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ON AN ELASTO-ACOUSTIC TRANSMISSION PROBLEM IN ANISOTROPIC, INHOMOGENEOUS MEDIA

RAINER PICARD

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Abstract. We consider a coupled system describing the interaction between acoustic and elastic regions, where the coupling occurs not via material properties but through an interaction on an interface separating the two regimes. Evolutionary well-posedness in the sense of Hadamard well-posedness supplemented by causal dependence is shown for a natural choice of generalized interface conditions. The results are obtained in a real Hilbert space setting incurring no regularity constraints on the boundary and almost none on the interface of the underlying regions.

1. Introduction and preliminaries

Similarities between various initial boundary value problems of mathematical physics have been noted as general observations throughout the literature. Indeed, the work by K. O. Friedrichs, [2, 3], already showed that the classical linear phenomena of mathematical physics belong — in the static case — to his class of *symmetric positive hyperbolic partial differential equations*, later referred to as *Friedrichs systems*, which are of the abstract form

$$(M_1 + A)\,u = f, \tag{1.1}$$

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with A at least formally, that is, on C_∞-vector fields with compact support in the underlying region Ω, a skew-symmetric differential operator and the L^∞-matrix-valued multiplication-operator M_1 satisfying the condition

$$\operatorname{sym}(M_1) \coloneqq \frac{1}{2}(M_1 + M_1^*) \geq c > 0$$

for some real number c. Indeed, a typical choice of boundary condition is, when A is skew-selfadjoint (A m-accretive would be sufficient). To assume A to be skew-selfadjoint is less restrictive than one might think. For this we note that for example typical dissipative boundary conditions actually give rise to natural skew-selfadjoint spatial operators A, [20]. That A is skew-selfadjoint, is a quite common assumption but may not be easily recognized. As a typical example in case we consider the popular transcription of the wave equation $\partial_0^2 - \Delta_D$, where Δ_D denotes the Laplacian with a homogeneous Dirichlet boundary condition in a bounded domain Ω, into a first order system of the form $\partial_0 + A$, where $A = \begin{pmatrix} 0 & \Delta_D \\ 1 & 0 \end{pmatrix}$ is indeed skew-selfadjoint due to the standard choice of Hilbert space setting. Problem (1.1) can be considered as the static problem associated with the dynamic problem (∂_0 denotes the time-derivative)

$$(\partial_0 M_0 + M_1 + A)\, u = f \tag{1.2}$$

with M_0 selfadjoint L^∞-multiplication-operator and $M_0 \geq 0$, which were also addressed in [3]. It is noteworthy, that even the temporal exponential weight factor, which plays a central role in the approach of [16], is introduced as an ad-hoc formal trick to produce a suitable M_1 for a well-posed static problem. For the so-called time-harmonic case, where ∂_0 is replaced by $\mathrm{i}\omega$, $\omega \in \mathbb{R}$, we replace A simply by $\mathrm{i}\omega M_0 + A$ to arrive at a system of the form (1.1).

Operators of the abstract Friedrichs type $\partial_0 M_0 + M_1 + A$ appearing in (1.2), can be generalized to obtain a fully time-dependent theory allowing for operator-valued coefficients, indeed, in the time-shift invariant case, for systems of the general form

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right) U = F \tag{Evo-Sys}$$

where A is — for simplicity — skew-selfadjoint and M an operator-valued — say — rational function (regular at 0) as an abstract coefficient. The meaning of the so-called material law operator $M\left(\partial_0^{-1}\right)$ is in terms of a suitable function calculus associated with the (normal) operator ∂_0, [19, Chapter 6]. This spacious class of operators allows for a large class of material laws including — the recently of great interest — *meta-materials*.

