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THE POLAR DECOMPOSITION FOR ADJOINTABLE OPERATORS ON HILBERT C^*-MODULES AND CENTERED OPERATORS

NA LIU, WEI LUO, and QINGXIANG XU*

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ABSTRACT. Let T be an adjointable operator between two Hilbert C^*-modules, and let T^* be the adjoint operator of T. The polar decomposition of T is characterized as $T = U(T^*T)^{\frac{1}{2}}$ and $\mathcal{R}(U^*) = \overline{\mathcal{R}(T^*)}$, where U is a partial isometry, $\mathcal{R}(U^*)$ and $\overline{\mathcal{R}(T^*)}$ denote the range of U^* and the norm closure of the range of T^*, respectively. Based on this new characterization of the polar decomposition, an application to the study of centered operators is carried out.

1. INTRODUCTION

Much progress has been made in the study of the polar decomposition for Hilbert space operators [3, 5, 6, 7, 15]. Let H and K be two Hilbert spaces, and let $\mathbb{B}(H,K)$ be the set of bounded linear operators from H to K. For any $T \in \mathbb{B}(H,K)$, let T^*, $\mathcal{R}(T)$, and $\mathcal{N}(T)$ denote the conjugate operator, the range, and the null space of T, respectively. It is well-known [6, 7] that every operator $T \in \mathbb{B}(H,K)$ has the unique polar decomposition

$$T = U|T| \text{ and } \mathcal{N}(T) = \mathcal{N}(U), \tag{1.1}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ and $U \in \mathbb{B}(H,K)$ is a partial isometry. An alternative expression of (1.1) is

$$T = U|T| \text{ and } \overline{\mathcal{R}(T^*)} = \mathcal{R}(U^*), \tag{1.2}$$

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*Corresponding author .
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since $\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}$ and $\mathcal{N}(U)^\perp = \overline{\mathcal{R}(U^*)} = \mathcal{R}(U^*)$ in the Hilbert space case. Note that if $H = K$, then $\mathbb{B}(H, H)$ abbreviated to $\mathbb{B}(H)$, is a von Neumann algebra. It follows from [14, Proposition 2.2.9] that the polar decomposition also works for elements in a von Neumann algebra. Nevertheless, it may be false for some elements in a general C^*-algebra; see [14, Remark 1.4.6].

Both Hilbert spaces and C^*-algebras can be regarded as Hilbert C^*-modules, so one might study the polar decomposition in the general setting of Hilbert C^*-modules. An adjointable operator between Hilbert C^*-modules may have no polar decomposition unless some additional conditions are satisfied; see Lemma 3.5 below for the details. The polar decomposition for densely defined closed operators and unbounded operators is also considered in some literatures; see [2, 4, 5], for example.

The purpose of this paper is, in the general setting of adjointable operators on Hilbert C^*-modules, to provide a new insight into the polar decomposition theory and its applications. We will prove in Lemma 3.6 that five equalities appearing in Lemma 3.5 (iii) can be in fact simplified to two equalities described in (3.2), which are evidently the same as that in (1.2) when the underlying spaces are Hilbert spaces. It is remarkable that (1.1) is a widely used characterization of the polar decomposition for Hilbert space operators. Nevertheless, Example 3.15 indicates that such a characterization of the polar decomposition is no longer true for adjointable operators on Hilbert C^*-modules. This leads us to figure out a modified version of (1.1), which is stated in Theorem 3.13. Note that the verification of the equivalence of Lemma 3.5 (i) and (ii) is trivial, so it is meaningful to give another interpretation of Lemma 3.5 (ii). We have managed to do that in Theorem 3.8 (iii).

One application of the polar decomposition is the study of centered operators on Hilbert spaces, which was initiated in [12] and generalized in [8, 9]. Based on the new characterization (3.2) of the polar decomposition for adjointable operators, some generalizations on centered operators are made in the framework of Hilbert C^*-modules.

The paper is organized as follows. Some elementary results on adjointable operators are provided in Section 2. In Section 3, we focus on the study of the polar decomposition for adjointable operators on Hilbert C^*-modules. As an application of the polar decomposition, centered operators are studied in Section 4.

2. Some elementary results on adjointable operators

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing inner products to take values in some C^*-algebras instead of the complex field. Let $\mathfrak{A}$ be a C^*-algebra. An inner-product $\mathfrak{A}$-module is a linear space E, which is a right $\mathfrak{A}$-module, together with a map $(x, y) \to \big\langle x, y\big\rangle : E \times E \to \mathfrak{A}$ such that for any $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$, the following conditions hold:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$;
(iv) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$.

An inner-product $\mathfrak{A}$-module E, which is complete with respect to the induced norm ($\|x\| = \sqrt{\|\langle x, x\rangle\|}$ for $x \in E$), is called a (right) Hilbert $\mathfrak{A}$-module.

Suppose that H and K are two Hilbert $\mathfrak{A}$-modules; let $\mathcal{L}(H, K)$ be the set of operators $T : H \to K$ for which there is an operator $T^* : K \to H$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \text{for any } x \in H \text{ and } y \in K.$$

We call $\mathcal{L}(H, K)$ the set of adjointable operators from H to K. For any $T \in \mathcal{L}(H, K)$, the range and the null space of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively. In case $H = K$, $\mathcal{L}(H, H)$, which is abbreviated to $\mathcal{L}(H)$, is a C^*-algebra. Let $\mathcal{L}(H)_{sa}$ and $\mathcal{L}(H)_+$ be the set of self-adjoint elements and positive elements in $\mathcal{L}(H)$, respectively.

