Adv. Oper. Theory 3 (2018), no. 4, 781–793
https://doi.org/10.15352/aot.1801-1300
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

ON BEHAVIOR OF FOURIER COEFFICIENTS AND UNIFORM CONVERGENCE OF FOURIER SERIES IN THE HAAR SYSTEM

M. G. GRIGORYAN* and A. KH. KOBELYAN

Communicated by A. Kostenko

ABSTRACT. Suppose that $\hat{b}_m \downarrow 0$, $\{\hat{b}_m\}_{m=1}^{\infty} \notin l^2$, and $b_n = 2^{-\frac{m}{2}}\hat{b}_m$ for all $n \in (2^m, 2^{m+1}]$. In this paper, it is proved that any measurable and almost everywhere finite function $f(x)$ on $[0,1]$ can be corrected on a set of arbitrarily small measure to a bounded measurable function $\widetilde{f}(x)$; so that the nonzero Fourier–Haar coefficients of the corrected function present some subsequence of $\{b_n\}$, and its Fourier–Haar series converges uniformly on $[0,1]$.

1. INTRODUCTION

At first we recall the definition of the Haar system normalized in $L^2[0,1]$(see [8]).

Set $\Delta_1 = (0,1)$. An interval $\Delta_{2^m+k} = (\frac{k-1}{2^m}, \frac{k}{2^m})$, $k = 1,2,\ldots,2^m$ and $m = 0,1,2,\ldots$, is called a dyadic interval. It is clear that

$$\Delta_n = \Delta_{2n-1} \cup \Delta_{2n} \cup \{\tfrac{2k-1}{2^{m+1}}\}.$$

The Haar function associated with Δ_1 is the function $h_1(x) = 1$, and the Haar function associated with Δ_n, $n = 2^m + k$, $k = 1,2,\ldots,2^m$, and $m = 0,1,2,\ldots$,

Date: Received: Jan. 21, 2018; Accepted: May 12, 2018.
* Corresponding author .
2010 *Mathematics Subject Classification.* Primary 42A65; Secondary 42A20, 43A50.
Key words and phrases. Haar system, Fourier–Haar coefficients, Uniform convergence.

is the following function:

$$h_n(x) = h_m^{(k)}(x) := \begin{cases} 2^{\frac{m}{2}} & \text{for } x \in \Delta_{2n-1}, \\ -2^{\frac{m}{2}} & \text{for } x \in \Delta_{2n}, \\ 0 & \text{for } x \notin [\frac{k-1}{2^m}, \frac{k}{2^m}]. \end{cases} \tag{1.1}$$

The values of functions in points of discontinuity are equal to the average of left and right limits at this point.

The Lebesgue measure of a measurable set E is denoted by mes$E$. The characteristic function of a set E is denoted by χ_E.

The notation $L^\infty[0,1]$ denotes the space of bounded measurable functions on $[0,1]$ with norm

$$||f||_\infty = \sup_{x\in[0,1]} \{|f(x)|\}.$$

The spectrum of $f(x)$ (denoted by $spec(f)$) is the support of $\{c_k(f)\}$; that is, the set of integers k for which $c_k(f)$ is nonzero, where $\{c_n(f)\}_{n=0}^\infty$ is the Fourier–Haar coefficients

$$c_n(f) = \int_0^1 f(x)h_n(x)dx \qquad (n \geq 1).$$

Note that the Haar system is a basis for all $L_p[0,1]$, $1 \leq p < \infty$ (see [10]); that is, each function $f(x) \in L^p[0,1]$ can be represented by a unique series

$$\sum_{n=1}^\infty c_n(f)h_n(x),$$

which converges to $f(x)$ in the $L^p[0,1]$-norm.

Note also that the Fourier–Haar series of any continuous function converges uniformly on $[0,1]$. But it is not true for functions in space $L^\infty[0,1]$. For example, it is not hard to see that the Fourier–Haar series of the function

$$f_0(x) = \sum_{n=0}^\infty c_n(f_0)h_n(x) = \sum_{k=1}^\infty \frac{1}{2^{\frac{k}{2}}} h_{2^k}^{(2^k-1)}(x) \in \ L^\infty[0,1]$$

does not converge uniformly on $[0,1]$ and

$$\{||c_n(f_0)h_n||_\infty : n \in spec(f_0)\} = 1.$$

The following question arises naturally: Is there a measurable set E of arbitrarily small measure such that a suitable change of the values of any function of class $L^\infty[0,1]$ on E leads to a new modified function $g(x) \in \ L^\infty[0,1]$, which the Fourier series in the Haar system converges uniformly on $[0,1]$ and the nonzero elements in the sequence $\{||c_n(g)h_n||_\infty\}_{n=0}^\infty$ are arranged in decreasing order?

In the present work, we prove that this question has a positive answer.

Theorem 1.1. *For any $\varepsilon > 0$ and each measurable and almost everywhere finite function f on $[0,1]$, there exists a function $\widetilde{f} \in L^\infty[0,1]$ with* mes$\{x \in [0,1] : \widetilde{f}(x) \neq f(x)\} < \varepsilon$ *such that the sequence $\{||c_n(\widetilde{f})h_n||_\infty : n \in spec(\widetilde{f})\}$ is monotonically decreasing and the Fourier series of function $\widetilde{f}$ in the Haar system converges uniformly on* $[0,1]$.