We shall refer to such systems as evo-systems (or evolutionary equations) to distinguish them from the special subclass of classical (explicit) evolution equations. In this paper we intend to study a particular transmission problem between two physical regimes, acoustics, and elasto-dynamics, within this general framework to establish its well-posedness, which for evo-systems entails not only Hadamard well-posedness; that is, *uniqueness*, *existence*, and *continuous dependence*, but also the crucial property of *causality*. For this we will only have to establish the skew-selfadjointness of a suitably constructed operator A. Then it is known that the requirement

$$\varrho M(0) + \operatorname{sym}(M'(0)) \geq c_0 > 0 \tag{1.3}$$

for some number c_0 and all sufficiently large $\varrho \in]0, \infty[$, yields the desired well-posedness; see the survey [22]. For the simple Friedrichs type case where we additionally assume that

$$M_0 = M(0) \geq c_0 > 0 \tag{1.4}$$

for some number c_0, which clearly implies (1.3), we may even use the commonly invoked semi-group theory to establish the desired well-posedness (note that in this case $M_1 = M'(0)$ and all higher derivatives of M vanish). Indeed, under these strong restrictions (1.2) is congruent to

$$\partial_0 + \sqrt{M_0^{-1}} M_1 \sqrt{M_0^{-1}} + \sqrt{M_0^{-1}} A \sqrt{M_0^{-1}}, \tag{1.5}$$

which amounts to having $M_0 = 1$ (M_1 replaced by the congruent $\sqrt{M_0^{-1}} M_1 \sqrt{M_0^{-1}}$) and using $\sqrt{M_0} U$ as the new unknown in the corresponding problem of the form (Evo-Sys). With $\sqrt{M_0^{-1}} A \sqrt{M_0^{-1}}$ inheriting its skew-selfadjointness from A we obtain indeed a one-parameter group $\left(\exp\left(t\sqrt{M_0^{-1}} A \sqrt{M_0^{-1}}\right)\right)_{t\in\mathbb{R}}$, which, by a simple perturbation argument, yields a group $(U(t))_{t\in\mathbb{R}}$ such that $\left(\chi_{[0,\infty[}(t)\, U(t)\right)_{t\in\mathbb{R}}$, with $\chi_{[0,\infty[}$ denoting the characteristic function of the interval $[0,\infty[$, is the fundamental solution associated with (1.5). Thus a fairly general solution can be obtained by convolution with this fundamental solution. Restricting this fundamental solution to its support yields a continuous, one-parameter semi-group $(U(t))_{t\in[0,\infty[}$. In any case we are justified to focus on the underlying skew-selfadjointness of the operator A as a central feature to obtain well-posedness for a large class of general material laws, since we shall be concerned with the interaction between the elastic and the acoustic regimes solely via the interface, not via material interactions through the material law; as for example in piezo-electrics, compare, for instance, [11] for a typical effect of the latter type. This specific focus also allows us in the interest of brevity to by-pass the intricacies of the time-dependent theory of [16].
Skew-selfadjointness of an operator A; that is,

$$A = -A^*, \tag{1.6}$$

in a real Hilbert space H results in

$$\langle u | Au \rangle_H = 0$$

for all $u \in D(A)$. Moreover, in typical cases skew-selfadjointness of A is a simple consequence of A being congruent to a block matrix of the form

$$\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

where $C : D(C) \subseteq H_0 \to H_1$ is a closed, densely defined,and linear operator between real Hilbert spaces H_0, H_1, which is clearly skew-selfadjoint in the direct sum Hilbert space $H = H_0 \oplus H_1$.

The interest of studying the coupling between acoustic and elasticity wave phenomena has a relatively long history in the engineering community, with [7], [8], and being earlier references. Originally motivated by submarine noise propagation, this coupling is also of interest in connection with loudspeaker and hearing aid design, as well as nondestructive testing. Near the close of the last century there has been a rekindled interest in these specific issues, [23], [9]. More recent publications are the numerical investigations [1], [24], [10], and the more mathematically oriented [5], [12], [6], [4], just to mention a few. Here we want to transcend the predominant constant coefficient and — with the notable exception of [4] — largely time-harmonic analysis and consider the time-dependent case in anisotropic and inhomogeneous media. Since we shall consider operator coefficients, this also includes media with nonlocal behavior. For sake of accessibility we restrict our attention to the autonomous case with classical block-diagonal material laws and no memory effects. We use a functional-analytical setting in real Hilbert space to obtain a well-posedness for this elasto-acoustic transmission problem.

We shall first establish the spatial operator of acoustics and elasticity, respectively, as intimately related skew-selfadjoint operators (mother-descendant mechanism) in a real Hilbert space framework based on the above-mentioned block structure with suitably introduced operators C. Then, in Section 3 we apply these observations to a particular interface coupling problem between the two regimes in adjacent regions via a refined mother-descendant mechanism. We emphasize that our setup allows for arbitrary open sets as underlying domains with no additional constraints on boundary regularity and almost no constraints on interface regularity. Indeed we only require the interface to be a Lebesgue null set to ensure the decomposition (3.1). The induced homogeneous boundary value constraints and transmission conditions are encoded — as customary — in suitable generalization as containment in the domain of the operator.