Definition 2.1. A closed submodule M of a Hilbert $\mathfrak{A}$-module E is said to be orthogonally complemented, if $E = M \dot{+} M^\perp$, where

$$M^\perp = \big\{x \in E : \langle x, y\rangle = 0 \text{ for any } y \in M\big\}.$$

In this case, the projection from H onto M is denoted by P_M.

Throughout the rest of this paper, $\mathfrak{A}$ is a C^*-algebra and E, H, and K are three Hilbert $\mathfrak{A}$-modules. Note that $\mathcal{L}(H)$ is a C^*-algebra; so we begin with an elementary result on C^*-algebras.

Definition 2.2. Let $\mathfrak{B}$ be a C^*-algebra. The set of positive elements of $\mathfrak{B}$ is denoted by $\mathfrak{B}_+$. For any $a, b \in \mathfrak{B}$, let $[a, b] = ab - ba$ be the commutator of a and b.

Proposition 2.3. *Let $\mathfrak{B}$ be a C^*-algebra , and let $a, b \in \mathfrak{B}$ be such that $a = a^*$ and $[a, b] = 0$. Then $[f(a), b] = 0$ whenever f is a continuous complex-valued function on the interval $[-\|a\|, \|a\|]$.*

Proof. We might as well assume that $\mathfrak{B}$ has a unit. Choose any sequence $\{p_n\}_{n=1}^\infty$ of polynomials such that $p_n(t) \to f(t)$ uniformly on the interval $[-\|a\|, \|a\|]$. Then $\|p_n(a) - f(a)\| \to 0$ as $n \to \infty$; hence

$$f(a)b = \lim_{n\to\infty} p_n(a)b = \lim_{n\to\infty} b\,p_n(a) = bf(a). \qquad \square$$

Next, we state some elementary results on the commutativity of adjointable operators. For any $\alpha > 0$, the function $f(t) = t^\alpha$ is continuous on $[0, +\infty)$; so a direct application of Proposition 2.3 yields the following proposition.

Proposition 2.4. *Let $S \in \mathcal{L}(H)$ and $T \in \mathcal{L}(H)_+$ be such that $[S, T] = 0$. Then $[S, T^\alpha] = 0$ for any $\alpha > 0$.*

The technical result of this section is as follows:

Proposition 2.5. *Let $T \in \mathcal{L}(H)_+$ be such that $\overline{\mathcal{R}(T)}$ is orthogonally complemented. Then*

$$\lim_{n\to\infty} \|T_n x - P_{\overline{\mathcal{R}(T)}} x\| = 0, \qquad \text{for all } x \text{ in } H, \tag{2.1}$$

where $T_n = \left(\frac{1}{n} I + T\right)^{-1} T$ for each $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, the continuous function f_n associated to the operator T_n is given by

$$f_n(t) = \frac{t}{\frac{1}{n} + t}, \qquad \text{for } t \in \mathrm{sp}(T) \subseteq [0, \|T\|],$$

where $\mathrm{sp}(T)$ is the spectrum of T. Then for each $n \in \mathbb{N}$,

$$\|T_n\| = \max\left\{\left|f_n(t)\right| : t \in \mathrm{sp}(T)\right\} \leq 1;$$
$$\|T_nT - T\| = \max\left\{\left|tf_n(t) - t\right| : t \in \mathrm{sp}(T)\right\} \leq \frac{1}{n}.$$

Now, given any $x \in H$ and any $\varepsilon > 0$, let $x = u + v$, where $u \in \overline{\mathcal{R}(T)}$ and $v \in \mathcal{N}(T) \subseteq \mathcal{N}(T_n)$ for any $n \in \mathbb{N}$. Choose $h \in H$ and $n_0 \in \mathbb{N}$ such that

$$\|u - Th\| < \frac{\varepsilon}{3} \text{ and } n_0 > \frac{3(\|h\| + 1)}{\varepsilon}.$$

Then for any $n \in \mathbb{N}$ with $n \geq n_0$, we have

$$\begin{aligned}
\|T_nx - P_{\overline{\mathcal{R}(T)}}x\| &= \|T_nu - P_{\overline{\mathcal{R}(T)}}u\| \\
&\leq \|T_nu - T_nTh\| + \|T_nTh - P_{\overline{\mathcal{R}(T)}}Th\| + \|P_{\overline{\mathcal{R}(T)}}Th - P_{\overline{\mathcal{R}(T)}}u\| \\
&\leq \|T_n(u - Th)\| + \|(T_nT)h - Th\| + \|P_{\overline{\mathcal{R}(T)}}(Th - u)\| \\
&\leq \|u - Th\| + \frac{1}{n}\|h\| + \|Th - u\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This completes the proof of (2.1). ☐

Based on Proposition 2.5, a result on the commutativity for adjointable operators can be provided as follows:

Proposition 2.6. *Let $S \in \mathcal{L}(H)$ and let $T \in \mathcal{L}(H)_+$ be such that $\overline{\mathcal{R}(T)}$ is orthogonally complemented. If $[S, T] = 0$, then $\left[S, P_{\overline{\mathcal{R}(T)}}\right] = 0$.*

Proof. Denote $P_{\overline{\mathcal{R}(T)}}$ simply by P. Since $[S, T] = 0$, we have $[S, T_n] = 0$, where T_n $(n \in \mathbb{N})$ are given in Proposition 2.5. It follows from (2.1) that

$$P(Sx) = \lim_{n\to\infty} T_n(Sx) = \lim_{n\to\infty} S(T_nx) = S(Px) \text{ for any } x \in H. \quad \square$$

We end this section by stating some range equalities for adjointable operators.