Note that the idea of correcting of the function to improve its properties is due to Luzin (see [9]). The classical Luzin correction theorem states that any measurable function can be made continuous by correcting its values on a set of arbitrarily small measure (see [9]).

This important result is generalized by numerous authors. These generalizations are related to the fact that the correcting function possesses some additional properties. In particular, the first generalization of that type belongs to Men'shov and states that the correcting function has a uniformly convergent Fourier series in the trigonometric system [11].

Further interesting results in this direction are obtained by many mathematicians (see [1],[4], [5], [13], [14], [15]).

Note that a number of papers (see [3], [4], [5],[7],[12]) have been devoted to the correction theorems, in which the absolute values of the nonzero Fourier coefficients (by Haar, Walsh, and Faber–Schauder systems) of the corrected function monotonically decrease. Note the result of paper [6].

There exists a function $U \in L^1[0,1)$ with the strictly decreasing Fourier–Walsh coefficients $\{c_k(U)\} \searrow$ such that, for every almost everywhere finite measurable function on $[0,1]$, one can find a function $g \in L^\infty[0,1)$ with $\mathrm{mes}\{x \in [0,1) : g \neq f\} < \epsilon$ such that $|c_k(g)| = c_k(U)$, for all $k \in spec(g)$, and the Fourier–Walsh series of $g(x)$ converges uniformly on $[0,1)$.

Let $\mathcal{B}$ denote the set of all sequences $\{b_n\}_{n=2}^{\infty}$ of the form

$$b_n = 2^{-\frac{m}{2}} \hat{b}_m \qquad \text{for } n \in (2^m, 2^{m+1}], \tag{1.2}$$

where $\{\hat{b}_m\}_{m=1}^{\infty}$ is a sequence with

$$\hat{b}_m \searrow 0 \text{ and } \sum_{m=0}^{\infty} \hat{b}_m^2 = +\infty. \tag{1.3}$$

Theorem 1.1 follows from a more general theorem.

Theorem 1.2. *Suppose that* $\{b_n\} \in \mathcal{B}$, $\varepsilon > 0$ *and that* f *is measurable and almost everywhere finite function on* $[0,1]$. *Then there exists a function* $\widetilde{f} \in L^\infty[0,1]$ *with the following properties:*

(1) $\mathrm{mes}\{x \in [0,1] : \widetilde{f}(x) \neq f(x)\} < \varepsilon$,
(2) $c_n(\widetilde{f}) = b_n, n \in spec(\widetilde{f})$,
(3) $\lim_{n\to\infty} || \sum_{n=1}^{\infty} c_n(\widetilde{f}) h_n - \widetilde{f}||_\infty = 0$.

The following result follows from Theorem 1.2.

Corollary 1.3. *For any* $\varepsilon > 0$ *and each measurable and almost everywhere finite function* f *on* $[0,1]$, *there exists a function* $\widetilde{f} \in L^\infty[0,1]$ *with* $\mathrm{mes}\{x \in [0,1] : \widetilde{f}(x) \neq f(x)\} < \varepsilon$ *such that the sequence* $\{||c_n(\widetilde{f})h_n||_\infty : n \in spec(\widetilde{f})\}$ *is monotonically decreasing and the Greedy algorithm of modified function converges uniformly on* $[0,1]$.

Recall that the Greedy algorithm is a method to approximate an element $f \in \mathcal{X}$ ($f = \sum_{k=1}^{\infty} a_k(f)\psi_k$) (see [2]) by the sequence $\{G_m(f,\psi)\}_{m=1}^{\infty}$,

$$G_m(f,\psi) = \sum_{k=1}^{m} a_{\sigma(k)}(f)\psi_{\sigma(k)},$$

where $\psi = \{\psi_k\}_{k=1}^{\infty}$ is a basis in a Banach space $\mathcal{X}$ and $\sigma = \{\sigma(k)\}_{k=1}^{\infty}$ is a permutation of the positive natural integers such that

$$||a_{\sigma(k)}(f)\psi_{\sigma(k)}||_{\mathcal{X}} \geq ||a_{\sigma(k+1)}(f)\psi_{\sigma(k+1)}||_{\mathcal{X}}, \qquad k = 1, 2, \ldots.$$

One can pose the following problems.

Problem 1.4. Can the values of any function $f \in L^{\infty}[0,1]$ be modified on a set of small measure such that the absolute values of the nonzero Fourier coefficients of corrected function with respect to trigonometric system are arranged in decreasing order?

Problem 1.5. Can the corrected function $\widetilde{f}(x)$ in Theorem 1.1 be chosen to be continuous on $[0,1]$?

2. Proof of main lemmas

A function g is called dyadic step function if $g(x) = \sum_{k=1}^{2^n} \gamma_k \chi_{\widehat{\Delta}_k}(x)$, where $\widehat{\Delta}_k = (\frac{k-1}{2^n}, \frac{k}{2^n})$, and the value of g in point $\frac{k}{2^n}$ is equal to the average of left and right limits ($k = 0, 1, \ldots, 2^n$). For each function g we will denote $g(x+0)$ and $g(x-0)$ right and left limits in point x, respectively (In the paper, we will assume that $g(-0) = g(+0)$ and $g(1+0) = g(1-0)$).

