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STABILITY OF A FUNCTIONAL EQUATION OF WHITEHEAD ON SEMIGROUPS

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ABSTRACT. Let S be a semigroup and X a Banach space. The functional equation $\varphi(xyz) + \varphi(x) + \varphi(y) + \varphi(z) = \varphi(xy) + \varphi(yz) + \varphi(xz)$ is said to be stable for the pair (X, S) if and only if $f : S \to X$ satisfying $\|f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(yz) - f(xz)\| \leq \delta$ for some positive real number δ and all $x, y, z \in S$, there is a solution $\varphi : S \to X$ such that $f - \varphi$ is bounded. In this paper, among others, we prove the following results: 1) this functional equation, in general, is not stable on an arbitrary semigroup; 2) this equation is stable on periodic semigroups; 3) this equation is stable on abelian semigroups; 4) any semigroup with left (or right) law of reduction can be embedded into a semigroup with left (or right) of reduction where this equation is stable. The main results of this paper generalize the works of Jung [J. Math. Anal. Appl. 222 (1998), 126–137], Kannappan [Results Math. 27 (1995), 368–372] and Fechner [J. Math. Anal. Appl. 322 (2006), 774–786].

1. INTRODUCTION

If $f : \mathbb{V} \to X$ is a function from a normed vector space $\mathbb{V}$ into a Banach space X, and $\|f(x+y) - f(x) - f(y)\| \leq \delta$ for some nonnegative real number δ, Hyers [16], answering a question of Ulam [24], proved that there exists an additive function $A : \mathbb{V} \to X$ such that $\|f(x) - A(x)\| \leq \delta$. Taking this result into account, the additive Cauchy functional equation is said to stable in the sense of Hyers-Ulam on $(\mathbb{V}, X)$ if for each function $f : \mathbb{V} \to X$ satisfying the inequality $\|f(x+y) - f(x) - f(y)\| \leq \delta$ for some $\delta \geq 0$ and for all $x, y \in \mathbb{V}$ there exists an additive function $A : \mathbb{V} \to X$ such that $f - A$ is bounded on $\mathbb{V}$. Since then,

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the stability problems of various functional equations have been studied by many authors (see the survey paper [22] and references therein). Among them, Skof [23] first considered the Hyers-Ulam stability of the quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) \tag{1.1}$$

where f maps a group G to an abelian group H. As usual, each solution of equation (1.1) is called a quadratic function. But Skof restricted herself to studying the case where f maps a normed space to a Banach space. In [2] Cholewa noticed that the theorem of Skof's is true if the relevant domain is replaced by an abelian group. The results of Skof and Cholewa were further generalized by Czerwik [3]. Further works on stability of the quadratic functional equation can be found in Fenyő [13], Czerwik [4], Czerwik and Dlutek [5, 6], Ger [15], Jung [18], and Jung and Sahoo [19].

Let $\mathbb{R}$ denote the set of real numbers. Let G be a group and X and Y be any two arbitrary Banach spaces over reals. Faiziev and Sahoo [8] proved that the quadratic functional equation is stable for the pair (G, X) if and only if it is stable for the pair (G, Y). In view of this result it is not important which Banach space is used on the range. Thus one may consider the stability of the quadratic functional equation on the pair $(G, \mathbb{R})$. Faiziev and Sahoo [8] proved that quadratic functional equation is not stable on the pair $(G, \mathbb{R})$ when G is any arbitrary group. It is well known (see Skof [23] and Cholewa [2]) that the quadratic functional equation is stable on the pair $(G, \mathbb{R})$ when G is an abelian group. Thus it is interesting to know on which noncommutative groups the quadratic functional equation is stable in the sense of Hyers-Ulam. Faiziev and Sahoo [8] proved that quadratic functional equation is stable on n-abelian groups and $T(2, \mathbb{K})$, where $\mathbb{K}$ is a commutative field. Further they also proved that every group can be embedded into a group in which the quadratic functional equation is stable. Yang [27] proved the stability of quadratic functional equation on amenable groups.

In an American Mathematical Society meeting, E. Y. Deeba of the University of Houston asked to find the general solution $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(x+z). \tag{1.2}$$

This functional equation is a variation of the quadratic functional equation and it was first appeared in a paper of Whitehead [25]. Kannappan [20] showed that the general solution $f : \mathbb{V} \to \mathbb{K}$ of the above functional equation is of the form

$$f(x) = B(x, x) + A(x)$$

where $B : \mathbb{V} \times \mathbb{V} \to \mathbb{K}$ is a symmetric biadditive function and $A : \mathbb{V} \to \mathbb{K}$ is an additive function, $\mathbb{V}$ is a vector space, and $\mathbb{K}$ is a field of characteristic different from two (or of characteristic zero).

The Hyers-Ulam stability of the equation (1.2) was investigated by Jung [17]. He proved the following theorem.

Theorem 1.1. *Suppose $\mathbb{V}$ is a real normed space and X a real Banach space. Let $f : \mathbb{V} \to X$ satisfy the inequalities*

$$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(y+z) - f(x+z)\| \le \delta \tag{1.3}$$

and

$$\|f(x) - f(-x)\| \leq \theta \tag{1.4}$$

for some $\delta, \theta \geq 0$ *and for all* $x, y, z \in \mathbb{V}$. *Then there exists a unique quadratic mapping* $Q : \mathbb{V} \to X$ *which satisfies*

$$\|f(x) - Q(x)\| \leq 3\,\delta \tag{1.5}$$

for all $x \in \mathbb{V}$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in \mathbb{V}$, *then* $Q(tx) = t^2\,Q(x)$ *for all* $x \in \mathbb{V}$ *and* $t \in \mathbb{R}$.

Jung [17] proved another theorem replacing the inequality $\|f(x) - f(-x)\| \leq \theta$ by $\|f(x) + f(-x)\| \leq \theta$. Fechner [12] proved the stability of the functional equation (1.2) on abelian group. For this functional equation (1.2), Kim [21] proved a generalized stability result in the spirit of Gavruta [14]. Chang and Kim [1] generalized the theorem of Jung [17] and proved the following theorem.

Theorem 1.2. *Suppose* $\mathbb{V}$ *is a real normed space and* X *a real Banach space. Let* $H : \mathbb{R}_+^3 \to \mathbb{R}_+$ *be a function such that* $H(tu, tv, tw) \leq t^p H(u, v, w)$ *for all* $t, u, v, w \in \mathbb{R}_+$ *and for some* $p \in \mathbb{R}$. *Further, let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $E(tx) \leq t^q E(x)$ *for all* $t, x \in \mathbb{R}_+$. *Let* $p, q < 1$ *be real numbers and let* $f : \mathbb{V} \to X$ *satisfy the inequalities*

$$\begin{aligned}\|f(x+y+z) &+ f(x) + f(y) + f(z)\\ &- f(x+y) - f(y+z) - f(z+x)\| \leq H(\|x\|, \|y\|, \|z\|)\end{aligned} \tag{1.6}$$

and

$$\|f(x) - f(-x)\| \leq E(\|x\|) \tag{1.7}$$

for some $\delta, \theta \geq 0$ *and for all* $x, y, z \in \mathbb{V}$. *Then there exists a unique quadratic mapping* $Q : \mathbb{V} \to X$ *which satisfies*

$$\|f(x) - Q(x)\| \leq \frac{H(\|x\|, \|x\|, \|x\|)}{2 - 2^p} + 2\|f(0)\| \tag{1.8}$$

for all $x \in \mathbb{V}$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in \mathbb{V}$, *then* $Q(tx) = t^2\,Q(x)$ *for all* $x \in \mathbb{V}$ *and* $t \in \mathbb{R}$.

Chang and Kim [1] also proved another similar theorem replacing the inequality $\|f(x) - f(-x)\| \leq E(\|x\|)$ by $\|f(x) + f(-x)\| \leq E(\|x\|)$.

The functional equation (1.2) is takes the form

$$f(xyz) + f(x) + f(y) + f(z) = f(xy) + f(yz) + f(xz) \tag{1.9}$$

on an arbitrary group G or on a semigroup S. In this sequel, we will write the arbitrary semigroup S in multiplicative notation. Similarly, the arbitrary group G will be written in multiplicative notation so that 1 will denote the identity element of G. This functional equation implies the Drygas functional equation $f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1})$ whose general solution was presented in Ebanks, Kannappan and Sahoo [7]. The stability of the Drygas functional equation was studied by Jung and Sahoo [19] and also by Yang [26]. The system of equations $f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1})$ and $f(yx) + f(y^{-1}x) = 2f(x) + f(y) + f(y^{-1})$ generalizes the Drygas functional equation on groups. The stability of this system of equation was investigated by Faiziev and Sahoo (see [9, 10, 11]) on nonabelian groups.

In the present paper, we consider the stability of the functional equation (1.9) for the pair (S, E) when S is an arbitrary semigroup and E is a real Banach space. If X is another real Banach space, then we prove that the functional equation (1.9) is stable for the pair (S, X) if and only if it is stable for the pair (S, E). We show that, in general, the equation (1.9) is not stable on semigroups. However, this equation (1.9) is stable on periodic semigroups as well as abelian semigroups. We also show that any semigroup with left (or right) cancellation law can be embedded into a semigroup with left (or right) cancellation law where the equation (1.9) is stable. The main results of this paper generalize the works of Jung [17], Kannappan [20], and Fechner [12].

2. Decomposition

Let S be a semigroup and X be a Banach space. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}$ be the set of integers.

Definition 2.1. A mapping $f : S \to X$ is said to be *a kannappan* mapping if it satisfies equation

$$f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) = 0 \tag{2.1}$$

for all $x, y, z \in S$.