Proposition 2.7. *Let $A \in \mathcal{L}(H, K)$ and $B, C \in \mathcal{L}(E, H)$ be such that $\overline{\mathcal{R}(B)} = \overline{\mathcal{R}(C)}$. Then $\overline{\mathcal{R}(AB)} = \overline{\mathcal{R}(AC)}$.*

Proof. Let $x \in E$ be arbitrary. Since $Bx \in \overline{\mathcal{R}(C)}$, there exists a sequence $\{x_n\}$ in E such that $Cx_n \to Bx$. Then $ACx_n \to ABx$, which means $ABx \in \overline{\mathcal{R}(AC)}$, and thus $\mathcal{R}(AB) \subseteq \overline{\mathcal{R}(AC)}$ and furthermore $\overline{\mathcal{R}(AB)} \subseteq \overline{\mathcal{R}(AC)}$. Similarly, we have $\overline{\mathcal{R}(AC)} \subseteq \overline{\mathcal{R}(AB)}$. ☐

Lemma 2.8. [17, Lemma 2.3] *Let $T \in \mathcal{L}(H)_+$. Then $\overline{\mathcal{R}(T^\alpha)} = \overline{\mathcal{R}(T)}$ for any $\alpha \in (0, 1)$.*

Proposition 2.9. *Let $T \in \mathcal{L}(H)_+$. Then $\overline{\mathcal{R}(T^\alpha)} = \overline{\mathcal{R}(T)}$ for any $\alpha > 0$.*

Proof. In view of Lemma 2.8, we might as well assume that $\alpha > 1$. Put $S = T^\alpha$. Then $S \in \mathcal{L}(H)_+$, so from Lemma 2.8 we have

$$\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(S^{\frac{1}{\alpha}})} = \overline{\mathcal{R}(S)} = \overline{\mathcal{R}(T^\alpha)}. \quad \square$$

3. The polar decomposition for adjointable operators

In this section, we study the polar decomposition for adjointable operators on Hilbert C^*-modules.

Definition 3.1. Recall that an element U of $\mathcal{L}(H,K)$ is said to be a partial isometry, if U^*U is a projection in $\mathcal{L}(H)$.

Proposition 3.2. [17, Lemma 2.1] *Let $U \in \mathcal{L}(H,K)$ be a partial isometry. Then U^* is also a partial isometry, which satisfies $UU^*U = U$.*

Lemma 3.3. [11, Proposition 3.7] *Let $T \in \mathcal{L}(H,K)$. Then $\overline{\mathcal{R}(T^*T)} = \overline{\mathcal{R}(T^*)}$ and $\overline{\mathcal{R}(TT^*)} = \overline{\mathcal{R}(T)}$.*

Definition 3.4. For any $T \in \mathcal{L}(H,K)$, let $|T|$ denote the square root of T^*T. That is, $|T| = (T^*T)^{\frac{1}{2}}$ and $|T^*| = (TT^*)^{\frac{1}{2}}$.

Lemma 3.5. [16, Proposition 15.3.7] *Let $T \in \mathcal{L}(E)$. Then the following statements are equivalent:*

(i) *$E = \mathcal{N}(|T|) \oplus \overline{\mathcal{R}(|T|)}$ and $E = \mathcal{N}(T^*) \oplus \overline{\mathcal{R}(T)}$;*
(ii) *Both $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(|T|)}$ are orthogonally complemented;*
(iii) *T has the polar decomposition $T = U|T|$, where $U \in \mathcal{L}(E)$ is a partial isometry such that*

$$\begin{aligned} &\mathcal{N}(U) = \mathcal{N}(T), \mathcal{N}(U^*) = \mathcal{N}(T^*), \\ &\mathcal{R}(U) = \overline{\mathcal{R}(T)}, \mathcal{R}(U^*) = \overline{\mathcal{R}(T^*)}. \end{aligned} \tag{3.1}$$

Lemma 3.6. *Let $T \in \mathcal{L}(H,K)$ be such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented, and let $U \in \mathcal{L}(H,K)$ be a partial isometry such that*

$$T = U|T| \text{ and } U^*U = P_{\overline{\mathcal{R}(T^*)}}. \tag{3.2}$$

Then $\overline{\mathcal{R}(T)}$ is also orthogonally complemented, and all equations in (3.1) are satisfied. Furthermore, the following equations are also valid:

$$T^* = U^*|T^*| \text{ and } UU^* = P_{\overline{\mathcal{R}(T)}}, \tag{3.3}$$

$$|T^*| = U|T|U^* \text{ and } U|T| = |T^*|U. \tag{3.4}$$

Proof. By Proposition 2.9 and Lemma 3.3, we have

$$\overline{\mathcal{R}(|T|)} = \overline{\mathcal{R}(T^*T)} = \overline{\mathcal{R}(T^*)} = \mathcal{R}(U^*U), \tag{3.5}$$