Lemma 2.1. *Let $\{b_n\}_{n=1}^{\infty}$ be a sequence satisfying the following conditions:*

$$\lim_{n\to\infty} ||b_n h_n||_{\infty} = 0, \tag{2.1}$$

$$\sum_{n=1}^{\infty} |b_n h_n(x)|^2 = +\infty, \ \text{almost everywhere on } [0,1], \tag{2.2}$$

and let numbers $\gamma \neq 0$, $\varepsilon > 0$, $\delta > 0$, $N_0 \in \mathbb{N}$, and dyadic interval $\Delta = (\alpha, \beta)$ be given. Then there exist a measurable set $E \subset \Delta$, a dyadic step function $g(x)$ and a polynomial in the Haar system $Q(x) = \sum_{n=N_0}^{M} \epsilon_n b_n h_n(x)$ (where $\epsilon_n = 0$ or 1) such that

(1) $\mathrm{mes}E > (1-\varepsilon)\mathrm{mes}\Delta$,
(2)

$$g(x) = \begin{cases} \gamma & \text{if } x \in E, \\ 0 & \text{if } x \notin [\alpha, \beta], \end{cases}$$

(3) $\|g - Q\|_{\infty} \leq \delta$,
(4) $Q(x) = 0$ *for all* $x \notin [\alpha, \beta]$,
(5) $\max_{N_0 \leq m \leq M} || \sum_{n=N_0}^{m} \epsilon_n b_n h_n||_{\infty} \leq \frac{4|\gamma|}{\varepsilon}$.

Proof. At first, we prove the lemma in the case when $\gamma > 0$.

By (2.1), (2.2), and Arutyunyan's theorem (see [1]), it follows that the following conditions are true almost everywhere on $[0,1]$:

$$\limsup_{L\to\infty} \sum_{n=0}^{L} b_n h_n(x) = +\infty, \tag{2.3}$$

$$\liminf_{L\to\infty} \sum_{n=0}^{L} b_n h_n(x) = -\infty. \tag{2.4}$$

We denote by $E_0 \subset \Delta$ the set of all points for which conditions (2.3) and (2.4) are true. Obviously $\mathrm{mes} E_0 = \mathrm{mes}\Delta$. We take a natural number $N > N_0$ such that $\mathrm{mes}\Delta_N < \mathrm{mes}\Delta$ and

$$|b_n h_n(x)| < \widetilde{\delta} \qquad \forall x \in E_0, \forall n \geq N, \tag{2.5}$$

where

$$\widetilde{\delta} := \min\left\{\frac{\delta}{2}, \frac{|\gamma|}{2}\right\}. \tag{2.6}$$

At first the numbers $\{N_s\}_{s=1}^{\infty}$ and sets $\{G_s\}_{s=1}^{\infty}$ will be constructed with some property. We denote by N_1 the least natural number $m \geq N$ for which

$$\sum_{n=N}^{N_1+1} b_n h_n(x) \notin [-\frac{4\gamma}{\varepsilon}, \gamma] \qquad \text{for some } x \in E_0.$$

Let

$$G_1 = \{x \in E_0 : \sum_{n=N}^{N_1+1} b_n h_n(x) \notin [-\frac{4\gamma}{\varepsilon}, \gamma]\}. \tag{2.7}$$

Since all polynomials in the Haar system are step functions, then, from the definition of N_1 by (2.5) and (2.7), we obtain $\mathrm{mes}\, G_1 > 0$, and

$$\text{if } x \in G_1, \text{ then } \left|\sum_{n=N}^{N_1} b_n h_n(x) - \gamma\right| < \widetilde{\delta} \text{ or } \left|\sum_{n=N}^{N_1} b_n h_n(x) + \frac{4\gamma}{\varepsilon}\right| < \widetilde{\delta}.$$

If $\mathrm{mes} E_0 \setminus G_1 = 0$, we assume that $N_s = N_1$ and $G_s = \emptyset$, for all $s > 1$ and that construction of numbers $\{N_s\}_{s=1}^{\infty}$ is completed. In the other case, we denote by N_2 the least natural number $m \geq N$ for which

$$\sum_{n=N}^{N_2+1} b_n h_n(x) \notin [-\frac{4\gamma}{\varepsilon}, \gamma] \qquad \text{for some } x \in E_0 \setminus G_1.$$

It is not hard to see that $N_2 > N_1$ and $\mathrm{mes}\, G_2 > 0$, where

$$G_2 = \{x \in E_0 \setminus G_1 : \sum_{n=N}^{N_2+1} b_n h_n(x) \notin [-\frac{4\gamma}{\varepsilon}, \gamma]\}.$$

From this and (2.5), we have

$$\sum_{n=N}^{k} b_n h_n(x) \in [-\frac{4\gamma}{\varepsilon}, \gamma] \qquad \forall k \leq N_2, \forall x \in E_0 \setminus G_1,$$

and

$$\text{if } x \in G_2, \text{ then } \left|\sum_{n=N}^{N_2} b_n h_n(x) - \gamma\right| < \widetilde{\delta} \text{ or } \left|\sum_{n=N}^{N_2} b_n h_n(x) + \frac{4\gamma}{\varepsilon}\right| < \widetilde{\delta}.$$

If mes $(E_0 \setminus (G_1 \cup G_2)) = 0$, we assume that $N_s = N_2$ and $G_s = \emptyset$, for all $s > 2$ and that construction of numbers $\{N_s\}_{s=1}^{\infty}$ is completed.