Definition 2.2. We will say that $f : S \to X$ is a quasikannappan mapping if there is $c > 0$ such that

$$\|f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)\| \le c \tag{2.2}$$

for all $x, y, z \in S$.

The set of kannappan and quasikannappan mappings will be denote by $K(S, X)$ and $KK(S, X)$, respectively.

Lemma 2.3. *If $f \in KK(S, X)$, then for any $n \ge 3$ and $x_1, \ldots, x_n \in S$ the inequality*

$$\left\| f(x_1 x_2 \cdots x_n) + (n-2) \sum_{i=1}^{n} f(x_i) - \sum_{1 \le i < j \le n} f(x_i x_j) \right\| \le \frac{(n-2)(n-1)}{2} c \tag{2.3}$$

holds.

Proof. We prove this lemma by induction. First we show that the inequality (2.3) is true for $n = 4$. Since $f \in KK(S, X)$, we obtain from (2.2)

$$\begin{aligned} \| f(x_1x_2x_3x_4) + f(x_1) + f(x_2) + f(x_3x_4) \\ - f(x_1x_2) - f(x_1x_3x_4) - f(x_2x_3x_4) \| \le c, \end{aligned}$$

$$\| f(x_2x_3x_4) + f(x_2) + f(x_3) + f(x_4) - f(x_2x_3) - f(x_2x_4) - f(x_3x_4) \| \le c,$$

and

$$\| f(x_1x_3x_4) + f(x_1) + f(x_3) + f(x_4) - f(x_1x_3) - f(x_1x_4) - f(x_3x_4) \| \le c.$$

Therefore from the above three inequalities we have

$$\begin{aligned}\| f(x_1x_2x_3x_4) &+ f(x_1) + f(x_2) + f(x_3x_4) - f(x_1x_2)\\ &+f(x_1) + f(x_3) + f(x_4) - f(x_1x_3) - f(x_1x_4) - f(x_3x_4)\\ &+f(x_2) + f(x_3) + f(x_4) - f(x_2x_3) - f(x_2x_4) - f(x_3x_4) \| \leq 3c.\end{aligned}$$

Simplifying we see that

$$\begin{aligned}\| f(x_1x_2x_3x_4) &+ 2[f(x_1) + f(x_2) + f(x_3) + f(x_4)] - f(x_1x_2)\\ &- f(x_1x_3) - f(x_1x_4) - f(x_2x_3) - f(x_2x_4) - f(x_3x_4) \| \leq 3c\end{aligned}$$

and this shows that inequality (2.3) holds for $n = 4$. We will rewrite the above inequality as

$$\begin{aligned}\| f(x_1x_2x_3x_4) &+ 2[f(x_1) + f(x_2) + f(x_3) + f(x_4)] - f(x_1x_2)\\ &- f(x_1x_3) - f(x_1x_4) - f(x_2x_3) - f(x_2x_4) - f(x_3x_4) \| \leq c_4\end{aligned}$$

where $c_4 = 3\,c$. Next suppose the above inequality holds for a positive integer n. That is

$$\left\| f(x_1x_2\cdots x_n) + (n-2)\sum_{i=1}^{n} f(x_i) - \sum_{1\leq i<j\leq n} f(x_ix_j) \right\| \leq c_n.$$

Consider

$$\left\| f(x_1x_2\cdots x_nx_{n+1}) + (n-1)\sum_{i=1}^{n+1} f(x_i) - \sum_{1\leq i<j\leq n+1} f(x_ix_j) \right\|.$$

By our supposition we have

$$\begin{aligned}\Bigg\| f(x_1x_2\cdots(x_nx_{n+1})) &+ (n-2)\left[\sum_{i=1}^{n-1} f(x_i) + f(x_nx_{n+1})\right]\\ &- \sum_{1\leq i<j\leq n-1} f(x_ix_j) - \sum_{1\leq i\leq n-1} f(x_ix_nx_{n+1}) \Bigg\| \leq c_n.\end{aligned}$$

Hence

$$\begin{aligned}\Bigg\| f(x_1x_2\cdots(x_nx_{n+1})) &+ (n-2)\left[\sum_{i=1}^{n-1} f(x_i) + f(x_nx_{n+1})\right] - \sum_{1\leq i<j\leq n-1} f(x_ix_j)\\ &+ \sum_{1\leq i\leq n-1} [f(x_i) + f(x_n) + f(x_{n+1}) - f(x_ix_n) - f(x_ix_{n+1}) - f(x_nx_{n+1})] \Bigg\|\\ \leq c_n &+ (n-1)\,c.\end{aligned}$$

The last inequality can be rewritten as

$$\Bigg\| f(x_1x_2\cdots(x_nx_{n+1})) + (n-1)\sum_{i=1}^{n+1} f(x_i) + (n-2)f(x_nx_{n+1}) - \sum_{1\le i<j\le n-1} f(x_ix_j) + \sum_{1\le i\le n-1}\big[-f(x_ix_n) - f(x_ix_{n+1}) - f(x_nx_{n+1})\big]\Bigg\|$$

$\le c_n + (n-1)\,c.$

Hence

$$\Bigg\| f(x_1x_2\cdots x_nx_{n+1}) + (n-1)\sum_{i=1}^{n+1} f(x_i) - \sum_{1\le i<j\le n+1} f(x_ix_j)\Bigg\| \le c_{n+1}$$

where $c_{n+1} = c_n + (n-1)\,c$ for $n \ge 3$.

From the recurrence relations $c_3 = c$ and $c_{n+1} = c_n + (n-1)\,c$ for $n \ge 3$, we get

$$c_{n+1} = \frac{n\,(n-1)}{2}.$$

Thus we have proved the inequality (2.3) for all positive integers n. □

The following lemma follows from the above lemma.

Lemma 2.4. *If $f \in KK(S,X)$, then for any $n \ge 3$, the inequality*

$$\left\| f(x^n) + (n-2)\,n\,f(x) - \frac{(n-1)\,n}{2} f(x^2)\right\| \le \frac{(n-2)\,(n-1)}{2}\,c \tag{2.4}$$

holds for all $x \in S$.

Proof. Letting $x_1 = x_2 = \cdots = x_n = x$ in the inequality (2.3), we have the asserted inequality (2.4). □

Lemma 2.5. *Let the function $\phi : S \to X$ be define by $\phi(x) = f(x^2)$.*

(1) *If $f \in KK(S,X)$, then $\phi \in KK(S,X)$.*
(2) *If $f \in K(S,X)$, then $\phi \in K(S,X)$.*

Proof. Since $f \in KK(S,X)$, we have

$$\|\, f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)\,\| \le c.$$

Consider

$$\|\,\phi(xyz) + \phi(x) + \phi(y) + \phi(z) - \phi(xy) - \phi(xz) - \phi(yz)\,\| = \|\, f(xyzxyz) + f(xx) + f(yy) + f(zz) - f(xyxy) - f(xzxz) - f(yzyz)\,\|.$$

We have

$$\|\, f((xy)z(xy)z) + 4f(xy) + 4f(z) - f(xyz) - f(xyxy) - f(xyz) - f(zxy) - f(zz) - f(xyz)\,\| \le 3c, \tag{2.5}$$

$$\|\, f((xy)z(xy)z) + 4f(xy) + 4f(z) - 3f(xyz) - f(xyxy) - f(zxy) - f(zz)\,\| \le 3c, \tag{2.6}$$

$$\| f(xzxz)+4f(x)+4f(z)-3f(xz)-f(zx)-f(x^2)-f(z^2)\| \le 3c, \quad (2.7)$$
$$\| f(yzyz)+4f(y)+4f(z)-3f(yz)-f(zy)-f(y^2)-f(z^2)\| \le 3c. \quad (2.8)$$

From (2.5)–(2.8) we have

$$\begin{aligned}
&\| f(xyzxyz)+f(xx)+f(yy)+f(zz)-f(xyxy)-f(xzxz)-f(yzyz)\| \\
&\quad = \| f(xyzxyz)+4f(xy)+4f(z)-3f(xyz)-f(xyxy)-f(zxy) \\
&\qquad -f(zz)-4f(xy)-4f(z)+3f(xyz)+f(xyxy)+f(zxy) \\
&\qquad +f(zz)+f(xx)+f(yy)+f(zz)-f(xyxy) \\
&\qquad -f(xzxz)-4f(x)-4f(z)+3f(xz)+f(x^2)+f(zx)+f(z^2) \\
&\qquad +4f(x)+4f(z)-3f(xz)-f(x^2)-f(zx)-f(z^2) \\
&\qquad -f(yzyz)-4f(y)-4f(z)+3f(yz)+f(y^2)+f(zy)+f(z^2) \\
&\qquad +4f(y)+4f(z)-3f(yz)-f(y^2)-f(zy)-f(z^2)\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\| f(xyzxyz)+f(xx)+f(yy)+f(zz)-f(xyxy)-f(xzxz)-f(yzyz)\| \\
&\quad \le \| f(xyzxyz)+4f(xy)+4f(z)-3f(xyz)-f(xyxy)-f(zxy)-f(zz)\| \\
&\qquad +\| -f(xzxz)-4f(x)-4f(z)+3f(xz)+f(x^2)+f(zx)+f(z^2)\| \\
&\qquad +\| -f(yzyz)-4f(y)-4f(z)+3f(yz)+f(y^2)+f(zy)+f(z^2)\| \\
&\qquad +\| -4f(xy)-4f(z)+3f(xyz)+f(xyxy)+f(zxy) \\
&\qquad\qquad +f(zz)+f(xx)+f(yy)+f(zz)-f(xyxy) \\
&\qquad\qquad +4f(x)+4f(z)-3f(xz)-f(x^2)-f(zx)-f(z^2) \\
&\qquad\qquad +4f(y)+4f(z)-3f(yz)-f(y^2)-f(zy)-f(z^2)\|.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\| -4f(xy)-4f(z)+3f(xyz)+f(xyxy)+f(zxy)+f(zz)+f(xx) \\
&\quad +f(yy)+f(zz)-f(xyxy)+4f(x)+4f(z)-3f(xz) \\
&\quad -f(x^2)-f(zx)-f(z^2)+4f(y)+4f(z) \\
&\quad -3f(yz)-f(y^2)-f(zy)-f(z^2)\ \| \\
&= \| 3f(xyz)+f(zxy)+4f(z)+4f(x)+4f(y)-4f(xy)-3f(xz)-3f(yz) \\
&\qquad -f(zx)-f(zy)\ \| \\
&\le \| f(zxy)+f(z)+f(x)+f(y)-f(xy)-f(zx)-f(zy)\ \| \\
&\quad +\| 3f(xyz)+3f(z)+3f(x)+3f(y)-3f(xy)-3f(xz)-3f(yz)\ \| \\
&\le 3c+9c=12c.
\end{aligned}$$