which gives by Proposition 2.7 that

$$\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(U|T|)} = \overline{\mathcal{R}(UU^*U)} = \mathcal{R}(UU^*);$$

hence $\overline{\mathcal{R}(T)}$ is orthogonally complemented such that the second equation in (3.3) is satisfied. Furthermore,

$$TT^* = U|T| \cdot |T|U^* = (U|T|U^*)^2;$$

hence the first equation in (3.4) is satisfied. As a result,

$$U^*|T^*| = (U^*U|T|)U^* = |T|U^* = (U|T|)^* = T^*,$$
$$U|T| = T = (T^*)^* = (U^*|T^*|)^* = |T^*|U.$$

This completes the proof of (3.3) and (3.4). Finally, equations stated in (3.1) can be derived directly from the second equations in (3.2) and (3.3), respectively. $\square$

Lemma 3.7. [17, Theorem 3.1] *Let $T \in \mathcal{L}(H,K)$ be such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented. If $\mathcal{R}(|T^*|) \subseteq \mathcal{R}(T)$, then the following statements are valid:*

(i) $\mathcal{R}(|T^*|) = \mathcal{R}(T)$;
(ii) $\mathcal{R}(|T|) = \mathcal{R}(T^*)$;
(iii) $\overline{\mathcal{R}(T)}$ *is orthogonally complemented.*

Theorem 3.8. *Let $T \in \mathcal{L}(H,K)$. Then the following statements are equivalent:*

(i) $\overline{\mathcal{R}(T)}$ *and* $\overline{\mathcal{R}(T^*)}$ *are both orthogonally complemented;*
(ii) $\overline{\mathcal{R}(T^*)}$ *is orthogonally complemented and* (3.2) *is satisfied for some partial isometry $U \in \mathcal{L}(H,K)$;*
(iii) $\overline{\mathcal{R}(T^*)}$ *is orthogonally complemented, $\mathcal{R}(|T|) = \mathcal{R}(T^*)$ and $\mathcal{R}(|T^*|) = \mathcal{R}(T)$.*

Proof. The implications of (ii)$\Longrightarrow$(i) and (iii)$\Longrightarrow$(i) follow from Lemmas 3.6 and 3.7, respectively.

"(i) $\Longrightarrow$ (ii)": Let $E = H \oplus K$ and $\widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in \mathcal{L}(E)$. Then both $\overline{\mathcal{R}(\widetilde{T})}$ and $\overline{\mathcal{R}(\widetilde{T}^*)}$ are orthogonally complemented; hence by Lemma 3.5 there exists a partial isometry $\widetilde{U} = \begin{pmatrix} U_{11} & U_{12} \\ U & U_{22} \end{pmatrix} \in \mathcal{L}(E)$ such that

$$\widetilde{T} = \widetilde{U}|\widetilde{T}|, \mathcal{R}(\widetilde{U}) = \overline{\mathcal{R}(\widetilde{T})} = \{0\} \oplus \overline{\mathcal{R}(T)} \text{ and } \mathcal{R}(\widetilde{U}^*) = \overline{\mathcal{R}(\widetilde{T}^*)} = \overline{\mathcal{R}(T^*)} \oplus \{0\},$$

which leads to $\widetilde{U} = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}$; hence U is a partial isometry satisfying (3.2).

"(ii) $\Longrightarrow$ (iii)": By (3.2)–(3.4), we have

$$T^* = (U|T|)^* = |T|U^* \text{ and } T^*U = U^*|T^*| \cdot U = U^* \cdot U|T|U^* \cdot U = |T|,$$

which obviously lead to $\mathcal{R}(|T|) = \mathcal{R}(T^*)$. Replacing T and U, respectively, with T^* and U^*, we obtain $\mathcal{R}(|T^*|) = \mathcal{R}(T)$. $\square$

Lemma 3.9. *Let $T \in \mathcal{L}(H,K)$ be such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented. If $U, V \in \mathcal{L}(H,K)$ are given such that $U|T| = V|T|$ and $U^*U = V^*V = P_{\overline{\mathcal{R}(T^*)}}$, then $U = V$.*

Proof. The equation $U|T| = V|T|$ together with (3.5) yields $UP_{\overline{\mathcal{R}(T^*)}} = VP_{\overline{\mathcal{R}(T^*)}}$; hence

$$U = U(U^*U) = UP_{\overline{\mathcal{R}(T^*)}} = VP_{\overline{\mathcal{R}(T^*)}} = V(V^*V) = V. \quad \square$$

Definition 3.10. The polar decomposition of $T\in\mathcal{L}(H,K)$ can be characterized as

$$T=U|T| \text{ and } U^*U=P_{\overline{\mathcal{R}(T^*)}},\tag{3.6}$$

where $U\in\mathcal{L}(H,K)$ is a partial isometry.

Remark 3.11. It follows from Theorem 3.8 and Lemma 3.9 that $T\in\mathcal{L}(H,K)$ has the (unique) polar decomposition if and only if $\overline{\mathcal{R}(T^*)}$ and $\overline{\mathcal{R}(T)}$ are both orthogonally complemented. In this case, $T^*=U^*|T^*|$ is the polar decomposition of T^*.