Hence we can define $\{N_s\}_{s=1}$ and sets $\{G_s\}_{s=1}$ by finite or infinite steps induction such that for each $s \geq 1$ satisfy the following conditions:

$$N < N_1 \leq N_2 \leq \cdots \leq N_s \leq \cdots$$

$$G_s = \{x \in E_0 \setminus (\cup_{j=1}^{s-1} G_j) : \sum_{n=N}^{N_s+1} b_n h_n(x) \notin [-\frac{4\gamma}{\varepsilon}, \gamma]\}, \tag{2.8}$$

$$\sum_{n=N}^{k} b_n h_n(x) \in [-\frac{4\gamma}{\varepsilon}, \gamma], \qquad \forall k \leq N_s, \forall x \in E_0 \setminus (\cup_{j=1}^{s-1} G_j), \tag{2.9}$$

$$\text{if } x \in G_s, \text{ then } \left|\sum_{n=N}^{N_s} b_n h_n(x) - \gamma\right| < \widetilde{\delta} \text{ or } \left|\sum_{n=N}^{N_s} b_n h_n(x) + \frac{4\gamma}{\varepsilon}\right| < \widetilde{\delta}. \tag{2.10}$$

Taking into account the relations (2.3),(2.4), and (2.8), we obtain

$$\text{mes}(E_0 \setminus \bigcup_{s=1}^{\infty} G_s) = 0,$$

$$\sum_{s=1}^{\infty} \text{mes } G_s = \text{mes} E_0.$$

We take a natural number $s_0 \geq 1$; so that

$$\text{mes}(\bigcup_{s=1}^{s_0} G_s) = \sum_{i=1}^{s_0} \text{mes } G_i \geq (1 - \frac{\varepsilon}{4}) \text{ mes} E_0 = \left(1 - \frac{\varepsilon}{4}\right) \text{ mes}\Delta.$$

Note that if we have a finite number steps in construction of numbers $\{N_s\}_{s=1}^{\infty}$, then we can take as s_0 the breaking step.

Let $E_1 = \bigcup_{s=1}^{s_0} G_s$. Obviously we have

$$\text{mes} E_1 > (1 - \frac{\varepsilon}{4}) \text{ mes}\Delta. \tag{2.11}$$

Now define a polynomial $Q(x)$ as follows:

$$Q(x) := \sum_{n=N_0}^{M} \epsilon_n b_n h_n(x), \tag{2.12}$$

where $M = N_{s_0}$ and

$$\epsilon_n := \begin{cases} 0 & \text{if } \Delta_n \bigcap \Delta = 0, \\ 0 & \text{if } n < N, \\ 0 & \text{if } (\Delta_n \cap E_0) \subset G_s, \text{ for some } s \in [1, s_0], \\ 1 & \text{otherwise.} \end{cases} \tag{2.13}$$

From (2.8), (2.9), (2.12), and (2.13), we get

$$|\sum_{n=N}^{m} \epsilon_n b_n h_n(x)| \le \frac{4\gamma}{\varepsilon}, \qquad \forall x \in [0,1], \forall m \in [N_0, M],$$

$$Q(x) = 0 \text{ if } x \notin [\alpha, \beta],$$

and

$$\text{if } x \in G_s \text{ and } x \in \Delta_n, \text{ for some } s \in [1, s_0] \text{ and } n > N_s, \text{ then } (\Delta_n \cap E_0) \subset G_s.$$

This immediately yields that if $x \in G_s$, $s \in [1, s_0]$, and $n > N_s$, then $\epsilon_n = 0$ or $h_n(x) = 0$. Clearly for each point $x \in G_s$, $s \in [1, s_0]$, we have

$$Q(x) = \sum_{n=N_0}^{M} \epsilon_n b_n h_n(x) = \sum_{n=N}^{N_s} b_n h_n(x) + \sum_{N_s+1}^{M} \epsilon_n b_n h_n(x) = \sum_{n=N}^{N_s} b_n h_n(x). \tag{2.14}$$

By (2.10) and (2.14), we have

$$\text{if } x \in E_1, \text{ then } |Q(x) - \gamma| < \widetilde{\delta} \text{ or } |Q(x) + \frac{4\gamma}{\varepsilon}| < \widetilde{\delta}.$$

Let

$$E_2 := \{x \in E_1 : |Q(x) - \gamma| < \widetilde{\delta}\}, \qquad A := \{x \in E_1 : |Q(x) + \frac{4\gamma}{\varepsilon}| < \widetilde{\delta}\}. \tag{2.15}$$