Hence

$$\begin{aligned}
&\| f(xyzxyz)+f(xx)+f(yy)+f(zz)-f(xyxy)-f(xzxz)-f(yzyz)\| \\
&\le 3c+3c+3c+12c=21c.
\end{aligned}$$

Thus from the last inequality we have

$$\| \phi(xyz)+\phi(x)+\phi(y)+\phi(z)-\phi(xy)-\phi(xz)-\phi(yz)\| \le 21c.$$

The proof of (2) follows similarly. □

Lemma 2.6. *Let $\{a_k\}_1^\infty$ be a sequence in X such that for any $m,k\in\mathbb{N}$*

$$\| a_{m+k}-2a_{k+1}+a_k\| \le \frac{d}{4^k} \quad (2.9)$$

holds. Then $\{a_k\}_1^\infty$ *is a Cauchy sequence.*

Proof. For any positive integers n, m we have

$$\| a_{m+k} - 2a_{k+1} + a_k \| \leq \frac{d}{4^k},$$

and

$$\| a_{n+k} - 2a_{k+1} + a_k \| \leq \frac{d}{4^k}.$$

Hence

$$\| a_{n+k} - a_{m+k} \| \leq \frac{2d}{4^k}.$$

The latter inequality implies that $\{a_k\}_1^\infty$ is a Cauchy sequence. □

Lemma 2.7. *Suppose $f \in KK(S, X)$. For any $x \in S$, the limit*

$$\lim_{n\to\infty} \frac{1}{4^n} f(x^{2^n}) = \widehat{f}(x), \tag{2.10}$$

exists and it satisfies the relations

$$\widehat{f}(x^n) = n^2\, \widehat{f}(x), \tag{2.11}$$

$$\left\| \widehat{f}(x) - \left[\frac{1}{2} f(x^2) - f(x)\right] \right\| \leq \frac{1}{2}\, c \tag{2.12}$$

for all $x \in S$ and $n \in \mathbb{N}$.

Proof. From (2.4) it follows that

$$\left\| \frac{1}{n^2}\, f(x^n) + \left(1 - \frac{2}{n}\right) f(x) - \frac{1}{2}\left(1 - \frac{1}{n}\right) f(x^2) \right\| \leq \frac{1}{2}\left(1 - \frac{3}{n} + \frac{2}{n^2}\right) c,$$

$$\left\| \frac{1}{n^2}\, f(x^n) - \left[\frac{1}{2} f(x^2) - f(x)\right] + \frac{1}{2n} f(x^2) - \frac{2}{n} f(x) \right\| \leq \frac{1}{2} c. \tag{2.13}$$

Therefore, there is an n_0 such that if $n > n_0$, then

$$\left\| \frac{1}{n^2} f(x^n) - \left[\frac{1}{2} f(x^2) - f(x)\right] \right\| \leq c. \tag{2.14}$$

Therefore, in (2.13) and (2.14) replacing n by 2^m, we have

$$\left\| \frac{1}{4^m}\, f(x^{2^m}) - \left[\frac{1}{2}\, f(x^2) - f(x)\right] \right\| \leq c, \tag{2.15}$$

and

$$\left\| \frac{1}{4^{m+k}}\, f(x^{2^{m+k}}) - \left[\frac{2}{4^{k+1}}\, f(x^{2^{k+1}}) - \frac{1}{4^k}\, f(x^{2^k})\right] \right\| \leq \frac{1}{4^k}\, c. \tag{2.16}$$

Denote $\frac{1}{4^k} f(x^{2^k})$ by a_k. Then from (2.16) we have

$$\| a_{m+k} - 2a_{k+1} + a_k \| \leq \frac{1}{4^k}\, c.$$

Now from Lemma 2.6 it follows that the sequence
$$\left\{ a_k := \frac{1}{4^k} f\left(x^{2^k}\right) \right\}_{k=1}^{\infty}$$
is a Cauchy sequence and thus has a limit. We denote this limit by $f_2(x)$. So
$$f_2(x) = \lim_{k\to\infty} \frac{1}{4^k}\, f\left(x^{2^k}\right).$$
Hence we have
$$\begin{aligned} f_2\left(x^{2^m}\right) &= \lim_{k\to\infty} \frac{1}{4^k}\, f\left(x^{2^{k+m}}\right) \\ &= \lim_{k\to\infty} \frac{4^m}{4^{k+m}}\, f\left(x^{2^{k+m}}\right) \\ &= 4^m \lim_{k\to\infty} \frac{1}{4^{k+m}}\, f\left(x^{2^{k+m}}\right) \\ &= 4^m\, f_2(x). \end{aligned}$$
From the relation (2.15) it follows that
$$\left\| f_2(x) - \left[\frac{1}{2} f(x^2) - f(x)\right] \right\| \le c.$$
Taking into account Lemma 2.5 we see that $f_2 \in KK(S,X)$.

Now let $m \ge 3$ be a positive integer. Then for any $x \in S$ we have
$$\left\| f_2\left(x^{m^n}\right) + (m^n-2)\, m^n\, f_2(x) - \frac{(m^n-1)\, m^n}{2} f_2\left(x^2\right) \right\| \le \frac{(m^n-2)\,(m^n-1)}{2}\, c.$$
Dividing the both sides of the last inequality by m^{2n} and simplifying, we have
$$\left\| \frac{1}{m^{2n}} f_2\left(x^{m^n}\right) + \left(1 - \frac{2}{m^n}\right) f_2(x) - \left(\frac{1}{2} - \frac{1}{2m^n}\right) f_2\left(x^2\right) \right\| \le \frac{(m^n-2)\,(m^n-1)}{2\, m^{2n}} c.$$
Hence we have
$$\left\| \frac{1}{m^{2n}}\, f_2\left(x^{m^n}\right) - \left[\frac{1}{2}\, f_2(x^2) - f_2(x)\right] - \frac{2}{m^n}\, f_2(x) + \frac{1}{2m^n}\, f_2(x^2) \right\| \le \frac{1}{2}\, c.$$
Therefore, there is an n_0 such that if $n \ge n_0$, then
$$\left\| \frac{1}{m^{2n}}\, f_2\left(x^{m^n}\right) - \left[\frac{1}{2} f_2\left(x^2\right) - f_2(x)\right] \right\| \le c.$$
From the later relation it follows that
$$\left\| \frac{1}{m^{2n}}\, f_2\left(x^{m^{n+k}}\right) - \left[\frac{1}{2}\, f_2\left(\left(x^{m^k}\right)^2\right) - f_2\left(x^{m^k}\right)\right] \right\| \le c.$$
Now dividing the both sides of the last inequality by m^{2k}, we obtain
$$\left\| \frac{1}{m^{2(n+k)}}\, f_2\left(x^{m^{n+k}}\right) - \frac{1}{m^{2k}}\left[\frac{4}{2}\, f_2\left(x^{m^k}\right) - f_2\left(x^{m^k}\right)\right] \right\| \le \frac{1}{m^{2k}}\, c$$
and thus
$$\left\| \frac{1}{m^{2(n+k)}}\, f_2\left(x^{m^{n+k}}\right) - \frac{1}{m^{2k}}\, f_2\left(x^{m^k}\right) \right\| \le \frac{1}{m^{2k}}\, c.$$

From the last relation it follows that there is a limit

$$f_m(x) = \lim_{n\to\infty} \frac{1}{m^{2k}} f_2\left(x^{m^k}\right).$$

It is clear that for any $q \in \mathbb{N}$ and $x \in S$ the following relations hold:

$$f_m\left(x^{m^q}\right) = m^{2q} f_m(x), \qquad f_m\left(x^{2^q}\right) = 4^q f_m(x).$$

Moreover we have

$$\left\| f_m(x) - \left[\frac{1}{2} f_2(x^2) - f_2(x)\right] \right\| \le c.$$

Taking into account relation $f_2(x^{2^k}) = 4^k f_2(x)$ we get

$$\| f_m(x) - f_2(x) \| \le 2c.$$

Now taking into account relation $f_m(x^{2^k}) = 4^k f_m(x)$ we get

$$f_m(x) = f_2(x) \quad \forall x \in S.$$

Now if we denote $f_2(x)$ by $\widehat{f}(x)$ we obtain $\widehat{f}(x^n) = n^2 \widehat{f}(x)$ and the proof of the lemma is now complete. □

Corollary 2.8. *If $f \in K(S, X)$, then the limit $\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{4^n} f(x^{2^n})$ exists and satisfies $\widehat{f}(x^n) = n^2 \widehat{f}(x)$ for all $x \in S$ and $n \in \mathbb{N}$. Moreover, $\widehat{f}(x) \in K(S, X)$ and $\widehat{f}(x) = \frac{1}{2} f(x^2) - f(x)$.*

Lemma 2.9. *Let the function $f : S \to X$ satisfy the condition*

$$\| f(x^2) - 2f(x) \| \le c$$

for some $c > 0$ and all $x \in S$. Then there is a limit

$$\widetilde{f}(x) = \lim_{k\to\infty} \frac{1}{2^k} f\left(x^{2^k}\right),$$

and for any $m \in \mathbb{N}$ and $x \in S$ the following relations

$$\widetilde{f}(x^m) = m \widetilde{f}(x),$$
$$\| \widetilde{f}(x) - f(x) \| \le c.$$

hold.