A slight generalization of (3.4) is as follows:

Lemma 3.12. *Let $T=U|T|$ be the polar decomposition of $T\in\mathcal{L}(H,K)$. Then for any $\alpha>0$, the following statements are valid:*

(i) $U|T|^\alpha U^*=(U|T|U^*)^\alpha=|T^*|^\alpha$;
(ii) $U|T|^\alpha=|T^*|^\alpha U$;
(iii) $U^*|T^*|^\alpha U=(U^*|T^*|U)^\alpha=|T|^\alpha$.

Proof. (i) Since $U^*U|T|=|T|$, we have

$$\big(U|T|U^*\big)^n=U|T|^nU^* \qquad \text{for any } n\in\mathbb{N}.\tag{3.7}$$

Let $f(t)=t^\alpha$, and choose any sequence $\{P_m\}_{m=1}^\infty$ of polynomials such that $P_m(0)=0$ ($\forall m\in\mathbb{N}$), and $P_m(t)\to f(t)$ uniformly on the interval $\big[0,\big\||T|\big\|\big]$. Then from (3.4) and (3.7), we have

$$\begin{aligned} U|T|^\alpha U^* &= U\,f(|T|)U^*=\lim_{m\to\infty}UP_m(|T|)U^*=\lim_{m\to\infty}P_m\big(U|T|U^*\big)\\ &= f\big(U|T|U^*\big)=(U|T|U^*)^\alpha=|T^*|^\alpha. \end{aligned}$$

(ii) By Proposition 2.9 and Lemma 3.3, we have

$$\overline{\mathcal{R}(|T|^\alpha)}=\overline{\mathcal{R}(T^*T)}=\overline{\mathcal{R}(T^*)},$$

and thus $U^*U|T|^\alpha=|T|^\alpha$. Taking $*$-operation, we get $|T|^\alpha=|T|^\alpha U^*U$. It follows from (i) that

$$U|T|^\alpha=U\big(|T|^\alpha U^*U\big)=\big(U|T|^\alpha U^*\big)U=|T^*|^\alpha U.$$

(iii) Since $T^*=U^*|T^*|$ is the polar decomposition of T^*, the conclusion follows immediately from (i) by replacing the pair (U,T) with (U^*,T^*). □

Before ending this section, we provide a criteria for the polar decomposition as follows:

Theorem 3.13. *Let $T\in\mathcal{L}(H,K)$ be such that $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented. Let $U\in\mathcal{L}(H,K)$ be a partial isometry which satisfies*

$$T=U|T| \text{ and } \mathcal{N}(T)\subseteq\mathcal{N}(U).\tag{3.8}$$

Then $T=U|T|$ is the polar decomposition of T.

Proof. By assumption, $Q = U^*U$ is a projection. For any $x \in H$, we have

$$\langle |T|x, |T|x\rangle = \langle Tx, Tx\rangle = \langle U|T|x, U|T|x\rangle = \langle Q|T|x, |T|x\rangle,$$

and thus

$$\big\|(I-Q)|T|x\big\|^2 = \big\|\langle (I-Q)|T|x, |T|x\rangle\big\| = 0;$$

hence $Q|T| = |T|$. It follows that $\overline{\mathcal{R}(T^*)} = \overline{\mathcal{R}(|T|)} \subseteq \mathcal{R}(Q)$. On the other hand, by assumption we have

$$\overline{\mathcal{R}(T^*)} = \mathcal{N}(T)^\perp \supseteq \mathcal{N}(U)^\perp = \mathcal{N}(Q)^\perp = \mathcal{R}(Q);$$

hence $\overline{\mathcal{R}(T^*)} = \mathcal{R}(Q)$, and thus $Q = P_{\overline{\mathcal{R}(T^*)}}$. □

Remark 3.14. Let $T \in \mathcal{L}(H, K)$, where H and K are both Hilbert spaces. In this case $\overline{\mathcal{R}(T^*)}$ is always orthogonally complemented, so if U is a partial isometry such that (3.8) is satisfied, then $U|T|$ is exactly the polar decomposition of T.

Unlike the assertion given in [10, P. 3400], the same is not true for general Hilbert C^*-modules H and K, since $\overline{\mathcal{R}(T^*)}$ can be not orthogonally complemented for some $T \in \mathcal{L}(H, K)$. Indeed, there exist a Hilbert C^*-module H, and an adjointable T and a partial isometry U on H such that (1.1) is satisfied, whereas T has no polar decomposition. Such an example is as follows:

Example 3.15. Let H be any countably infinite-dimensional Hilbert space, and let $\mathcal{L}(H)$ and $\mathcal{C}(H)$ be the set of bounded linear operators and compact operators on H, respectively. Given any orthogonal normalized basis $\{e_n : n \in \mathbb{N}\}$ for H, let $S \in \mathcal{C}(H)$ be defined by

$$S(e_n) = \frac{1}{n} e_n \qquad \text{for any } n \in \mathbb{N}.$$

Clearly, S is a positive element in $\mathcal{C}(H)$. Let $K = \mathfrak{A} = \mathcal{L}(H)$. With the inner product given by

$$\big\langle X, Y\big\rangle = X^*Y \text{ for any } X, Y \in K,$$

K is a Hilbert $\mathfrak{A}$-module.