2. The connection of the spatial operators of acoustics and elasticity

2.1. **Basic ideas.** Without loss of generality we may and will assume that all Hilbert spaces used in the following are real. Note that every complex Hilbert space X is a real Hilbert space choosing only real numbers as multipliers and

$$(\phi, \psi) \mapsto \mathfrak{Re} \langle \phi | \psi \rangle_X$$

as new inner product. With this choice ϕ and $\mathrm{i}\phi$ are always orthogonal. Moreover, for any skew-symmetric operator A, we have

$$x \perp Ax$$

for all $x \in D(A)$. Indeed, since $\langle x|y\rangle - \langle y|x\rangle = 0$ (symmetry), we have

$$\langle x|Ax\rangle - \langle Ax|x\rangle = 0$$

or by skew-symmetry

$$\begin{aligned} 0 &= \langle x|Ax\rangle - \langle Ax|x\rangle \\ &= 2\langle x|Ax\rangle \end{aligned}$$

for all $x \in D(A)$.
In many practical cases the desired skew-selfadjointness of the spatial operator A is evident from its structure as a block operator matrix of the form

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

with $H = H_0 \oplus H_1$ and $C : D(C) \subseteq H_0 \to H_1$ is a closed, densely defined, and linear operator. We shall start our exploration by focusing for simplicity and definiteness on the Cartesian situation and on the case of the so-called Dirichlet boundary condition. For this, we initially take C as the closure $\mathring{\mathrm{grad}}$ of the classical differential operator

$$\begin{aligned} \mathring{C}_1\left(\Omega, \mathbb{R}^3\right) \subseteq L^2\left(\Omega, \mathbb{R}^3\right) &\to L^2\left(\Omega, \mathbb{R}^{3\times 3}\right), \\ u &\mapsto u', \end{aligned}$$

where u' is the derivative (in matrix language the Jacobian) of the vector field u and $\mathring{C}_1(\Omega, \mathbb{R}^3)$ denotes the space of continuously differentiable vector fields in $\mathbb{R}^3$ with compact support in Ω. The negative adjoint is the weak extension of the classical divergence operator on matrix fields

$$\mathrm{div} := -\left(\mathring{\mathrm{grad}}\right)^*.$$

Thus, the operator of our initial interest is

$$A = \begin{pmatrix} 0 & \mathrm{div} \\ \mathring{\mathrm{grad}} & 0 \end{pmatrix}$$

as a skew-selfadjoint operator in $H = L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega, \mathbb{R}^{3\times 3})$. Here $\mathbb{R}^{3\times 3}$ is equipped with the standard Frobenius inner product. As an illustration, let us consider

$$\left(\partial_0 \begin{pmatrix} \varrho_* & 0 \\ 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{div} \\ -\mathring{\mathrm{grad}} & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

as an associated dynamic problem for finding a solution $\begin{pmatrix} v \\ T \end{pmatrix} \in L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega, \mathbb{R}^{3\times 3})$.
Here $\varrho_* : L^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$ and $C : L^2(\Omega, \mathbb{R}^{3\times 3}) \to L^2(\Omega, \mathbb{R}^{3\times 3})$ are assumed to be strongly positive definite mappings in order to obtain well-posedness in the sense of our introductory exposition. This type of system can be understood as modeling asymmetric elasticity theory in the sense of [13, 14, 15].

2.2. **Symmetric elasticity as a descendant of asymmetric elasticity.** To illustrate the mother-descendant mechanism, as introduced in [18], see also [17, 21], we first perform the transition to classical (symmetric) elasticity using this concept.
We recall from [18] the following simple but crucial lemma.