Proof. The proof is similar to the proof of the previous lemma. □

Lemma 2.10. *For any $f \in KK(S, X)$, the function $\varphi = f - \widehat{f}$ satisfies inequality*

$$\| \varphi(x^2) - 2\varphi(x) \| \le c \tag{2.17}$$

for some positive c and any $x \in S$.

Proof. Let $f \in KK(S, X)$. Then f satisfies relation (2.2). Hence from (2.12), we get

$$\| 2\widehat{f}(x) - f(x^2) + 2f(x) \| \le c.$$

Now we obtain

$$\begin{aligned}\| \varphi(x^2) - 2\varphi(x) \| &= \| f(x^2) - \widehat{f}(x^2) - 2f(x) + 2\widehat{f}(x) \| \\ &= \| f(x^2) - 4\widehat{f}(x) - 2f(x) + 2\widehat{f}(x) \| \\ &= \| f(x^2) - 2f(x) - 2\widehat{f}(x) \| \\ &\le c\end{aligned}$$

and the proof of the lemma is complete. □

From above lemma and Corollary 2.8 we get the following Corollary.

Corollary 2.11. *If $f \in K(S,X)$, then the function defined by $\phi = f - \widehat{f}$ satisfies $\phi(x^2) = 2\,\phi(x)$ for all $x \in S$ and belongs to $K(S,X)$.*

Denote by $PK_4(S,X)$ and $PK_2(S,X)$ the subspaces of $KK(S,X)$ consisting of functions f satisfying

$$f(x^k) = k^2 f(x) \;\; \forall\, k \in \mathbb{N}, \;\; \forall\, x \in S,$$

and

$$f(x^k) = k f(x) \;\; \forall\, k \in \mathbb{N}, \;\; \forall\, x \in S,$$

respectively.

Theorem 2.12. *For any semigroup S we have the following decomposition:*

$$KK(S,X) = PK_4(S,X) \oplus PK_2(S,X) \oplus B(S,X),$$

where $B(S,X)$ denotes the space of all bounded mappings from S to X.

Proof. It is clear that $KK(S,X)$ is the direct sum of $PK_4(S,X)$, $PK_2(S,X)$ and $B(S,X)$. To see this, let f be a quasikannappan function satisfying inequality (2.2). Then function $\varphi = f - \widehat{f}$ belongs to $KK(S,X)$ and satisfies relation (2.17). Now from Lemma 2.9 and Lemma 2.10 it follows that $\widetilde{\varphi} \in KK(S,X)$ and

$$\| \widetilde{\varphi}(x) - \varphi(x) \| \le c$$

for any $x \in S$. So, the function $\delta(x) = f(x) - \widehat{f}(x) - \widetilde{\varphi}(x)$ is bounded. We can rewrite the last relation as $f(x) = \widehat{f}(x) + \widetilde{\varphi}(x) + \delta(x)$ and hence $KK(S,X) = PK_4(S,X) \oplus PK_2(S,X) \oplus B(S,X)$. □

3. Stability

Definition 3.1. Let S be a semigroup and X be a Banach space. The functional equation (2.1) is said to be stable for the pair (S,X) if for any $f : S \to X$ satisfying inequality

$$\| f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) \| \le d \tag{3.1}$$

for some positive real number d and all $x, y, z \in S$, then there is a solution φ of (2.1) such that the difference $f - \varphi$ is a bounded mapping.

The subspace of $K(S,X)$ consisting of functions belonging to $PK_4(S,X)$ will be denoted by $K_4(S,X)$. In other words $K_4(S,X)$ consists of solutions of (2.1) satisfying the additional condition

$$f(x^k) = k^2 f(x) \ \ \forall\, k \in \mathbb{N}, \ \forall\, x \in S.$$

The subspace of $K(S,X)$ consisting of functions belonging to $PK_2(S,X)$ will be denoted by $K_2(S,X)$. In other words $K_2(S,X)$ consists of solutions of (2.1) satisfying the additional condition

$$f(x^k) = k\, f(x) \ \ \forall\, k \in \mathbb{N}, \ \forall\, x \in S.$$

Proposition 3.2. *$K(S,X) = K_4(S,X) \oplus K_2(S,X)$ for any semigroup S and any Banach space X.*

Proof. It is clear that $K_4(S,X) \cap K_2(S,X) = \{0\}$. Let f be a solution of (2.1). From Lemma 2.5, Corollary 2.8 and Corollary 2.11 it follows that $f = \widehat{f} + \varphi$, where $\widehat{f} \in K_4(S,X)$ and $\varphi \in K_2(S,X)$. □

Proposition 3.3. *The equation (2.1) is stable for the pair (S,X) if and only if $PK_4(S,X) = K_4(S,X)$ and $PK_2(S,X) = K_2(S,X)$.*

Proof. Suppose that the equation (2.1) is stable for the pair (S,X), and assume that $PK_4(S,X) \neq K_4(S,X)$. Let $f \in PK_4(S,X) \setminus K_4(S,X)$. Then by Proposition 3.2 there are $\varphi_4 \in K_4(S,X)$ and $\varphi_2 \in K_2(S,X)$ such that for some positive d we have $|f(x) - \varphi_4(x) - \varphi_2(x)| \leq d$ for all $x \in S$. Thus the function $\psi(x) = f(x) - \varphi_4(x) - \varphi_2(x)$ is bounded. Therefore we get $\widehat{\psi} = \widehat{f} - \widehat{\varphi}_4(x) - \widehat{\varphi}_2(x) \equiv 0$. Now taking into account $\widehat{f} = f$, $\widehat{\varphi}_4(x) = \varphi_4(x)$, $\widehat{\varphi}_2(x) \equiv 0$ we obtain $f = \widehat{\varphi}_4(x) = \varphi_4(x)$. Thus we obtain a contradiction to the assumption $f \in PK_4(S,X) \setminus K_4(S,X)$.

Now assume that $PK_2(S,X) \neq K_2(S,X)$. Let $f \in PK_2(S,X) \setminus K_2(S,X)$. Then by the last proposition there are $\varphi_4 \in K_4(S,X)$ and $\varphi_2 \in K_2(S,X)$ such that for some positive d we have $|f(x) - \varphi_4(x) - \varphi_2(x)| \leq d$ for all $x \in S$. The function $\psi(x) = f(x) - \varphi_4(x) - \varphi_2(x)$ is bounded. Therefore we obtain $\widehat{\psi} = \widehat{f} - \widehat{\varphi}_4(x) - \widehat{\varphi}_2(x) \equiv 0$. Now taking into account $\widehat{f} = 0$, $\widehat{\varphi}_4(x) = \varphi_4(x), \widehat{\varphi}_2(x) \equiv 0$ we get $0 = \widehat{\varphi}_4(x) = \varphi_4(x)$. Hence $|f(x) - \varphi_2(x)| \leq d$. The latter relation implies $k\,|f(x) - \varphi_2(x)| = |f(x^k) - \varphi_2(x^k)| \leq d$ for all $k \in \mathbb{N}$ and thus we see that $f \equiv \varphi_2$. So, we obtain a contradiction to the assumption $f \in PK_2(S,X) \setminus K_2(S,X)$.

Therefore if equation (2.1) is stable for the pair (S,X), then

$$PK_4(S,X) = K_4(S,X) \quad \text{and} \quad PK_2(S,X) = K_2(S,X).$$

Now suppose that $PK_4(S,X) = K_4(S,X)$ and $PK_2(S,X) = K_2(S,X)$. Let us verify that equation (2.1) is stable for the pair (S,X). If f satisfies (3.1), then $f \in KK(S,X)$ and there are $f_4 \in PK_4(S,X)$, $f_2 \in PK_2(S,X)$ and bounded function δ such that $f = f_4 + f_2 + \delta$. Now from the relations $PK_4(S,X) = K_4(S,X)$ and $PK_2(S,X) = K_2(S,X)$ we get that $\varphi = f_4 + f_2$ is a solution of (2.1) such that $f - \varphi$ is a bounded function. This means that equation (2.1) is stable for the pair (S,X). This completes the proof of the proposition. □

Theorem 3.4. *Let S be a semigroup, and X and E be two Banach spaces. Then equation (2.1) is stable for the pair (S, X) if and only it is stable for the pair (S, E).*

Proof. It is clear that we can only consider the case when E is the set of real numbers $\mathbb{R}$. Suppose that the equation (2.1) is stable for the pair (S, X). Suppose that (2.1) is not stable for the pair $(S, \mathbb{R})$, then either $PK_4(S, \mathbb{R}) \neq K_4(S, \mathbb{R})$ or $PK_2(S, \mathbb{R}) \neq K_2(S, \mathbb{R})$. First, consider the case $PK_4(S, \mathbb{R}) \neq K_4(S, \mathbb{R})$. Let $f \in PK_4(S, \mathbb{R}) \setminus K_4(S, \mathbb{R})$.

