=1}^n \gamma_s \xi_s \Big) e^{-(y,\xi)} e^{-i(t,\xi)}\, d\xi = -\int_{\Gamma^*} (\gamma, \xi) e^{-(y,\xi)} e^{-i(t,\xi)}\, d\xi. \end{aligned} \tag{2.9}$$

Iterating (2.9), we obtain, for an arbitrary set $\gamma^{(1)}, \ldots, \gamma^{(k)} \in \Gamma$, that

$$\begin{aligned} D_{\gamma^{(1)}} D_{\gamma^{(2)}} \ldots D_{\gamma^{(k)}} K_\Gamma(iy - t) &= (-1)^k \int_{\Gamma^*} \prod_{j=1}^k (\gamma^{(j)}, \xi) e^{-(y,\xi)} e^{-i(t,\xi)}\, d\xi \\ &= (-1)^k \int_{\mathbb{R}^n} \Big(\prod_{j=1}^k (\gamma^{(j)}, \xi) e^{-(y,\xi)} \chi_{\Gamma^*}(\xi) \Big) e^{-i(t,\xi)}\, d\xi. \end{aligned} \tag{2.10}$$

For fixed $y \in \Gamma$ and for $\theta \in \Gamma$, we define the function E_θ by

$$E_\theta(\xi) = (\theta, \xi) e^{-(y,\xi)} \chi_{\Gamma^*}(\xi), \qquad \xi \in \mathbb{R}^n.$$

Since Γ is an open cone, it is easy to see that there exists $\delta = \delta(y) > 0$ such that

$$(y, \xi) \geq \delta |\xi|_2 \qquad \text{for all} \quad \xi \in \Gamma^*$$

(see, e.g., [14, p. 104]). Hence, E_θ is a nonnegative bounded and integrable function on $\mathbb{R}^n$. Note that the function

$$\xi \to \prod_{j=1}^k (\gamma^{(j)}, \xi) e^{-(y,\xi)} \chi_{\Gamma^*}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{2.11}$$

which we used in (2.10), is equal to

$$\prod_{j=1}^k E_{\gamma^{(j)}}(\xi).$$

Hence, the function (2.11) is also nonnegative bounded and integrable on $\mathbb{R}^n$. Thus, applying the Bochner theorem to the right side of (2.10), we see that, for any fixed $y \in \Gamma$ and for any choice of $\gamma^{(1)}, \dots, \gamma^{(k)} \in \Gamma$, the function

$$(-1)^k D_{\gamma^{(1)}} D_{\gamma^{(2)}} \dots D_{\gamma^{(k)}} K_\Gamma(iy - t) \tag{2.12}$$

is continuous positive definite as a function of $t \in \mathbb{R}^n$. Moreover, by Lemma 2.1, the function $K_\Gamma(z)$ and its derivative (2.12) are analytic on T_Γ. Hence, (2.12) belongs to $\mathcal{E}(\mathbb{R}^n)$ as a function of t.

Suppose now that $f \in \mathcal{E}'(\mathbb{R}^n)$ and that f is positive definite. Using Lemma 2.3, we see that, for any fixed choice of $y \in \Gamma$, $\gamma^{(1)}, \dots, \gamma^{(k)} \in \Gamma$, there exists a sequence (ψ_m) of positive definite functions $\psi_m \in D(\mathbb{R}^n)$, $m = 1, 2, \dots,$ such that

$$\lim_{m\to\infty} \psi_m(t) = (-1)^k D_{\gamma^{(1)}} D_{\gamma^{(2)}} \dots D_{\gamma^{(k)}} K_\Gamma(iy - t)$$

in the topology of $\mathcal{E}(\mathbb{R}^n)$. Then by Lemma 2.2, we get

$$(-1)^k \Big(f_t,\ D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k} K_\Gamma(iy - t)) \Big) = \lim_{m\to\infty} (f_t, \psi_m(t)) \geq 0.$$

Combining this with (1.7) and (2.3), we see that

$$(-1)^k D_{\gamma^{(1)}} D_{\gamma^{(2)}} \dots D_{\gamma^{(k)}} K_\Gamma(f)(iy) \geq 0.$$

This shows that $y \to K_\Gamma(f)(iy)$ is a completely monotonic function on Γ. Necessity of Theorem 1.2 is proved.

We will use the following lemma (see, e.g., [3, p. 211]), which gives an analytic Cauchy representation for any distribution with compact support.

Lemma 2.4. *Suppose that $\{\Gamma_k\}_1^m$ is a family of regular cones such that $\{\Gamma_k^*\}_1^m$ covers exactly $\mathbb{R}^n$. Let $y^{(k)} \in \Gamma_k$, $k = 1, \dots, m$. If $f \in \mathcal{E}'(\mathbb{R}^n)$, then*

$$\lim_{\max \|y^{(k)}\|_2 \to 0} \sum_{k=1}^m \int_{\mathbb{R}^n} K_{\Gamma_k}(f)(x + iy^{(k)}) \omega(x)\, dx = (f, \omega)$$

for all $\omega \in D(\mathbb{R}^n)$.