Lemma 2.1. *Let $C : D(C) \subseteq H \to Y$ be a closed densely-defined linear operator between Hilbert spaces H, Y. Moreover, let $B : Y \to X$ be a continuous linear operator into another Hilbert space X. If $C^* B^*$ is densely defined, then*

$$\overline{BC} = (C^* B^*)^* .$$

Proof. It is

$$C^* B^* \subseteq (BC)^* .$$

If $\phi \in D\left((BC)^*\right)$, then

$$\langle BCu|\phi\rangle_X = \langle u|\,(BC)^*\,\phi\rangle_H$$

for all $u \in D(C)$. Thus, we have

$$\langle Cu|B^*\phi\rangle_Y = \langle BCu|\phi\rangle_X = \langle u|\,(BC)^*\,\phi\rangle_H$$

for all $u \in D(C)$, and we read off that $B^*\phi \in D(C^*)$ and

$$C^* B^* \phi = (BC)^*\,\phi.$$

Thus we have

$$(BC)^* = C^* B^*.$$

If now $C^* B^*$ is densely defined, we have, for its adjoint operator,

$$(C^* B^*)^* = \overline{BC}.$$

□

As a consequence we have that the *descendant*

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}} = \begin{pmatrix} 0 & -C^* B^* \\ \overline{BC} & 0 \end{pmatrix}$$

indeed inherits its skew-selfadjointness from its *mother* $\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ (with C replaced by $\overline{BC}$).

Remark 2.2. Clearly, the role of the components can be interchanged; so that

$$\overline{\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\begin{pmatrix} D^* & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 0 & -\overline{DC^*} \\ CD^* & 0 \end{pmatrix}$$

with $D : H \to Y$ such that CD^* is densely defined, is also a valid descendant construction.

These constructions can be combined. In general, a repeated application of the mother-descendant mechanism may, however, depend on the order in which they are carried out. This fact has been overlooked in [18]. An illuminating example is choosing C as the weak $L^2(\mathbb{R})$-derivative ∂ and $B = D$ as the cut-off by the characteristic function $\chi_{]-1/2,1/2[}$ of the symmetric unit interval $]-1/2,1/2[$ yielding

$$\begin{pmatrix} 0 & \overline{\chi_{]-1/2,1/2[}\left(\partial \chi_{]-1/2,1/2[}\right)} \\ \overline{\chi_{]-1/2,1/2[}\partial \chi_{]-1/2,1/2[}} & 0 \end{pmatrix} \tag{2.1}$$

if first the construction with B and then with D is carried out. In reverse order we obtain

$$\left(\begin{array}{cc} 0 & \overline{\chi_{]-1/2,1/2[}\partial\chi_{]-1/2,1/2[}} \\ \overline{\chi_{]-1/2,1/2[}\left(\partial\chi_{]-1/2,1/2[}\right)} & 0 \end{array}\right). \tag{2.2}$$

In comparison, (2.1) models vanishing at $\pm\frac{1}{2}$ for the second component, whereas (2.2) leads to no boundary condition for the second component (implying in turn vanishing at $\pm\frac{1}{2}$ of the first component).

As a convenient mother operator to start from, we take the above-mentioned theory of asymmetric elasticity of Nowacki, [13, 15]. Indeed, the classical (symmetric) elasticity theory can be considered as a descendant in the above sense of the form

$$\left(\begin{array}{cc} 0 & -\operatorname{Div} \\ -\mathring{\operatorname{Grad}} & 0 \end{array}\right), \tag{2.3}$$

where

$$\mathring{\operatorname{Grad}} := \overline{\iota_{\mathrm{sym}}^{*}\mathring{\operatorname{grad}}}$$

and

$$\operatorname{Div} := \operatorname{div}\iota_{\mathrm{sym}}$$

with

$$\begin{aligned} \iota_{\mathrm{sym}} : L^2\left(\Omega, \operatorname{sym}\left[\mathbb{R}^{3\times3}\right]\right) &\to L^2\left(\Omega, \mathbb{R}^{3\times3}\right), \\ T &\mapsto T, \end{aligned}$$

where $\operatorname{sym}[\mathbb{R}^{3\times3}]$ denotes the image of $\mathbb{R}^{3\times3}$ under the mapping sym; that is, we have in the descendant construction $B = \iota_{\mathrm{sym}}^{*}$. Note that

$$\iota_{\mathrm{sym}}^{*}T = \operatorname{sym}(T)$$

for all $T \in L^2(\Omega, \mathbb{R}^{3\times3})$.