It is clear that $E_1 = E_2 \bigcup A$ and $E_2 \bigcap A = \emptyset$.
From (2.6), (2.11), and (2.15), we have

$$\begin{aligned}
0 = \int_0^1 Q(x)dx &= \int_\Delta Q(x)dx = \int_{\Delta \setminus E_1} Q(x)dx + \int_{E_1} Q(x)dx \\
&= \int_{\Delta \setminus E_1} Q(x)dx + \int_{E_2} Q(x)dx + \int_A Q(x)dx \\
&\le \frac{4\gamma}{\varepsilon} \text{ mes}(\Delta \setminus E_1) + (\gamma + \widetilde{\delta}) \text{ mes} E_2 + (-\frac{4\gamma}{\varepsilon} + \widetilde{\delta}) \text{ mes} A \\
&\le \gamma \text{ mes}\Delta + \gamma \text{ mes}\Delta + \widetilde{\delta}(\text{ mes}E_2 + \text{mes}A) - \frac{4\gamma}{\varepsilon} \text{ mes}A \\
&= 3\gamma \text{ mes}\Delta - \frac{4\gamma}{\varepsilon} \text{ mes}A.
\end{aligned}$$

Clearly

$$\text{mes}A \le \frac{3\varepsilon}{4} \text{ mes}\Delta.$$

Then (see (2.11) and (2.16))

$$\text{mes}E_2 \ge (1 - \varepsilon) \text{ mes}\Delta.$$

Let

$$D := \{x = \frac{i}{2^j} \in \Delta : i \le 2^j \ s.t. \ Q(x+0) \ne Q(x-0)\},$$

and let

$$E := E_2 \backslash D, \qquad \text{mes}E = \text{mes}E_2 \ge (1 - \varepsilon) \text{ mes}\Delta.$$

Define

$$g(x) := \begin{cases} \gamma & \text{if } x \in E, \\ Q(x) & \text{if } x \notin E \bigcup D, \\ \frac{g(x+0)+g(x-0)}{2} & \text{if } x \in D. \end{cases}$$

From the definition of function $g(x)$, we deduce that $g(x)$ is a dyadic step function and

$$\sup_{x\in D}|g(x)-Q(x)| = \sup_{x\in D}|\frac{g(x+0)+g(x-0)}{2} - \frac{Q(x+0)+Q(x-0)}{2}| < \delta.$$

From this and (2.15), we have

$$||g-Q||_\infty = \sup_{x\in[0,1]}|g(x)-Q(x)| = \max\{\sup_{x\in E}|\gamma - Q(x)|, \sup_{x\in D}|g(x)-Q(x)|\} < \delta.$$

The lemma is proved in the case $\gamma > 0$.

In the case when $\gamma < 0$, we consider the sequence $\{a_n\}_{n=0}^\infty$ with $a_n = -b_n$ and number $\gamma' = -\gamma$. It is easy to see that the sequence $\{a_n\}_{n=0}^\infty$ also satisfies conditions (2.1) and (2.2) and $\gamma' > 0$. Applying Lemma 2.1 for sequence $\{a_n\}_{n=0}^\infty$, numbers $\gamma' > 0$, ε, $\delta, N_0 \in \mathbb{N}$, and dyadic interval $\Delta = (\alpha, \beta)$, one can find a measurable set $E \subset \Delta$, a dyadic step function $g'(x)$, and a polynomial in the Haar system $Q'(x) = \sum_{n=N_0}^M \epsilon_n a_n h_n(x)$ (where $\epsilon_n = 0$ or 1) such that

(1) $\text{mes}E > (1-\varepsilon)\text{mes}\Delta$,
(2)
$$g'(x) = \begin{cases} \gamma' & \text{if } x \in E, \\ 0 & \text{if } x \notin [\alpha, \beta], \end{cases}$$
(3) $\|g' - Q'\|_\infty \leq \delta$,
(4) $Q'(x) = 0$ for all $x \notin [\alpha, \beta]$,
(5) $\max_{N_0\leq m\leq M} ||\sum_{n=N_0}^m \epsilon_n a_n h_n||_\infty \leq \frac{4|\gamma|}{\varepsilon}$.

If we choose $g(x) = -g'(x)$ and $Q(x) = -Q'(x) = -\sum_{n=N_0}^M \epsilon_n a_n h_n(x) = \sum_{n=N_0}^M \epsilon_n b_n h_n(x)$, then, from the above, we get the truthfulness of Lemma 2.1 in the case when $\gamma < 0$.

Lemma 2.1 is proved. □

Lemma 2.2. *Let $\{b_n\}_{n=1}^\infty \in \mathcal{B}$, and let numbers $N_0 > 1$, $\varepsilon > 0$, $\delta > 0$ and dyadic step function $f(x)$ be given. Then one can find a dyadic step function $g(x)$, measurable set $E \subset [0,1]$, and a polynomial in the Haar system of the form $Q(x) = \sum_{n=N_0}^M \epsilon_n b_n h_n(x)$, where $\epsilon_n = 0$ or 1, such that*

(1) $\text{mes}E > 1 - \varepsilon$,
(2) $g(x) = f(x)$ *for all* $x \in E$,
(3) $||g - Q||_\infty < \delta$,
(4) $\max_{N_0\leq m\leq M} ||\sum_{n=N_0}^m \epsilon_n b_n h_n||_\infty \leq \frac{8||f||_\infty}{\varepsilon}$.