Let $e \in X$ and $\|e\| = 1$. Consider the function $\varphi : S \to X$ given by the formula $\varphi(x) = f(x) \cdot e$. Then from relation

$$\begin{aligned}
&\| \varphi(xyz) + \varphi(x) + \varphi(y) + \varphi(z) - \varphi(xy) - \varphi(xz) - \varphi(yz) \| \\
&= \| f(xyz) \cdot e + f(x) \cdot e + f(y) \cdot e + f(z) \cdot e - f(xy) \cdot e - f(xz) \cdot e - f(yz) \cdot e \| \\
&= \| [f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)] \cdot e \| \\
&= | f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) | \cdot \|e\|
\end{aligned}$$

it follows that $\varphi \in PK_4(S, X) \setminus K_4(S, X)$ which contradicts the fact that the equation (2.1) is stable for the pair (S, X). Similarly we verify that $PK_2(S, \mathbb{R}) = K_2(S, \mathbb{R})$. So, the equation (2.1) is stable for the pair $(S, \mathbb{R})$.

Now suppose that the equation (2.1) is stable for the pair $(S, \mathbb{R})$, that is

$$PK_4(S, \mathbb{R}) = K_4(S, \mathbb{R}) \qquad \text{and} \qquad PK_2(S, \mathbb{R}) = K_2(S, \mathbb{R}).$$

Denote by X^* the space of linear bounded functionals on X endowed by functional normed topology. It is clear that for any $\psi \in PK_i(S, X)$ and any $\lambda \in X^*$ the function $\lambda \circ \psi$ belongs to the space $PK_i(S, \mathbb{R})$, $i = 2, 4$. Indeed, let for some $c > 0$ and any $x, y, z \in S$ we have

$$\| \psi(xyz) + \psi(x) + \psi(y) + \psi(z) - \psi(xy) - \psi(xz) - \psi(yz) \| \leq c.$$

Hence

$$\begin{aligned}
&\|\lambda \circ \psi(xyz) + \lambda \circ \psi(x) + \lambda \circ \psi(y) + \lambda \circ \psi(z) - \lambda \circ \psi(xy) \\
&\qquad - \lambda \circ \psi(xz) - \lambda \circ \psi(yz)\| \\
&\quad = \| \lambda[\psi(xyz) + \psi(x) + \psi(y) + \psi(z) - \psi(xy) - \psi(xz) - \psi(yz)] \| \leq c\|\lambda\|.
\end{aligned}$$

Obviously, $\lambda \circ \psi(x^n) = n^2 \lambda \circ \psi(x)$ if $\psi \in PK_4(S, X)$ and $\lambda \circ \psi(x^n) = n\lambda \circ \psi(x)$ if $\psi \in PK_2(S, X)$ for any $x \in S$ and for any $n \in \mathbb{N}$.

Hence the function $\lambda \circ \psi$ belongs to the space $PK_4(S, \mathbb{R}) \oplus PK_2(S, \mathbb{R})$. Let $f : S \to X$ belongs to the set $[PK_4(S, X) \oplus PK_2(S, X)] \setminus [K_4(S, X) \oplus K_2(S, X)]$. Then there are $x, y, z \in S$ such that $f(xyz)+f(x)+f(y)+f(z)-f(xy)-f(xz)-f(yz) \neq 0$. Hahn–Banach Theorem implies that there is an $\ell \in X^*$ such that $\ell(f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)) \neq 0$, and we see that $\ell \circ f$ belongs to the set $[PK_4(S, \mathbb{R}) \oplus PK_2(S, \mathbb{R})] \setminus [K_4(S, \mathbb{R}) \oplus K_2(S, \mathbb{R})]$. This contradiction proves the theorem. □

In view of Theorem 3.4, it is not important which Banach space is used on the range. Thus one may consider the stability of the functional equation (2.1) on the pair $(S, \mathbb{R})$. Let us simplify the following notations: In the case $X = \mathbb{R}$ the spaces $K(S,\mathbb{R})$, $KK(S,\mathbb{R})$, $KK_4(S,\mathbb{R})$, $KK_2(S,\mathbb{R})$, $PK_4(S,\mathbb{R})$, $PK_2(S,\mathbb{R})$ will be denoted by $K(S)$, $KK(S)$, $KK_4(S)$, $KK_2(S)$, $PK_4(S)$, $PK_2(S)$, respectively.

Theorem 3.5. *In general, the functional equation* (2.1) *is not stable on semigroups.*

Proof. Let $\mathcal{F}$ be a free semigroup of rank two with free generators a, b. For any word $w \in \mathcal{F}$. Denote by $\eta(w)$ the number of occurrences of a^2b^2 in w. It is easy to verify that for any $u, v \in \mathcal{F}$

$$\eta(uv) - \eta(u) - \eta(v) \in \{\, 0, 1 \,\}.$$

So

$$\eta(uvw) - \eta(u) - \eta(v) - \eta(w) \in \{\, 0, 1, 2 \,\},$$

and

$$|\, \eta(xyz) + \eta(x) + \eta(y) + \eta(z) - \eta(xy) - \eta(xz) - \eta(xz) \,| \leq 5.$$

Thus we see that $\eta \in KK(\mathcal{F})$, and

$$|\, \eta(x^2) - 2\,\eta(x) \,| \leq 1 \qquad \forall\, x \in \mathcal{F}.$$

Therefore, function $\widetilde{\eta}$ defined by

$$\widetilde{\eta}(x) = \lim_{n\to\infty} \frac{1}{2^n}\,\eta(x^{2^n})$$

belongs to $PK_2(\mathcal{F})$. Let us verify that $\widetilde{\eta}$ dos not belong to $K(\mathcal{F})$. Indeed, it is clear that

$$\eta(aab^2) = 1,\ \eta(a) = \eta(b) = \eta(b^2) = \eta(a^2) = \eta(ab^2) = 0,$$
$$\widetilde{\eta}(aab^2) = 1,\ \widetilde{\eta}(a) = \widetilde{\eta}(b) = \widetilde{\eta}(b^2) = \widetilde{\eta}(a^2) = \widetilde{\eta}(ab^2) = 0.$$

Therefore letting $x = a$, $y = a$, $z = b^2$, we get

$$\begin{aligned}&\widetilde{\eta}(xyz) + \widetilde{\eta}(x) + \widetilde{\eta}(y) + \widetilde{\eta}(z) - \widetilde{\eta}(xy) - \widetilde{\eta}(xz) - \widetilde{\eta}(xz)\\ &\quad = \widetilde{\eta}(aab^2) + \widetilde{\eta}(a) + \widetilde{\eta}(a) + \widetilde{\eta}(b^2) - \widetilde{\eta}(aa) - \widetilde{\eta}(ab^2) - \widetilde{\eta}(ab^2) = 1 \neq 0.\end{aligned}$$

So $PK_2(\mathcal{F}) \neq K_2(\mathcal{F})$ and equation (2.1) is not stable on $\mathcal{F}$. □

Definition 3.6. An element x of a semigroup S is said to be *periodic* if there are $n, m \in \mathbb{N}$ such that $n \neq m$ and $x^n = x^m$. We shall say that the semigroup is periodic if every element of S is periodic.

Theorem 3.7. *The equation* (2.1) *is stable for any periodic semigroup.*

Proof. It is clear that if S is a periodic semigroup, then $PK_4(S) = \{0\}$ and $PK_2(S) = \{0\}$. Therefore by Theorem 2.12 we have $KK(S) = B(S)$, and equation (2.1) is stable on S. □

Now let us show that equation (2.1) is stable on any abelian semigroup S. It is clear that for any abelian group A and any real-valued symmetric bimorphism $B(x, y)$ of $A \times A$, the function $x \to B(x, x)$ belongs to $K_4(A)$. Denote by $BM(A)$ the set of all real-valued functions f on A defined by the rule $f(x) = B(x, x)$, where $B(.,.)$ is a symmetric bimorphism.

Lemma 3.8. *Let A_3 be an abelian free semigroup of rank three. Then $PK_4(A_3) = K_4(A_3) = BM(A_3)$.*

Proof. Let A_3 be a free abelian semigroup of rank three with free generators a, b, c. The space of symmetric bimorphisms on A_3 is six dimensional. For $f \in K_4(A_3)$, we choose a symmetric bimorphism $B(x,y)$ such that $B(a,a) = f(a)$, $B(b,b) = f(b)$, $B(c,c) = f(c)$, $B(a,b) = \frac{1}{2}[f(ab) - f(a) - f(b)]$, $B(a,c) = \frac{1}{2}[f(ac) - f(a) - f(c)]$, $B(b,c) = \frac{1}{2}[f(bc) - f(b) - f(c)]$.