Let $T : K \to K$ be defined by $T(X) = SX$ for any $X \in K$. Clearly, $T \in \mathcal{L}(K)_+$ and $\mathcal{R}(T) \subseteq \mathcal{C}(H)$. Given any $n \in \mathbb{N}$, let P_n be the projection from H onto the linear subspace spanned by $\{e_1, e_2, \cdots, e_n\}$. Let $X_n \in K$ be defined by

$$X_n(e_j) = \begin{cases} je_j, & \text{if } 1 \le j \le n, \\ 0, & \text{otherwise}. \end{cases}.$$

It is obvious that $T(X_n) = P_n$, which implies that $\overline{\mathcal{R}(T)} = \mathcal{C}(H)$, hence $\overline{\mathcal{R}(T)}^\perp = \{0\}$, therefore $\overline{\mathcal{R}(T)}$ fails to be orthogonally complemented. By Remark 3.11, we conclude that T has no polar decomposition. Furthermore, given any $X \in K$ such that $T(X) = SX = 0$, then $X = 0$ since S is injective. It follows that $\mathcal{N}(T) = \{0\}$.

Now, let U be the identity operator on K. Then since T is positive, we have $T = U|T|$ and $\mathcal{N}(U) = \mathcal{N}(T)$, whereas T has no polar decomposition.

4. Characterizations of centered operators

In this section, we study centered operators in the general setting of Hilbert C^*-modules.

Definition 4.1. [12] An element $T \in \mathcal{L}(H)$ is said to be centered if the following sequence

$$\ldots, T^3(T^3)^*, T^2(T^2)^*, TT^*, T^*T, (T^2)^*T^2, (T^3)^*T^3, \ldots$$

consists of mutually commuting operators.

We begin with a cancellation law introduced in [8, Lemma 3.7]. Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(H)$. Suppose that $n \in \mathbb{N}$ is given such that

$$\big[U^k|T|(U^k)^*, |T|\big] = 0 \qquad \text{for } 1 \le k \le n. \tag{4.1}$$

Then by Proposition 2.6, we have

$$\big[U^k|T|(U^k)^*, U^*U\big] = 0 \qquad \text{for } 1 \le k \le n; \tag{4.2}$$

hence for any $s, t \in \mathbb{N}$ with $1 \le t \le s \le n+1$, we have

$$U^s|T|(U^s)^*U^t = U^s|T|(U^{s-t})^* \text{ and } (U^t)^*U^s|T|(U^s)^* = U^{s-t}|T|(U^s)^*. \tag{4.3}$$

Indeed, by (4.1) and (4.2), we have

$$\begin{aligned} U^s|T|(U^s)^*U^t &= U \cdot U^{s-1}|T|(U^{s-1})^* \cdot U^*U \cdot U^{t-1} \\ &= U \cdot U^*U \cdot U^{s-1}|T|(U^{s-1})^* \cdot U^{t-1} \\ &= U \cdot U^{s-1}|T|(U^{s-1})^* \cdot U^{t-1} \\ &= U^2 \cdot U^{s-2}|T|(U^{s-2})^* \cdot U^{t-2} \\ &= \cdots = U^s|T|(U^{s-t})^*. \end{aligned}$$

Taking $*$-operation, we get the second equation in (4.3).

Now we are ready to state the technical lemma of this section, which is a modification of [8, Lemma 4.2].

Lemma 4.2. *Suppose that $T = U|T|$ is the polar decomposition of $T \in \mathcal{L}(H)$. Let $n \in \mathbb{N}$ be given such that*

$$\big[U^k|T|(U^k)^*, |T^l|\big] = 0 \quad \textit{for any } k, l \in \mathbb{N} \textit{ with } k + l \le n+1. \tag{4.4}$$

Then the following statements are equivalent:

(i) $\big[U^s|T|(U^s)^*, |T^t|\big] = 0$ *for some $s, t \in \mathbb{N}$ with $s + t = n + 2$;*
(ii) $\big[U^s|T|(U^s)^*, |T^t|\big] = 0$ *for any $s, t \in \mathbb{N}$ with $s + t = n + 2$.*

Proof. Let $s, t \in \mathbb{N}$ be such that $s + t = n + 2$. Put

$$A_t = |T^t|^2 \cdot U^s|T|(U^s)^* \text{ and } B_t = U^s|T|(U^s)^* \cdot |T^t|^2. \tag{4.5}$$

Then clearly,

$$\big[U^s|T|(U^s)^*, |T^t|\big] = 0 \Longleftrightarrow A_t = B_t. \tag{4.6}$$

Substituting $k = 1$ and $U|T|U^* = |T^*|$ into (4.4) yields

$$\big[|T^*|, |T^l|\big] = 0, \qquad \text{for } 1 \le l \le n,$$

which leads by Proposition 2.6 to

$$\big[UU^*, |T^l|\big] = 0 \qquad \text{for } 1 \le l \le n. \tag{4.7}$$

Note that $t = n + 2 - s \le n + 1$; so if $t \ge 2$, then by (4.4),

$$\begin{aligned} |T^t|^2 &= T^* \cdot (T^{t-1})^* T^{t-1} \cdot T = |T|U^* \cdot |T^{t-1}|^2 \cdot U|T| \\ &= U^* \cdot U|T|U^* \cdot |T^{t-1}|^2 \cdot U|T| = U^* \cdot |T^{t-1}|^2 \cdot U|T|U^* \cdot U|T| \\ &= U^* \cdot |T^{t-1}|^2 \cdot U|T|^2. \end{aligned} \tag{4.8}$$