Proof. Taking into account (1.1)–(1.3), we obtain that the sequence $\{b_n\}_{n=1}^\infty$ satisfies conditions (2.1) and (2.2).

Let

$$D := \left\{ \frac{i}{2^j} : i \le 2^j, j = 1, 2, \dots \right\},$$

and let

$$f(x) = \sum_{n=1}^{n_0} \gamma_n \cdot \chi_{\widehat{\Delta}_n}(x), \qquad \text{for } x \in [0,1] \setminus D, \tag{2.16}$$

where $\widehat{\Delta}_n = (\alpha_n, \beta_n)$ are dyadic intervals that satisfy $\sum_{n=1}^{n_0} \mathrm{mes}\widehat{\Delta}_n = 1$ and $\widehat{\Delta}_j \bigcap \widehat{\Delta}_i = \emptyset \ (i \ne j)$. We can assume that $\gamma_n \ne 0, n = 1, \dots, n_0$.

Successive applications of Lemma 2.1 yield measurable sets $E_n \subset \widehat{\Delta}_n$, dyadic step function $g_n(x)$, and polynomials

$$Q_n(x) = \sum_{k=N_{n-1}}^{N_n - 1} \epsilon_k b_k h_k(x), \qquad n \ge 1, \ N_n \nearrow, \quad \epsilon_k = 1 \text{ or } 0,$$

satisfy, for all $n = 1, \dots, n_0$,

$$Q_n(x) = 0 \qquad \text{if} \quad x \notin [\alpha_n, \beta_n], \tag{2.17}$$

$$g_n(x) = \begin{cases} \gamma_n & \text{if } x \in \ E_n, \\ 0 & \text{if } x \notin [\alpha_n, \beta_n], \end{cases} \tag{2.18}$$

$$||g_n - Q_n||_\infty < 2^{-n}\delta, \tag{2.19}$$

$$\mathrm{mes}E_n > (1-\varepsilon)\ \mathrm{mes}\widehat{\Delta}_n \tag{2.20}$$

and

$$\max_{N_{n-1} \le m < N_n} || \sum_{k=N_{n-1}}^{m} \epsilon_k b_k h_k ||_\infty \le \frac{4|\gamma_n|}{\varepsilon}. \tag{2.21}$$

Let

$$Q = \sum_{n=1}^{n_0} Q_n = \sum_{k=N_0}^{M} \epsilon_k b_k h_k \ \text{ where } M = N_{n_0} - 1, \tag{2.22}$$

and let

$$E = \bigcup_{n=1}^{n_0} E_n \qquad g = \sum_{n=1}^{n_0} g_n. \tag{2.23}$$

The definition of intervals $\widehat{\Delta}_n$, (2.20), and (2.23) immediately yield the assertion (1) in the statement of lemma.

By using (2.17)–(2.23), we obtain that $g(x)$ is a dyadic step function and

$$||g - Q||_\infty < \delta, \qquad g(x) = f(x) \qquad \text{for } x \in E.$$

For a given $m \in [N_0, M]$, there is a unique $\bar{n} \in [1, n_0]$ such that $m \in [N_{\bar{n}}, N_{\bar{n}+1})$. Thus

$$\sum_{k=N_0}^{m} \epsilon_k b_k h_k(x) = \sum_{n=1}^{\bar{n}} Q_n(x) + \sum_{k=N_{\bar{n}}}^{m} \epsilon_k b_k h_k(x).$$

From this, (2.16), (2.17), and (2.21), we have

$$\max_{N_0\le m\le M}||\sum_{k=N_0}^{m}\epsilon_k b_k h_k||_\infty\le\frac{8||f||_\infty}{\varepsilon}.$$

Lemma 2.2 is proved. □

3. Proof of Theorem 1.2.

Proof. Let $\varepsilon\in(0,1)$, and let $f(x)$ be an arbitrary measurable and almost everywhere finite function on $[0,1]$. By Luzin's theorem (see [9]) one can find a continuous function $g(x)$, defined on $[0,1]$, such that

$$\mathrm{mes}\{x\in[0,1]:f(x)\neq g(x)\}<\varepsilon/2.$$

One can find a sequence of dyadic step function $\{f_n(x)\}_{n=1}^{\infty}$ such that

$$\lim_{N\to\infty}||\sum_{n=1}^{N}f_n-g||_\infty=0\qquad ||f_n||_\infty\le\varepsilon\cdot 2^{-8(n+1)},\quad n\ge 2. \tag{3.1}$$