Hence, the function $\varphi(x) = f(x) - B(x,x)$ belongs to $PK_4(A_3)$, and

$$\varphi(a) = \varphi(b) = \varphi(c) = \varphi(ab) = \varphi(ac) = \varphi(bc) = 0.$$

We have $\varphi(a^k) = \varphi(b^k) = \varphi(c^k) = 0$ for any $k \in \mathbb{N}$. Let

$$|\ \varphi(xyz) + \varphi(x) + \varphi(y) + \varphi(z) - \varphi(xy) - \varphi(xz) - \varphi(yz)\ | \leq \delta.$$

Then for any $p, q, k \in \mathbb{N}$ we have

$$|\ \varphi(a^{pk}b^{2qk}) + \varphi(a^{pk}) + 2\varphi(b^{qk}) - 2\varphi(a^{pk}b^{qk}) - \varphi(b^{2qk})\ | \leq \delta$$

which simplifies to

$$|\ \varphi(a^{pk}b^{2qk}) - 2\varphi(a^{pk}b^{qk})\ | \leq \delta.$$

Hence

$$k^2\ |\ \varphi(a^p b^{2q}) - 2\varphi(a^p b^q)\ | \leq \delta$$

which is

$$|\ \varphi(a^p b^{2q}) - 2\varphi(a^p b^q)\ | \leq \frac{1}{k^2}\delta.$$

Thus $k \to \infty$, we obtain $\varphi(a^p b^{2q}) = 2\varphi(a^p b^q)$. Similarly, for any $p, q, k, \ell \in \mathbb{N}$, we have

$$\left|\ \varphi(a^{pk}b^{\ell qk}) + (\ell-1)[\varphi(a^{pk}) + \ell\varphi(b^{qk})] - \ell\varphi(a^{pk}b^{qk}) - \frac{\ell(\ell-1)}{2}\varphi(b^{2qk})\ \right| \leq \frac{\ell(\ell-1)}{2}\delta$$

which is

$$|\ \varphi(a^{pk}b^{\ell qk}) - \ell\varphi(a^{pk}b^{qk})\ | \leq \frac{\ell(\ell-1)}{2}\,\delta.$$

Hence

$$k^2\ |\ \varphi(a^p b^{\ell q}) - \ell\varphi(a^p b^q)\ | \leq \frac{\ell(\ell-1)}{2}\,\delta$$

which is

$$|\ \varphi(a^p b^{\ell q}) - \ell\varphi(a^p b^q)\ | \leq \frac{\ell(\ell-1)}{2k^2}\,\delta.$$

Therefore as $k \to \infty$, we get $\varphi(a^p b^{\ell q}) = \ell\varphi(a^p b^q)$. Similarly, we obtain $\varphi(a^{\ell p} b^q) = \ell\varphi(a^p b^q)$. So, for any $n, m \in \mathbb{N}$, we get $\varphi(a^n b^m) = nm\,\varphi(ab) = 0$.

The same way we obtain equalities $\varphi(a^n c^m) = nm\,\varphi(ac) = 0$ and $\varphi(b^n c^m) = nm\,\varphi(bc) = 0$.

Now for any $n, m, k, \ell \in \mathbb{N}$ we have

$$|\ \varphi(a^{pk}b^{qk}c^{\ell k}) - \varphi(a^{pk}) - \varphi(b^{qk}) - \varphi(c^{\ell k}) - \varphi(a^{pk}b^{qk}) - \varphi(a^{pk}c^{\ell k}) - \varphi(b^{qk}c^{\ell k})\ | \leq \delta.$$

Hence $|\ \varphi(a^{pk}b^{qk}c^{\ell k})\ | \leq \delta$, and we have $k^2\ |\ \varphi(a^p b^q c^\ell)\ | \leq \delta$. Thus

$$|\ \varphi(a^p b^q c^\ell)\ | \leq \frac{1}{k^2}\,\delta.$$

By taking the limit as $k \to \infty$, we see that $\varphi(a^p b^q c^\ell) = 0$. It means that

$$f(x) = B(x, x) \in BM(A_3)$$

and the proof of the lemma is now finished. □

For any group G, we will denote by $X(G)$ the set of real-valued additive characters of G.

Lemma 3.9. *Let A_3 be an abelian free semigroup of rank three. Then $PK_2(A_3) = K_2(A_3) = X(A_3)$.*

Proof. Let $f \in PK_2(A_3)$ and $f(a) = p$, $f(b) = q$, $f(c) = r$. Further, let ψ be an additive character of A_3 such that $\psi(a) = p$, $\psi(b) = q$, $\psi(c) = r$. Then the function $\varphi(x) = f(x) - \psi(x)$ belongs to $PK_2(A_3)$ and satisfies the condition $\varphi(a) = \varphi(b) = \varphi(c) = 0$. Let us show that $\varphi \equiv 0$.

Let δ be a positive number such that for any $x, y, z \in A_3$

$$|\,\varphi(xyz) + \varphi(x) + \varphi(y) + \varphi(z) - \varphi(xy) - \varphi(xz) - \varphi(yz)\,| \le \delta.$$

Then for any $p, q, k, \ell \in \mathbb{N}$ we have

$$\left|\,\varphi(a^{pk}b^{\ell qk}) + (\ell-1)[\varphi(a^{pk}) + \ell\varphi(b^{qk})] - \ell\varphi(a^{pk}b^{qk}) - \frac{\ell(\ell-1)}{2}\varphi(b^{2qk})\,\right| \le \frac{\ell(\ell-1)}{2}\delta$$

which is

$$|\,\varphi(a^{pk}b^{\ell qk}) - \ell\varphi(a^{pk}b^{qk})\,| \le \frac{\ell(\ell-1)}{2}\,\delta.$$

Hence

$$k\,|\,\varphi(a^p b^{\ell q}) - \ell\varphi(a^p b^q)\,| \le \frac{\ell(\ell-1)}{2}\,\delta$$

which is

$$|\,\varphi(a^p b^{\ell q}) - \ell\varphi(a^p b^q)\,| \le \frac{\ell(\ell-1)}{2k}\,\delta.$$

Therefore as $k \to \infty$, we get $\varphi(a^p b^{\ell q}) = \ell\varphi(a^p b^q)$. Similarly we obtain $\varphi(a^{\ell p} b^q) = \ell\varphi(a^p b^q)$. So, for any $n, m \in \mathbb{N}$, we get $\varphi(a^n b^m) = nm\,\varphi(ab)$. It follows that for $u = a^n b^m$ we have $\varphi(u^k) = \varphi(a^{kn} b^{km}) = k^2 nm\varphi(ab) = k^2\varphi(u)$ But $\varphi(x) \in PK_2(A_3)$, therefore we have $\varphi(u^k) = k\varphi(u) = k^2\varphi(u)$. The last relation implies $\varphi(u) = 0$.

The same way we obtain equalities $\varphi(a^n c^m) = 0$ and $\varphi(b^n c^m) = 0$ for any $n, m \in \mathbb{N}$.

Now for any $n, m, k, \ell \in \mathbb{N}$ we have

$$|\,\varphi(a^{pk}b^{qk}c^{\ell k}) - \varphi(a^{pk}) - \varphi(b^{qk}) - \varphi(c^{\ell k}) - \varphi(a^{pk}b^{qk}) - \varphi(a^{pk}c^{\ell k}) - \varphi(b^{qk}c^{\ell k})\,| \le \delta.$$

Hence $|\,\varphi(a^{pk}b^{qk}c^{\ell k})\,| \le \delta$, and we have $k\,|\,\varphi(a^p b^q c^\ell)\,| \le \delta$. Thus

$$|\,\varphi(a^p b^q c^\ell)\,| \le \frac{1}{k^2}\,\delta.$$

By taking the limit as $k \to \infty$, we see that $\varphi(a^p b^q c^\ell) = 0$.

Therefore, $\varphi \equiv 0$ and $f \equiv \psi \in X(A_3)$. □

Theorem 3.10. *Let A be any abelian group. Then $PK(A) = K_4(A) \oplus K_2(A)$.*

Proof. Let us show that $PK_4(A) = K_4(A)$ and $PK_2(A) = K_2(A)$. Suppose that $PK_4(A) \neq K_4(A)$. In this case there are $f \in PK_4(A)$ and $x, y, z \in A$ such that

$$| f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) | = d > 0.$$

Denote by B the subsemigroup of A generated by three elements x, y, z. Let τ be an epimorphism of A_3 onto B given by the rule $\tau(a) = x,\ \tau(b) = y,\ \tau(c) = z$. So, if we consider function $g(t) = f(\tau(t))$ we get an element of $PK_4(A_3)$ such that

$$| g(abc) + g(a) + g(b) + g(c) - g(ab) - g(ac) - g(bc) | = d > 0$$

which contradicts Lemma 3.8.

Similarly, we come to a contradiction if we suppose that $PK_2(A) \neq K_2(A)$. Hence $PK_4(A) = K_4(A)$, and $PK_2(A) = K_2(A)$. □

Corollary 3.11. *Suppose A is an abelian group. Then*

$$K(A) = PK(A) = K_4(A) \oplus K_2(A).$$

Now from Proposition 3.3 we get the following corollary.

Corollary 3.12. *The equation (2.1) is stable on any abelian semigroup A.*

For any group G, let $Q(G)$ be the set of solutions of the quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y).$$

Moreover, we denote by $PK^+(G)$ and by $PK^-(G)$ subspaces of $PK(G)$ consisting of functions f such that $f(x^{-1}) = f(x)$ and $f(x^{-1}) = -f(x)$, respectively.

Lemma 3.13. *For any group G,*

$$PK_4(G) = PK^+(G) \quad \text{and} \quad PK_2(G) = PK^-(G)$$

hold.

Proof. It is clear that $PK_4(G) \subseteq PK^+(G)$, and $PK_2(G) \subseteq PK^-(G)$. Let us show that $PK^+(G) \subseteq PK_4(G)$, and $PK^-(G) \subseteq PK_2(G)$, respectively. Let $f \in PK(G)$, then there are $\varphi \in PK_4(G)$ and $\psi \in PK_2(G)$ such that $f(x) = \varphi(x) + \psi(x)$. If $f \in PK^+(G)$, then

$$f(x) = f(x^{-1}) = \varphi(x^{-1}) + \psi(x^{-1}) = \varphi(x) - \psi(x),$$

so $\varphi(x) + \psi(x) = \varphi(x) - \psi(x)$ and we see that $\psi(x) \equiv 0$ and $f(x) = \varphi(x) \in PK_4(G)$. Therefore $PK^+ \subseteq PK_4(G)$.