2.3. **Acoustics as a descendant of asymmetric elasticity.** The spatial operator used in the acoustics model can also be introduced as a descendant of asymmetric elasticity. It is actually the scalar version corresponding to the asymmetric elasticity case.
Indeed, classical acoustics can be considered as a descendant of the form

$$\left(\begin{array}{cc} 0 & \operatorname{grad} \\ \mathring{\operatorname{div}} & 0 \end{array}\right),$$

where we reuse the classical notations by letting

$$\mathring{\operatorname{div}} := \overline{\operatorname{trace}\,\mathring{\operatorname{grad}}}$$

and

$$\operatorname{grad} := \operatorname{div}\ \operatorname{trace}^{*}$$

with

$$\begin{aligned} \operatorname{trace} : L^2\left(\Omega,\mathbb{R}^{3\times3}\right) &\to L^2\left(\Omega,\mathbb{R}\right), \\ T = \left(T_{ij}\right)_{i,j} &\mapsto \operatorname{trace} T \coloneqq \sum_i T_{ii}; \end{aligned}$$

that is, $B = \operatorname{trace}$. Note that

$$\operatorname{trace}^* p = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

for all $p \in L^2\left(\Omega,\mathbb{R}\right)$.

Remark 2.3. The acoustic system can also be constructed by applying $B = \operatorname{trace}$ to the symmetric elasticity operator (2.3). Note that the pressure distribution p is in both cases obtained from the stress as

$$p \coloneqq -\operatorname{trace} T.$$

3. An interface coupling between acoustics and elasticity

We will now combine the two descendant constructions above to obtain an interface coupling set-up for the skew-selfadjoint operator A. We assume that $\Omega_0 \cup \Omega_1 \subseteq \Omega$, such that the orthogonal decomposition

$$L^2\left(\Omega,\mathbb{R}\right) \quad = \quad L^2\left(\Omega_0,\mathbb{R}\right) \oplus L^2\left(\Omega_1,\mathbb{R}\right) \tag{3.1}$$

holds; that is, the interface $\Omega \cap \dot{\Omega}_0 \cap \dot{\Omega}_1$ is a Lebesgue null set. Here $/dot\Omega_k$ denotes the set of boundary points of Ω_k, $k = 0, 1$. Consequently, we also have

$$\begin{aligned} L^2\left(\Omega,\mathbb{R}^{3\times3}\right) &= L^2\left(\Omega_0,\mathbb{R}^{3\times3}\right) \oplus L^2\left(\Omega_1,\mathbb{R}^{3\times3}\right), \\ L^2\left(\Omega,\mathbb{R}^{3}\right) &= L^2\left(\Omega_0,\mathbb{R}^{3}\right) \oplus L^2\left(\Omega_1,\mathbb{R}^{3}\right). \end{aligned}$$

Then, with the respective canonical embeddings into $L^2\left(\Omega,\mathbb{R}^{3\times3}\right)$, we obtain

$$\begin{aligned} B : L^2\left(\Omega,\mathbb{R}^{3\times3}\right) &\to L^2\left(\Omega_0,\operatorname{sym}\left[\mathbb{R}^{3\times3}\right]\right) \oplus L^2\left(\Omega_1,\mathbb{R}\right), \\ T &\mapsto \begin{pmatrix} \iota^*_{L^2(\Omega_0,\operatorname{sym}[\mathbb{R}^{3\times3}])}\iota^*_{\operatorname{sym}} T \\ -\iota^*_{L^2(\Omega_1,\mathbb{R})} \operatorname{trace} T \end{pmatrix}, \end{aligned}$$

and so

$$B = \begin{pmatrix} \iota^*_{L^2(\Omega_0,\operatorname{sym}[\mathbb{R}^{3\times3}])}\iota^*_{\operatorname{sym}} \\ -\iota^*_{L^2(\Omega_1,\mathbb{R})} \operatorname{trace} \end{pmatrix}.$$

With this, we get as a descendant construction

$$A \quad = \quad \overline{\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & -\operatorname{div} \\ -\overset{\circ}{\operatorname{grad}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}} \tag{3.2}$$

$$\subseteq \begin{pmatrix} 0 & \begin{pmatrix} -\operatorname{Div}_{\Omega_0} & \operatorname{grad}_{\Omega_1} \end{pmatrix} \\ \begin{pmatrix} -\operatorname{Grad}_{\Omega_0} \\ \operatorname{div}_{\Omega_1} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{3.3}$$

and

$$M\left(\partial_0^{-1}\right) = M(0) = \begin{pmatrix} \varrho_{*,\Omega_0} + \kappa_{\Omega_1}^{-1} & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} C_{\Omega_0}^{-1} & 0 \\ 0 & c_{\Omega_1} \end{pmatrix} \end{pmatrix}.$$