Proof of Theorem 1.2. (Sufficiency). Let Λ be a regular cone in $\mathbb{R}^n$. Let $g \in \mathcal{E}'(\mathbb{R}^n)$ and suppose that the function $y \to K_\Lambda(g)(iy)$ is completely monotonic on Λ. We claim that for any fixed $y \in \Gamma$, the function

$$x \to K_\Lambda(g)(x + iy)$$

is continuous and positive definite on $\mathbb{R}^n$. Since Λ is convex, it follows that Λ is an additive semigroup. Fix a point $\delta \in \Lambda$. Because Λ is open, it is easy to see that

$$\delta + \overline{\Lambda} \subset \Lambda. \tag{2.13}$$

Define the function

$$G(y) = K_\Lambda(g)\Big(i(\delta + y)\Big). \tag{2.14}$$

By (2.13), this function is well defined for all $y \in \overline{\Lambda}$. Of course, it is completely monotonic on Λ. Moreover, using (2.13), we see that G is continuous on $\overline{\Lambda}$. Then (see [6, p. 172]) there exists a nonnegative measure $\mu_{\delta,\Lambda}$ on Λ^* such that

$$G(y) = \int_{\Lambda^*} e^{-(y,\zeta)}\, d\mu_{\delta,\Lambda}(\zeta) \tag{2.15}$$

for all $y \in \overline{\Lambda}$. Since $0 \in \overline{\Lambda}$ and G is continuous on $\overline{\Lambda}$, then

$$G(0) = \int_{\Lambda^*} d\mu_{\delta,\Lambda}(\zeta).$$

Hence, $\mu_{\gamma,\Lambda}$ is a finite measure. Therefore, the function G can be extended analytically on T_Λ as the Laplace transform of $\mu_{\gamma,\Lambda}$, i.e., for $z = x + iy \in T_\Gamma$, we can set

$$G(z) = \int_{\Lambda^*} e^{i(z,\zeta)}\, d\mu_{\gamma,\Lambda}(\zeta).$$

Note that this integral converges absolutely.

By (2.14), the function $G(z)$ coincides with $K_\Lambda(g)\Big(i\delta + z\Big)$ for $z = iy$, $y \in \overline{\Lambda}$. We will show that this is true also for all $z \in T_\Lambda$. To this end, we use the following identity theorem (see e.g., [10, p.16-17]): if H is an analytic function on an open connected domain D in $\mathbb{C}^n$, $a \in D$, and $H(a+x) = 0$ for all x in a neighborhood of 0 in $\mathbb{R}^n$, then $H \equiv 0$ on D. Of course, a similar statement is valid also in the case if we replace a real neighborhood of a by any imaginary neighborhood, i.e., if we have $H(a + iy) = 0$ for all y in a neighborhood of 0 in $\mathbb{R}^n$, then also $H \equiv 0$ on D. Now fix any $z_0 = iy_0 \in i\Lambda \subset T_\Lambda$. Then using (2.13)and (2.14), we see that $G(z)$ and $K_\Lambda(g)(i\gamma + z)$ coincide for all z in an imaginary neighborhood $I_{z_0} = \{z = x + iy \in \mathbb{C}^n :\ |y - y_0| < r,\ x = x_0\}$ of z_0 such that $I_{z_0} \subset T_\Lambda$. Hence, $G(z) = K_\Lambda(g)\Big(i\delta + z\Big)$ for all $z \in T_\Lambda$. Moreover, by (2.14) and (2.15), we have that

$$K_\Lambda(g)(i\delta + z) = G(z) = \int_{\Lambda^*} e^{i(z,\zeta)}\, d\mu_{\gamma,\Lambda}(\zeta) = \int_{\Lambda^*} e^{i(x,\zeta)} e^{-(y,\zeta)}\, d\mu_{\gamma,\Lambda}(\zeta) \qquad (2.16)$$

for all $z = x + iy \in T_\Lambda$. Using (2.16) and having the Bochner theorem for continuous positive definite functions on $\mathbb{R}^n$, we obtain that for any fixed $y \in \Gamma$, the functions

$$x \to G(x + iy) \qquad \text{and} \qquad x \to K_\Lambda(g)(x + i(\delta + y))$$

are continuous and positive definite on $\mathbb{R}^n$. Thus, since Λ is open and δ is an arbitrary point of Λ, we obtain that the function $x \to K_\Lambda(g)(x + iy)$ also is continuous and positive definite on $\mathbb{R}^n$. This proves our claim.

Let $\{\Gamma_k\}_{k=1}^m$ be as in the Theorem 1.2 and suppose that for $f \in \mathcal{E}'(\mathbb{R}^n)$, the functions $y \to K_{\Gamma_k}(f)(iy)$, $y \in \Gamma_k$, $k = 1, 2, \dots, m$ are completely monotonic. Fix $y^{(1)} \in \Gamma_1, \dots, y^{(m)} \in \Gamma_m$, and define

$$F(x) = \sum_{k=1}^{m} K_{\Gamma_k}(f)(x + iy^{(k)}) \qquad (2.17)$$

for $x \in \mathbb{R}^n$. We just proved that each $x \to K_{\Gamma_k}(f)(x + iy^{(k)})$, $k = 1, \dots, m$, is a continuous and positive definite function on $\mathbb{R}^n$. Hence, the same is still true for (2.17). If $\omega \in D(\mathbb{R}^n)$, then $F \cdot \omega$ is integrable on $\mathbb{R}^n$, and by Lemma 2.2, we have that

$$\sum_{k=1}^{m} \int_{\mathbb{R}^n} K_{\Gamma_k}(f)(x + iy^{(k)})\omega(x)\, dx = \int_{\mathbb{R}^n} F(x)\omega(x)\, dx \geq 0. \qquad (2.18)$$

Letting now $\max \|y^{(k)}\|_2 \to 0$. Then (2.18) this, together with Lemma 2.4, proves that $(f, \omega) \geq 0$ for all $\omega \in D(\mathbb{R}^n)$. Thus, f is a positive definite distribution.

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