Now if $f \in PK^-(G)$, then

$$f(x) = -f(x^{-1}) = -\varphi(x^{-1}) - \psi(x^{-1}) = -\varphi(x) + \psi(x),$$

so $\varphi(x)+\psi(x) = -\varphi(x)+\psi(x)$ and we see that $\varphi \equiv 0$ and $f(x) = \psi(x) \in PK_2(G)$. Therefore $PK^-(G) \subseteq PK_2(G)$. □

Lemma 3.14. *Let G be an arbitrary group and $f \in Q(G)$, then for any $x, y, z \in G$ we have*

$$f(xyz) + f(xzy) = 2f(xy) + 2f(xz) + 2f(yz) - 2f(x) - 2f(y) - 2f(z). \quad (3.2)$$

Proof. Since $f \in Q(G)$, we have

$$f(xyz) + f(xyz^{-1}) = 2\,f(xy) + 2\,f(z),$$
$$f(xyz^{-1}) + f(xzy^{-1}) = 2\,f(x) + 2\,f(yz^{-1}),$$
$$f(xzy^{-1}) + f(xzy) = 2\,f(xz) + 2\,f(y)$$

for all $x, y, z \in G$. Therefore

$$f(xyz) + f(xzy) = 2f(xy) + 2f(z) - 2f(x) - 2f(yz^{-1}) + 2f(xz) + 2f(y).$$

Using the equation

$$f(yz) + f(yz^{-1}) = 2f(y) + 2f(z)$$

in the last equality, we get

$$\begin{aligned} f(xyz) &+ f(xzy) \\ &= 2f(xy) + 2f(z) - 2f(x) - 4f(y) - 4f(z) + 2f(yz) + 2f(xz) + 2f(y) \\ &= 2f(xy) + 2f(xz) + 2f(yz) - 2f(x) - 2f(y) - 2f(z). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.15. *Let $\bar{A}_3$ be an abelian free group of rank 3. Then $Q(\bar{A}_3) = BM(\bar{A}_3)$.*

Proof. Let $\bar{A}_3$ be a free abelian group of rank three with free generators a, b, c. It is clear that $BM(\bar{A}_3) \subseteq Q(\bar{A}_3)$. The space of symmetric bimorphisms on $\bar{A}_3$ is six dimensional. For $f \in Q(\bar{A}_3)$, we choose a symmetric bimorphism $B(x, y)$ such that $B(a, a) = f(a)$, $B(b, b) = f(b)$, $B(c, c) = f(c)$, $B(a, b) = \frac{1}{2}[f(ab) - f(a) - f(b)]$, $B(a, c) = \frac{1}{2}[f(ac) - f(a) - f(c)]$, $B(b, c) = \frac{1}{2}[f(bc) - f(b) - f(c)]$.

Hence, the function $\varphi(x) = f(x) - B(x, x)$ belongs to $Q(\bar{A}_3)$, and

$$\varphi(a) = \varphi(b) = \varphi(c) = \varphi(ab) = \varphi(ac) = \varphi(bc) = 0. \tag{3.3}$$

We have $\varphi(a^k) = \varphi(b^k) = \varphi(c^k) = 0$ for any $k \in \mathbb{Z}$.

Since A_3 is an abelian group, from (3.2), we get

$$\varphi(xyz) = \varphi(xy) + \varphi(xz) + \varphi(yz) - \varphi(x) - \varphi(y) - \varphi(z) \tag{3.4}$$

for all $x, y, z \in A_3$. From the equality (3.4), we obtain

$$\varphi(a^n b^m b^k) = \varphi(a^n b^m) + \varphi(a^n b^k) + \varphi(b^m b^k) - \varphi(a^n) - \varphi(b^m) - \varphi(b^k)$$

and therefore

$$\varphi(a^n b^m b^k) = \varphi(a^n b^m) + \varphi(a^n b^k).$$

Similarly, from the equality (3.4), we can obtain

$$\varphi(a^n a^k b^m) = \varphi(a^n b^m) + \varphi(a^k b^m).$$

for all $n, mk \in \mathbb{Z}$. The last two relations imply

$$\varphi(a^n b^m) = n\,m\,\varphi(ab)$$

for any $n, m \in \mathbb{Z}$. Taking into account (3.3) we get $\varphi(a^n b^m) = 0$. Now from (3.4) we get

$$\varphi(a^n b^m c^k) = \varphi(a^n b^m) + \varphi(a^n c^k) + \varphi(b^m c^k) - \varphi(a^n) - \varphi(b^m) - \varphi(c^k) = 0.$$

It means that

$$f(x) = B(x, x) \in BM(\bar{A}_3)$$

and the proof of the lemma is now finished. □

Lemma 3.16. *Let $\bar{A}_3$ be an abelian group of rank three, then $K_2(\bar{A}_3) = X(\bar{A}_3)$.*

Proof. The proof is similar to the proof of Lemma 3.15. □

Proposition 3.17. *Let A be an abelian group, then*

$$PK(A) = PK_4(A) \oplus PK_2(A) = K_4(A) \oplus K_2(A) = Q(A) \oplus X(A).$$

Another words general solution $f : A \to R$ of equation (2.1) is of the form

$$f(x) = B(x, x) + \psi(x),$$

where $B(x, y) \in BM(A)$, $\psi \in X(A)$.

Proof. First we verify equality $K_4(A) = Q(A)$. We have $K_4(A) \subseteq Q(A)$. Suppose that there is $f \in Q(A) \setminus K_4(A)$, then there are $x, y, z \in A$ such that

$$f(xyz) + f(x) + f(y) + f(z) \neq f(xy) + f(xz) + f(yz). \tag{3.5}$$

Now let $\bar{A}_3$ be a free abelian group with free generators a, b, c. Let B be a subgroup of A generated by elements x, y, z and let $\pi : \bar{A}_3 \to B$ be an epimorphism such that $\pi(a) = x$, $\pi(b) = y$, $\pi(c) = z$. Then the function $\omega(t) = f(\pi(t))$ is an element of $Q(\bar{A}_3)$. By Lemma 3.15 we have $\omega \in BM(\bar{A}_3)$. But this contradicts to (3.5) because

$$\begin{aligned}
&\omega(abc) + \omega(a) + \omega(b) + \omega(c) - \omega(ab) - \omega(ac) - \omega(bc) \\
&\qquad = f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) \neq 0.
\end{aligned}$$

Therefore, $f \in K_4(A)$. Similar way we verify that $K_2(A) = X(A)$. □

As a first corollary of Proposition 3.17 we obtain the following corollary that generalizes Kannappan's result [20] (see Section 1, Introduction) in the case $\mathbb{K} = \mathbb{R}$.

Corollary 3.18. *Let A be an abelian group, then general solution $f : A \to \mathbb{R}$ of equation (2.1) is of the form*

$$f(x) = B(x, x) + \psi(x),$$

where $B(x, y)$ is an symmetric bimorphism and $\psi \in X(A)$.

From Proposition 3.17 we obtain the following two theorems that generalize the results of Jung [17] mentioned in the Introduction.

Theorem 3.19. *Suppose that A is an abelian group, and X a real Banach space. Let $f : A \to X$ satisfies the inequalities*

$$\begin{aligned}
\| f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) \| &\leq d \\
\| f(x) - f(-x) \| &\leq \theta
\end{aligned}$$

for some $d, \theta > 0$ and for all $x, y, z \in A$. Then there exists a unique quadratic mapping $q : A \to X$ which satisfies

$$\| f(x) - q(x) \| \leq \delta$$

for some positive δ *and all* $x \in A$.

Proof. According to the Theorem 3.4 we can assume that $X = \mathbb{R}$. From Theorem 2.12 and Proposition 3.17 it follows that there are $q(x) \in Q(A)$, $\psi \in X(A)$ and $\gamma \in B(A)$ such that $f(x) = q(x) + \psi(x) + \gamma(x)$. Therefore,

$$\begin{aligned} |\, f(x) - f(x^{-1})\,| &= |\, q(x) + \psi(x) + \gamma(x) - q(x^{-1}) - \psi(x^{-1}) - \gamma(x^{-1})\,| \\ &= |\, 2\psi(x) + \gamma(x) - \gamma(x^{-1})\,| \leq \theta, \end{aligned}$$

and we see that $\psi(x)$ is a bounded function. Hence, $\psi \equiv 0$ and $f(x) = q(x)+\gamma(x)$. If δ is a positive real number such that $|\gamma(x)| \leq \delta$ for all $x \in A$, then we have $|f(x) - q(x)| \leq \delta$ for all $x \in A$. The proof is now complete. □

Theorem 3.20. *Suppose that* A *is an abelian group, and* X *a real Banach space. Let* $f : A \to X$ *satisfies the inequalities*

$$\begin{aligned} \|\, f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)\,\| &\leq d \\ \|\, f(x) + f(-x)\,\| &\leq \theta \end{aligned}$$

for some $d, \theta > 0$ *and for all* $x, y, z \in A$. *Then there exists a unique additive mapping* $\psi : A \to X$ *which satisfies*

$$\|\, f(x) - \psi(x)\,\| \leq \delta$$

for some positive δ *and all* $x \in A$.

Proof. The proof is similar to that of the previous theorem. □

4. Embedding

Remark 4.1. If S is a semigroup with zero and $f \in KK(S)$, then f is bounded.