Assume now that $t \ge 2$. Then $s + 1 = (n + 2 - t) + 1 \le n + 1$; hence by (4.5), (4.8), (4.4) with $l = 1$, and (4.3), we have

$$\begin{aligned} A_t &= U^*|T^{t-1}|^2U \cdot |T|^2 \cdot U^s|T|(U^s)^* \\ &= U^*|T^{t-1}|^2U \cdot U^s|T|(U^s)^* \cdot |T|^2 \qquad (4.9) \\ &= U^* \cdot A_{t-1} \cdot U|T|^2. \qquad (4.10) \end{aligned}$$

Similarly,

$$\begin{aligned} B_t &= U^s|T|(U^s)^* \cdot U^*UU^*|T^{t-1}|^2U|T|^2 \\ &= U^*U \cdot U^s|T|(U^s)^*U^*|T^{t-1}|^2U|T|^2 \\ &= U^* \cdot U^{s+1}|T|(U^{s+1})^*|T^{t-1}|^2 \cdot U|T|^2 \\ &= U^* \cdot B_{t-1} \cdot U|T|^2. \end{aligned} \tag{4.11}$$

It follows from (4.10) and (4.11) that $A_t = B_t$ whenever $A_{t-1} = B_{t-1}$. Suppose on the contrary that $A_t = B_t$. Then by (4.9), (4.7), and (4.5), we have

$$\begin{aligned} UA_t &= UU^* \cdot |T^{t-1}|^2U \cdot U^s|T|(U^s)^* \cdot |T|^2 = |T^{t-1}|^2 \cdot UU^*U \cdot U^s|T|(U^s)^*|T|^2 \\ &= |T^{t-1}|^2 \cdot U \cdot U^s|T|(U^s)^* \cdot |T|^2 = |T^{t-1}|^2 \cdot U^{s+1}|T|(U^{s+1})^* \cdot U|T|^2 \\ &= A_{t-1} \cdot U|T|^2 \end{aligned}$$

Furthermore, it can be deduced directly from (4.11) and (4.5) that

$$UB_t = B_{t-1} \cdot U|T|^2.$$

As a result, we obtain

$$A_{t-1} \cdot U|T|^2 = B_{t-1} \cdot U|T|^2,$$

which gives

$$A_{t-1} = A_{t-1}UU^* = B_{t-1}UU^* = B_{t-1},$$

since $\overline{\mathcal{R}(U|T|^2)} = \mathcal{R}(UU^*)$ and $\big[|T^{t-1}|^2, UU^*\big] = 0$.

Letting $t = 2, 3, \ldots, n + 1$, respectively, we conclude that

$$A_1 = B_1 \Longleftrightarrow A_2 = B_2 \Longleftrightarrow \cdots \Longleftrightarrow A_{n+1} = B_{n+1}.$$

In view of (4.6), the proof of the equivalence of (i) and (ii) is complete. $\square$

In view of Lemma 4.2, we introduce the terms of restricted sequence and the commutativity of an operator along a restricted sequence as follows:

Definition 4.3. A sequence $\{t_n\}_{n=1}^\infty$ is called restricted if $t_n \in \{1,2,\dots,n\}$ for each $n\in\mathbb{N}$, and an operator $T\in\mathcal{L}(H)$ is called commutative along this restricted sequence, if T has the polar decomposition $T=U|T|$ such that

$$\Big[U^{t_n}|T|\big(U^{t_n}\big)^*, \big|T^{n+1-t_n}\big|\Big]=0 \qquad \text{for any } n\in\mathbb{N}.$$

A direct application of Lemma 4.2 and Definition 4.3 gives the following corollary:

Corollary 4.4. *Let $T\in\mathcal{L}(H)$ have the polar decomposition $T=U|T|$. Then the following statements are equivalent:*

(i) $\big[U^s|T|(U^s)^*, |T^t|\big]=0$ *for any $s,t\in\mathbb{N}$;*
(ii) *T is commutative along any restricted sequence;*
(iii) *T is commutative along some restricted sequence.*

Lemma 4.5. [8, Lemma 4.3] *Suppose that $T=U|T|$ is the polar decomposition of $T\in\mathcal{L}(H)$. Let $n\in\mathbb{N}$ be given such that (4.1) is satisfied. Then for $1\le k\le n+1$,*

$$|(T^k)^*| = U|T|U^*\cdot U^2|T|(U^2)^*\cdot\ \cdots\ \cdot\ U^k|T|(U^k)^*. \tag{4.12}$$

Proof. This lemma was given in [8, Lemma 4.3], where H is a Hilbert space and $T\in\mathbb{B}(H)$. Checking the proof of [8, Lemma 4.3] carefully, we find out that the same is true for an adjointable operator on a Hilbert C^*-module. □

The main result of this section is as follows:

Theorem 4.6. (cf. [8, Theorem 4.1]) *Let $T\in\mathcal{L}(H)$ have the polar decomposition $T=U|T|$. Then the following statements are equivalent:*