Further, by using induction and Lemma 2.2, we construct sequences of dyadic step function $\{\widetilde{g}_n(x)\}_{n=1}^{\infty}$, a sequence of polynomials

$$Q_n(x)=\sum_{k=m_{n-1}}^{m_n-1}\epsilon_k b_k h_k(x),\quad (\epsilon_k=0 \text{ or } 1, n=1,2,\dots)$$

and a sequence of sets $\{E_n\}_{n=1}^{\infty}$ satisfying the following conditions:

$$||\widetilde{g}_n||_\infty\le 2^{-2n}\ (n\ge 2), \tag{3.2}$$

$$\max_{m_{n-1}\le m\le m_n-1}||\sum_{k=m_{n-1}}^{m}\epsilon_k b_k h_k||_\infty\le 2^{-3(n-1)}, \tag{3.3}$$

$$\mathrm{mes}E_n>1-\varepsilon 2^{-n-1}, \tag{3.4}$$

$$\widetilde{g}_n(x)=f_n(x)\qquad\forall x\in E_n, \tag{3.5}$$

$$\left|\left|\sum_{j=1}^{n}(Q_j-\widetilde{g}_j)\right|\right|_\infty\le\varepsilon\ 2^{-5(n+1)}. \tag{3.6}$$

At first step, let us apply Lemma 2.2 for $f_1(x)$ and find set E_1, dyadic step function $g_1(x)$, and polynomial in the Haar system of the form

$$Q_1(x)=\sum_{k=1}^{m_1-1}\epsilon_k b_k h_k(x),\qquad \epsilon_k=0 \text{ or } 1,$$

satisfying the following conditions:

$$\widetilde{g}_1(x)=g_1(x)=f_1(x),\qquad x\in E_1,$$

$$\mathrm{mes}E_1>1-\varepsilon 2^{-2},$$

$$||\widetilde{g}_1-Q_1||_\infty\le\varepsilon 2^{-10},$$

$$||\widetilde{g}_1||_\infty\le 8\varepsilon^{-1}||f_1||_\infty+\varepsilon 2^{-10}.$$

It is easy to see that the conditions (3.4)–(3.6) are true for $n=1$.

Assume that dyadic step functions $\widetilde{g}_n(x)$, $1 \le n \le q-1$, sets $E_1, E_2, \dots, E_{q-1}$, and polynomials

$$Q_n(x) = \sum_{k=m_{n-1}}^{m_n-1} \epsilon_k b_k h_k(x), \qquad 1 \le n \le q-1, m_0 = 1,$$

are chosen in such way that the conditions (3.4)–(3.6) are fulfilled for all $n \le q-1$. Now we construct function $\widetilde{g}_q(x)$, a polynomial $Q_q(x)$, and set E_q and show that the conditions (3.2)–(3.6) hold for $n = q$.

Let

$$H_q(x) = f_q(x) - \sum_{n=1}^{q-1} \left[Q_n(x) - \widetilde{g}_n(x) \right]. \tag{3.7}$$

From above, (3.1), and (3.6), one can find

$$||H_q||_\infty \le \varepsilon \ 2^{-4q-1}.$$

It is not hard to see that the function $H_q(x)$ is a dyadic step function. By applying Lemma 2.2, we obtain set E_q, dyadic step function $g_q(x)$, and polynomial in the Haar system of the form

$$Q_q(x) = \sum_{k=m_{q-1}}^{m_q-1} \epsilon_k b_k h_k(x),$$

where ϵ_k is equal to 0 or 1, which satisfy the following conditions:

$$g_q(x) = H_q(x), \qquad x \in E_q, \tag{3.8}$$

$$\mathrm{mes} E_q > 1 - \varepsilon 2^{-q-1}, \tag{3.9}$$

$$||g_q - Q_q||_\infty \le \varepsilon 2^{-5(q+1)}, \tag{3.10}$$

$$\max_{m_{q-1} \le m \le m_q - 1} || \sum_{k=m_{q-1}}^{m} \epsilon_k b_k h_k ||_\infty \le \varepsilon^{-1} \ \ 2^{q+4} ||H_q||_\infty \le 2^{-3(q-1)}. \tag{3.11}$$

We put

$$\widetilde{g}_q(x) = f_q(x) - [H_q(x) - g_q(x)] = \sum_{j=1}^{q-1} (Q_j(x) - \widetilde{g}_j(x)) + g_q(x). \tag{3.12}$$

By (3.8), we have

$$\widetilde{g}_q(x) = f_q(x), \qquad \forall x \in E_q. \tag{3.13}$$

By employing the relations (3.6), (3.7), and (3.10)–(3.12), we find that

$$||\widetilde{g}_q||_\infty \le \left|\left| \sum_{j=1}^{q-1} (Q_j - \widetilde{g}_j) \right|\right|_\infty + ||g_q||_\infty \le 2^{-3q+4}. \tag{3.14}$$

From this and (3.7), (3.10), and (3.12), we get

$$\left|\left|\sum_{j=1}^{q}(Q_j-\widetilde{g}_j)\right|\right|_\infty = \left|\left|\sum_{j=1}^{q-1}(Q_j-\widetilde{g}_j)+Q_q-\widetilde{g}_q\right|\right|_\infty$$
$$= \left|\left|H_q-\left(f_q-\sum_{n=1}^{q-1}[Q_n-\widetilde{g}_n]\right)+Q_q-g_q\right|\right|_\infty$$
$$= ||Q_q-g_q||_\infty \leq \varepsilon\, 2^{-5(q+1)}. \tag{3.15}$$

Clearly we prove that the statements (3.2)–(3.6) are true for $n=q$. (see (3.9), (3.11), (3.13)–(3.15)).