The indexes Ω_k, $k = 0, 1$, are used to denote the respective supports of the quantities. The coefficients κ_{Ω_1} and c_{Ω_1} describe the acoustic properties of the media in Ω_1. In the spirit of the mother and descendant mechanism they may be considered as resulting from suitable corresponding (artificial) elastic coefficients $\varrho_{*,\Omega_1}, C_{\Omega_1}$ in Ω_1:

$$\begin{aligned} \kappa_{\Omega_1}^{-1} &= \varrho_{*,\Omega_1}, \\ c_{\Omega_1} &= \operatorname{trace} C_{\Omega_1}^{-1} \operatorname{trace}^*. \end{aligned}$$

The unknowns are now of the form

$$\begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \end{pmatrix} \in H = L^2\left(\Omega, \mathbb{R}^3\right) \oplus \left(L^2\left(\Omega_0, \operatorname{sym}\left[\mathbb{R}^{3\times 3}\right]\right) \oplus L^2\left(\Omega_1, \mathbb{R}\right)\right),$$

where the first component is to be understood in the sense of (3.1). From the inclusion (3.2) and (3.3), we read off that the resulting evo-system

$$\left(\partial_0 M(0) + A\right) \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} f_{\Omega_0} + f_{\Omega_1} \\ \begin{pmatrix} F_{\Omega_0} \\ g_{\Omega_1} \end{pmatrix} \end{pmatrix} \tag{3.4}$$

indeed yields

$$\partial_0 \left(\varrho_{*,\Omega_0} + \kappa_{\Omega_1}^{-1}\right)\left(v_{\Omega_0} + v_{\Omega_1}\right) - \operatorname{Div}_{\Omega_0} T_{\Omega_0} + \operatorname{grad}_{\Omega_1} p_{\Omega_1} = f_{\Omega_0} + f_{\Omega_1},$$

which in turn — according to (3.1) — splits into equations in Ω_0 and in Ω_1

$$\begin{aligned} \partial_0 \varrho_{*,\Omega_0} v_{\Omega_0} - \operatorname{Div}_{\Omega_0} T_{\Omega_0} &= f_{\Omega_0}, \\ \partial_0 \kappa_{\Omega_1}^{-1} v_{\Omega_1} + \operatorname{grad}_{\Omega_1} p_{\Omega_1} &= f_{\Omega_1}. \end{aligned}$$

The second block row yields another pair of equations

$$\begin{aligned} \partial_0 C_{\Omega_0}^{-1} T_{\Omega_0} - \operatorname{Grad} v_{\Omega_0} &= F_{\Omega_0}, \\ \partial_0 c_{\Omega_1} p_{\Omega_1} + \operatorname{div} v_{\Omega_1} &= g_{\Omega_1}. \end{aligned}$$

The actual system models now generalized natural transmission conditions on the common boundary part $\Omega \cap \dot{\Omega}_0 \cap \dot{\Omega}_1$ (the interface) and the homogeneous Dirichlet boundary condition on $\dot{\Omega}_0 \setminus \dot{\Omega}_1$ and the standard homogeneous Neumann boundary condition on $\dot{\Omega}_1 \setminus \dot{\Omega}_0$ without assuming any smoothness of the boundary via containment of the solution $U = \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \end{pmatrix}$ in the operator domain $D\left(\overline{\partial_0 M(0) + A}\right)$. Due to lack of boundary and interface regularity, we have chosen $\dot{\Omega}_k$, to denote the set of boundary points, rather than the boundary manifold notation $\partial\Omega_k$, $k=0,1$. Since we do not have maximal regularity in this case, this does not mean that $U \in D(A)$, but we do have

$$\partial_0^{-1} U \in D(A)$$

as a form of expressing *generalized* boundary constraints and transmission conditions.