(i) *T is a centered operator;*
(ii) $\big[|T^n|, |(T^m)^*|\big]=0$ *for any $m,n\in\mathbb{N}$;*
(iii) $\big[|T^n|, |T^*|\big]=0$ *for any $n\in\mathbb{N}$;*
(iv) *T is commutative along any restricted sequence;*
(v) *T is commutative along some restricted sequence;*
(vi) $\big[U^m|T|(U^m)^*, |T^n|\big]=0$ *for any $m,n\in\mathbb{N}$;*
(vii) $\big[U^n|T|(U^n)^*, |T|\big]=0$ *for any $n\in\mathbb{N}$;*
(viii) $\big[|(T^n)^*|, |T|\big]=0$ *for any $n\in\mathbb{N}$;*
(ix) $\big[(U^n)^*|T^*|U^n, |T^*|\big]=0$ *for any $n\in\mathbb{N}$;*
(x) $\big[(U^m)^*|T^*|U^m, |(T^n)^*|\big]=0$ *for any $m,n\in\mathbb{N}$;*
(xi) *The operators in $\{|T|, U|T|U^*, U^*|T|U, U^2|T|(U^2)^*, (U^2)^*|T|U^2,\cdots\}$ commute with one another.*

Proof. The proof of (i)$\Longleftrightarrow$(ii) is the same as that given in [8, Theorem 4.1]. "(ii)$\Longrightarrow$(iii)" is clear by putting $m=1$ in (ii).

"(iii)$\Longleftrightarrow$(vii)": Putting $t_n=1$ and $s_n=n$ for any $n\in\mathbb{N}$. Then T is commutative along $\{t_n\}$ $\Longleftrightarrow$ (iii) is satisfied, and T is commutative along $\{s_n\}$ $\Longleftrightarrow$ (vii) is satisfied. The equivalence of (iii)–(vii) then follows from Corollary 4.4.

"(vi)$\Longrightarrow$(ii)": Let m and n be any in $\mathbb{N}$. From (vii) and Lemma 4.5, we know that $|(T^m)^*|$ has the form (4.12) with k therein be replaced by m. Now, each term in (4.12) commutes with $|T^n|$ by (vi), so $\big[|T^n|, |(T^m)^*|\big]=0$.

The proof of the equivalence of (i)–(vii) is therefore complete. Since T is centered if and only if T^* is centered, the equivalent conditions (viii), (ix), and (x) are then obtained by replacing T and U with T^*, and U^*, respectively.

It is obvious that (xi)$\Longrightarrow$(vii).

"(vii)+(ix)$\Longrightarrow$(xi)": From (vii), (ix), and Proposition 2.6, we get

$$\big[U^k|T|(U^k)^*, U^*U\big] = \big[(U^k)^*|T^*|U^k, UU^*\big] = 0 \qquad \text{for any } k \in \mathbb{N}.$$

We prove that the operators in

$$\Omega = \big\{|T|, U|T|U^*, U^*|T|U, U^2|T|(U^2)^*, (U^2)^*|T|U^2, \dots\big\} \tag{4.13}$$

commute with one another. That is, $[A, B] = 0$ for any $A, B \in \Omega$. To this end, four cases are considered as follows:

Case 1: $A = U^t|T|(U^t)^*$ and $B = U^s|T|(U^s)^*$ with $1 \le t < s$. In this case, we have

$$\begin{aligned} AB &= U^t \cdot |T| \cdot U^{s-t}|T|(U^{s-t})^* \cdot (U^t)^* \\ &= U^t \cdot U^{s-t}|T|(U^{s-t})^* \cdot |T| \cdot (U^t)^* = BA. \end{aligned}$$

Case 2: $A = (U^t)^*|T|U^t$ and $B = (U^s)^*|T|U^s$ with $1 \le t < s$. In this case, we have $[A, B] = 0$ as shown in Case 1 by replacing U and T, respectively, with U^* and T^*, since A and B can be expressed alternately as $A = (U^{t+1})^*|T^*|U^{t+1}$ and $B = (U^{s+1})^*|T^*|U^{s+1}$.

Case 3: $A = U^t|T|(U^t)^*$ and $B = (U^s)^*|T|U^s$ with $t, s \in \mathbb{N}$. In this case, we have

$$\begin{aligned} BA &= (U^s)^*|T|U^{s+t}|T|(U^t)^* = (U^s)^* \cdot |T| \cdot U^{s+t}|T|(U^{s+t})^* \cdot U^s \\ &= (U^s)^* \cdot U^{s+t}|T|(U^{s+t})^* \cdot |T| \cdot U^s = U^t|T|(U^{s+t})^* \cdot |T| \cdot U^s = AB. \end{aligned}$$

Case 4: $A = |T|$ and $B = (U^s)^*|T|U^s$ with $s \in \mathbb{N}$. In this case, we have

$$\begin{aligned} AB &= U^* \cdot U|T|U^* \cdot (U^{s-1})^*|T|U^{s-1} \cdot U \\ &= U^* \cdot (U^{s-1})^*|T|U^{s-1} \cdot U|T|U^* \cdot U \\ &= (U^s)^*|T|U^s \cdot |T|U^*U = BA. \end{aligned}$$

This completes the proof that any two elements in Ω are commutative. □

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Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China.

E-mail address: liunana0616@163.com; luoweipig1@163.com; qingxiang_xu@126.com