We put

$$\widetilde{f}(x)=\sum_{n=1}^{\infty}\widetilde{g}_n(x).$$

Obviously (see (3.1),(3.2),(3.4),(3.5))

$$\widetilde{f}(x)\in L^\infty[0,1],$$

$$\widetilde{f}(x)=g(x),\ \text{ for } x\in\bigcap_{n=1}^{\infty}E_n,\ \text{ and }\ \operatorname{mes}(\bigcap_{n=1}^{\infty}E_n)\geq 1-\frac{\varepsilon}{2}.$$

Consequently,

$$\operatorname{mes}\{x\in[0,1]:\widetilde{f}(x)=f(x)\}\geq \operatorname{mes}(\bigcap_{n=1}^{\infty}E_n)-\operatorname{mes}\{x\in[0,1]:f(x)\neq g(x)\}\geq 1-\varepsilon.$$

From (3.3) and (3.6) it follows that the series

$$\sum_{k=1}^{\infty}\epsilon_k b_k h_k(x)=\sum_{q=1}^{\infty}\sum_{k=m_{q-1}}^{m_q-1}\epsilon_k b_k h_k(x)$$

converges to the function $\widetilde{f}(x)$ uniformly in $[0,1]$. Therefore

$$\epsilon_k b_k=\int_0^1\widetilde{f}(x)h_k(x)dx=c_k(\widetilde{f}),\qquad k=1,2,\dots.$$

and Theorem 1.2 is proved. □

Remark 3.1. Note that Lemma 2.2 is true with assumptions (2.1) and (2.2), which are weaker than assumption $\{b_n\}_{n=1}^{\infty}\in\mathcal{B}$. That would yield a more general result.

Suppose that the sequence $\{b_n\}_{n=1}^{\infty}$ satisfies the conditions (2.1),(2.2), and that f is measurable and almost everywhere finite function on $[0,1]$. Then there exists a function $\widetilde{f}\in L^\infty[0,1]$ with the following properties:

(1) $\operatorname{mes}\{x\in[0,1]:\widetilde{f}(x)\neq f(x)\}<\varepsilon$,
(2) $c_n(\widetilde{f})=b_n, n\in spec(\widetilde{f})$,
(3) $\lim_{n\to\infty}||\sum_{n=1}^{\infty}c_n(\widetilde{f})h_n-\widetilde{f}||_\infty=0$.

References

1. F. G. Arutyunyan, *On series in the Haar system*, ANA SSR Dokl. **42** (1966), no.3, 134–140.
2. R. A. DeVore, V. N. Temlyakov, *Some remarks on greedy algorithms*, Adv. Comput. Math. **5** (1966) no. 2-3, 173–187.
3. S. A. Episkoposyan, *On the existence of universal series by the generalized Walsh system* , Banach J. Math. Anal. **10** (2016), no. 2, 415–429.
4. L. N. Galoyan, M. G. Grigoryan, and A. Kh. Kobelyan, *Convergence of Fourier series in classical systems*, Sbornik Mathematics **206** (2015), no. 7-8, 941–979. Matematicheski Sbornik **206** (2015), no. 7, 55–94.
5. M. G. Grigorian, *On the convergence of Fourier series in the metric of* L^1, Anal. Math. **17** (1991), 211–237.
6. M. G. Grigoryan, *On the universal and strong* (L^1, L^∞)*-property related to Fourier–Walsh series*, Banach J. Math. Anal. **11** (2017), no. 3, 698–712.
7. M. G. Grigoryan, V. G. Krotov, *Luzin's Correction Theorem and the Coefficients of Fourier Expansions in the Faber-Schauder System*, Math. Notes **93** (2013), no. 2, 11–17.
8. A. Haar, *Zur Theorie der orthogonalen Funktionensysteme* , Math. Ann. **69** (1910), no. 3, 331–371.
9. N. N. Luzin, *On the basic theorem of integral calculus*, Mat. Sbornik **28** (1912), 266–294.
10. J. Marcinkiewicz, *Quelques thoremes sur les sries orthogonales*, Ann. Soc. Polon. Math., **16** (1937), 84–96 (pp. 307–318 of the Collected Papers).
11. D. E. Menchoff, *Sur la convergence uniforme des séries de Fourier* , Mat. Sbornik **53** (1942), no. 1-2, 67–96.(French)
12. K. A. Navasardyan, A. A. Stepanyan, *Series by Haar system*, Izv. Nats. Akad. Nauk Armenii mat. **42** (2007), no. 4, 53–66.
13. A. M. Olevskii, *Modification of functions and Fourier series*, Uspekhi Mat. Nauk, 40 (1985), no. 3(243), 157–193.
14. K. I. Oskolkov, *The uniform modulus of continuity of integrable functions on sets of positive measure*, Dokl. Akad. Nauk SSSR **229** (1976), 304–306.
15. J. J. Price, *Walsh series and adjustment of functions on small sets*, Illinois J. Math. **13** (1969), no. 1,131–136.

Higer mathematics, Faculty of Physics, Yerevan State University, Alex Manukian st. 1,0025 Yerevan, Armenia.

E-mail address: gmarting@ysu.am

E-mail address: a_kobelyan@ysu.am