If, however, we assume sufficient regularity of the boundary and solution one can easily motivate that the model yields a generalization of classical transmission conditions on $\dot{\Omega}_0 \cap \dot{\Omega}_1$. Indeed, with

$$\begin{pmatrix} u_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} S_{\Omega_0} \\ q_{\Omega_1} \end{pmatrix} \end{pmatrix}, \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \end{pmatrix} \in D(A)$$

we have (noting for the smooth exterior unit normal vector fields $n_{\dot{\Omega}_0}$, $n_{\dot{\Omega}_1}$ on the boundaries of Ω_0 and Ω_1, respectively, that $n_{\dot{\Omega}_0} = -n_{\dot{\Omega}_1}$ on $\dot{\Omega}_0 \cap \dot{\Omega}_1$) with

$$\widetilde{A} = \begin{pmatrix} 0 & \begin{pmatrix} -\operatorname{Div}_{\Omega_0} & \operatorname{grad}_{\Omega_1} \end{pmatrix} \\ \begin{pmatrix} -\operatorname{Grad}_{\Omega_0} \\ \operatorname{div}_{\Omega_1} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\begin{aligned}
0 &= \left\langle \begin{pmatrix} u_{\Omega_0} + u_{\Omega_1} \\ \begin{pmatrix} S_{\Omega_0} \\ q_{\Omega_1} \end{pmatrix} \end{pmatrix} \middle| \widetilde{A} \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \end{pmatrix} \right\rangle \\
&\quad + \left\langle \widetilde{A} \begin{pmatrix} u_{\Omega_0} + u_{\Omega_1} \\ \begin{pmatrix} S_{\Omega_0} \\ q_{\Omega_1} \end{pmatrix} \end{pmatrix} \middle| \begin{pmatrix} v_{\Omega_0} + v_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ p_{\Omega_1} \end{pmatrix} \end{pmatrix} \right\rangle \\
&= -\left\langle u_{\Omega_0} \middle| \operatorname{Div}_{\Omega_0} T_{\Omega_0} \right\rangle - \left\langle S_{\Omega_0} \middle| \operatorname{Grad}_{\Omega_0} v_{\Omega_0} \right\rangle + \\
&\quad + \left\langle q_{\Omega_1} \middle| \operatorname{div}_{\Omega_1} v_{\Omega_1} \right\rangle + \left\langle u_{\Omega_1} \middle| \operatorname{grad}_{\Omega_1} p_{\Omega_1} \right\rangle + \\
&\quad - \left\langle \operatorname{Grad}_{\Omega_0} u_{\Omega_0} \middle| T_{\Omega_0} \right\rangle - \left\langle \operatorname{Div}_{\Omega_0} S_{\Omega_0} \middle| v_{\Omega_0} \right\rangle + \\
&\quad + \left\langle \operatorname{grad}_{\Omega_1} q_{\Omega_1} \middle| v_{\Omega_1} \right\rangle + \left\langle \operatorname{div}_{\Omega_1} u_{\Omega_1} \middle| p_{\Omega_1} \right\rangle \\
&= -\int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} u_{\Omega_0}^{\top} T_{\Omega_0} n_{\dot{\Omega}_0} \, do - \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} v_{\Omega_0}^{\top} S_{\Omega_0} n_{\dot{\Omega}_0} \, do + \\
&\quad + \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} n_{\dot{\Omega}_1}^{\top} \left(p_{\Omega_1} u_{\Omega_1} \right) do + \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} n_{\dot{\Omega}_1}^{\top} \left(q_{\Omega_1} v_{\Omega_1} \right) do.
\end{aligned}$$

We want to arrive at point-wise transmission conditions by the fundamental lemma of variational calculus. Since $(u_{\Omega_0} + u_{\Omega_1}) \in D\left(\overset{\circ}{\operatorname{Grad}}\right) = D\left(\overset{\circ}{\operatorname{grad}}\right) = D\left(B\,\overset{\circ}{\operatorname{grad}}\right) \subseteq D\left(\overline{B\,\overset{\circ}{\operatorname{grad}}}\right)$ is, by construction, admissible, we may assume that $u_{\Omega_0} = u_{\Omega_1}$ on the interface and conclude with $S_{\Omega_0} = 0$, $q_{\Omega_1} = 0$ that

$$T_{\Omega_0} n_{\dot{\Omega}_0} + p_{\Omega_1} n_{\dot{\Omega}_0} = 0 \tag{3.5}$$

is a needed transmission condition. In particular, we see that

$$n_{\dot{\Omega}_0} \times T_{\Omega_0} n_{\dot{\Omega}_0} = 0.$$

Now letting $u_{\Omega_0} = u_{\Omega_1} = 0$ on the interface and noting that due to skew-selfadjointness of $\widetilde{A}$ also our test elements S_{Ω_0}, q_{Ω_1} must satisfy the explicit transmission condition (3.5) (now with S_{Ω_0} replacing T_{Ω_0} and q_{Ω_1} instead of p_{Ω_1}) yields

$$\begin{aligned} 0 &= \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} \left(v_{\Omega_0} - v_{\Omega_1}\right)^\top \left(q_{\Omega_1} n_{\dot{\Omega}_0}\right) do \\ &= \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} q_{\Omega_1} n_{\dot{\Omega}_0}^\top \left(v_{\Omega_0} - v_{\Omega_1}\right) do \end{aligned}$$

which, with q_{Ω_1} being arbitrary, now implies

$$n_{\dot{\Omega}_0}^\top v_{\Omega_0} = n_{\dot{\Omega}_0}^\top v_{\Omega_1}. \tag{3.6}$$

That is, the continuity of the normal components

$$v_{\Omega_0,\mathrm{n}} = v_{\Omega_1,\mathrm{n}},$$

as a complementing transmission condition. Note that although we were allowed to choose a test element $(u_{\Omega_0} + u_{\Omega_1})$ in the dense subspace $D\left(\mathring{\operatorname{grad}}\right)$ of $D\left(\overline{B\,\mathring{\operatorname{grad}}}\right)$ for general $(v_{\Omega_0} + v_{\Omega_1}) \in D\left(\overline{B\,\mathring{\operatorname{grad}}}\right)$ we are left with the weaker transmission condition (3.6).

Conversely, transmission conditions (3.5) and (3.6), which are recovering the classical transmission conditions, compare, for example, [4, 5, 9, 10] and the literature quoted there, warrant the vanishing of the above interface integral. These more or less heuristic considerations motivate to take the above evo-system as a appropriate generalization to cases, where the boundary does *not* have a reasonable normal vector field.

All in all, we summarize our findings in the following well-posedness result.

Theorem 3.1. *If $\varrho_{*,\Omega_0}, C_{\Omega_0}$ and $\kappa_{\Omega_1}, c_{\Omega_1}$ are selfadjoint, strictly positive definite, and continuous operators on $L^2(\Omega_0, \mathbb{R}^3)$, $L^2(\Omega_0, \operatorname{sym}[\mathbb{R}^{3\times 3}])$, and on $L^2(\Omega_1, \mathbb{R}^3)$, $L^2(\Omega_1, \mathbb{R})$, respectively, and the interface $\Omega \cap \dot{\Omega}_0 \cap \dot{\Omega}_1$ is a Lebesgue null set, then the evo-system (3.4) is Hadamard well-posed. Moreover, the solution depends causally on the data.*

The operator character of the coefficients in our well-posedness result even allows for nonlocal material behavior. If, however, it is assumed that the coefficients ϱ_{*,Ω_0}, C_{Ω_0}, κ_{Ω_1}, and c_{Ω_1} are bounded measurable multiplication operators of the appropriate type; that is, coefficients in the more colloquial sense, then, since they are scalar, tensor, matrix, and scalar-valued, respectively, we see that the inhomogeneous and anisotropic media case is covered in great generality. This is particularly interesting, since commonly only the isotropic case and mostly the isotropic, constant coefficient case has been considered; see, for example, [4, 5, 9, 10].

Remark 3.2.

(1) Since $M(0) \gg 0$, we could construct a fundamental solution of $\partial_0 + \sqrt{M(0)}^{-1} A \sqrt{M(0)}^{-1}$, which in turn is obtained from the unitary group

$$\left(\exp\left(-t\sqrt{M(0)}^{-1} A \sqrt{M(0)}^{-1}\right)\right)_{t\in\mathbb{R}}$$

as described above.

(2) We note that we may actually allow for completely general — say, for simplicity, rational — material laws as long as condition (1.3) is warranted. The above simple choice has been used as a more approachable illustrating example, which links up more explicitly with cases considered elsewhere.

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Institute for Analysis, Department of Mathematics, Technische Universität Dresden, D-01062, Dresden, Germany.
E-mail address: rainer.picard@tu-dresden.de