Proof. Since $f \in KK(S)$, the function f satisfies

$$|\, f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)\,| \leq d,$$

for all $x, y, z \in S$ and for some $d > 0$. If we put $y = z = 0$ in the above inequality, we obtain

$$|\, f(0) + f(x) + f(0) + f(0) - f(0) - f(0) - f(0)\,| \leq d.$$

Therefore $|\, f(x)\,| \leq d$. So f is a bounded function. □

The following corollary follows from the Remark 4.1.

Corollary 4.2. *Let* S_0 *be a semigroup obtained by adjoining the zero to the arbitrary semigroup* S. *Then* S *can be embedded into the semigroup* S_0 *such that the equation* (2.1) *is stable on* S_0.

Proof. From Remark 4.1, we have $KK(S_0) = B(S_0)$. Hence the equation (2.1) is stable on S_0. □

Definition 4.3. We shall say that in a semigroup S a *left law of reduction* is fulfilled if any equality $xy = xz$ in S implies $y = z$. Similarly, we shall say that in a semigroup S a *right law of reduction* is fulfilled if any equality $yx = zx$ in S implies $y = z$.

Obviously in a semigroup with zero neither left nor right law of reduction is fulfilled.

The embedding presented in Corollary 4.2 does not preserve some important properties of semigroup. For instance, if S is a group S_0 is not necessarily a group. Similarly, if S is a semigroup with law of reduction, then S_0 does not have the same property.

Our main goal in this section is to construct another embedding preserving properties of semigroups such as laws of reduction and the axioms of a group. From now on let S be an arbitrary semigroup with unit e.

Lemma 4.4. *Let $f \in KK(S)$ so that*

$$| f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) | \leq d \tag{4.1}$$

for any $x, y, z \in S$ and for some $d > 0$. Further, let c be an element of order two. Then

$$| f(u) - f(cu) | \leq 2\,d, \tag{4.2}$$

$$| f(u) - f(uc) | \leq 2\,d, \tag{4.3}$$

$$| f(u^c) - f(u) | \leq 8\,d$$

for any $u \in S$.

Proof. Letting $x = y = z = e$ in (4.1), we obtain $|f(e)| \leq d$. Similarly, letting $x = y = z = c$ in (4.1), we have $| f(ccc) + 3f(c) - 3f(cc) | \leq d$. Since c is an element of order two, the last inequality reduces to $| 4f(c) - 3f(e) | \leq d$. Hence we have $| 4f(c) | \leq d + 3|f(e)|$ and consequently

$$| f(c) | \leq \frac{1}{4}\,d + \frac{3}{4}\,f(e) \leq d.$$

Next substituting $x = c$, $y = c$ and $z = u$ in (4.1), we have

$$| f(ccu) + f(c) + f(c) + f(u) - f(cc) - f(cu) - f(cu) | \leq d.$$

Since c is an element of order two, the last inequality yields

$$| 2f(u) + 2f(c) - f(e) - 2f(cu) | \leq d$$

and hence we have $| 2f(u) - 2f(cu) | \leq d + 3\,d$. Therefore simplifying, we see that

$$| f(u) - f(cu) | \leq 2\,d$$

which is (4.2).

Similarly, letting $x = u$, $y = c$ and $z = c$ in (4.1), we get

$$| f(ucc) + f(c) + f(c) + f(u) - f(uc) - f(uc) - f(cc) | \leq d.$$

Using the fact that c is of order two, we have

$$| 2f(u) + 2f(c) - 2f(uc) - f(e) | \leq d.$$

This last inequality yields $| 2f(u) - 2f(uc) | \leq d + 3\,d$. Simplifying, we get

$$| f(u) - f(uc) | \leq 2\,d$$

which is (4.3).

Again, substituting $x = c$, $y = u$ and $z = c$ in (4.1), we obtain

$$| f(cuc) + f(c) + f(c) + f(u) - f(cu) - f(uc) - f(cc) | \leq d.$$

Using the fact that c is of order two and simplifying, we have

$$| f(u^c) + f(u) - f(cu) - f(uc) | \leq d + 3\,d.$$

Using (4.2) we obtain

$$\begin{aligned} | f(u^c) - f(uc) | &= | f(u^c) + f(u) - f(cu) - f(uc) + f(cu) - f(u) | \\ &\leq | f(u^c) + f(u) - f(cu) - f(uc) | + | f(cu) - f(u) | \\ &\leq 4\,d + 2\,d = 6\,d. \end{aligned}$$

Similarly using (4.3) we have

$$| f(u^c) - f(cu) | \leq 6\,d.$$

Now taking into account the last inequality and (4.2), we get

$$\begin{aligned} | f(u^c) - f(u) | &= | f(u^c) - f(cu) + f(cu) - f(u) | \\ &= | f(u^c) - f(cu) | + | f(cu) - f(u) | \\ &\leq 6\,d + 2\,d = 8\,d. \end{aligned}$$

The proof of the lemma is now complete. □

Now consider semidirect product $H = K \rtimes S$ of semigroup S and a group K, where elements of K act on S by automorphisms. Also we suppose that every non unit element of K has order two.

Lemma 4.5. *Suppose that $f \in PK_4(G)$ and satisfies condition (4.1) on H. Let $b, c, bc \in K$ be the elements of order two. Suppose for $u \in S$ the elements u^{bc}, u^c, u generate an abelian subsemigroup, then*

$$f(u^{bc}u^cu) = 9\,f(u) \qquad \forall\, u \in S.$$

Proof. Using Lemma 2.3 with $n = 5$ and $x_1 = u, x_2 = b, x_3 = u, x_4 = c, x_5 = u$, we get

$$\begin{aligned} | f(ubucu) + 3[3f(u) + f(b) + f(c)] - 3f(u^2) - f(ub) \\ - 2f(uc) - 2f(bu) - f(cu) - f(bc) | \leq 6d. \end{aligned}$$

Now taking into account relations $|f(b)| \leq d$, $|f(c)| \leq d$, and $|f(bc)| \leq d$, we obtain

$$| f(ubucu) + 9f(u) - 3f(u^2) - f(ub) - 2f(uc) - 2f(bu) - f(cu) | \leq 13\,d$$

which is

$$| f(bcu^{bc}u^cu) + 9f(u) - 3f(u^2) - f(ub) - 2f(uc) - 2f(bu) - f(cu) | \leq 13\,d.$$

Now using (4.3) and (4.2) we get

$$| f(u^{bc}u^cu) + 9f(u) - 3f(u^2) - f(u) - 2f(u) - 2f(u) - f(u) | \leq 13\,d + 14\,d = 27\,d.$$

Using $f(u^2) = 4\,f(u)$, the last inequality yields

$$| f(u^{bc}u^cu) - 9f(u) | \leq 27\,d.$$

Therefore for any $n \in \mathbb{N}$ we have

$$\begin{aligned} n^2|f(u^{bc}u^cu) - 9f(u)| &= |f((u^{bc}u^cu)^n) - 9f(u^n)| \\ &= |f((u^n)^{bc}(u^n)^cu^n) - 9f(u^n)| \leq 27d. \end{aligned}$$

Thus we have

$$f(u^{bc}u^cu) = 9\,f(u)$$

and the proof of the lemma is complete. □

Lemma 4.6. *Let f be an element of $PK_2(G)$ satisfying condition (4.1) on H. Let $b, c, bc \in K$ be the elements of order two. Suppose for $u \in S$ the elements u^{bc}, u^c, u generate an abelian subsemigroup, then*

$$f(u^{bc}u^cu) = 3\,f(u) \qquad \forall\, u \in S.$$

Proof. Using Lemma 2.3 with $n = 5$ and $x_1 = u, x_2 = b, x_3 = u, x_4 = c, x_5 = u$, we get

$$\begin{aligned} |\ f(ubucu) + 3[3f(u) + f(b) + f(c)] - 3f(u^2) - f(ub) \\ -\ 2f(uc) - 2f(bu) - f(cu) - f(bc)\ | \leq 6d. \end{aligned}$$

Now taking into account relations $|f(b)| \leq d$, $|f(c)| \leq d$, and $|f(bc)| \leq d$, we obtain

$$|\ f(ubucu) + 9f(u) - 3f(u^2) - f(ub) - 2f(uc) - 2f(bu) - f(cu)\ | \leq 13\,d$$

which is

$$|\ f(bcu^{bc}u^cu) + 9f(u) - 3f(u^2) - f(ub) - 2f(uc) - 2f(bu) - f(cu)\ | \leq 13\,d.$$

Now using (4.3) and (4.2) we get

$$|\ f(u^{bc}u^cu) + 9f(u) - 3f(u^2) - f(u) - 2f(u) - 2f(u) - f(u)\ | \leq 13\,d + 14\,d = 27\,d.$$

Using $f(u^2) = 2\,f(u)$, the last inequality yields

$$|\ f(u^{bc}u^cu) - 3f(u)\ | \leq 27\,d.$$

Therefore for any $n \in \mathbb{N}$ we have

$$\begin{aligned} n^2|f(u^{bc}u^cu) - 3f(u)| &= |f((u^{bc}u^cu)^n) - 3f(u^n)| \\ &= |f((u^n)^{bc}(u^n)^cu^n) - 3f(u^n)| \leq 27d. \end{aligned}$$

Thus we have

$$f(u^{bc}u^cu) = 3\,f(u)$$

and the proof of the lemma is complete. □

Let S be an arbitrary semigroup with unit and B a group. For each $b \in B$ denote by $S(b)$ a group that is isomorphic to S under isomorphism $a \to a(b)$. Denote by $H = S^{(B)} = \prod_{b\in B} S(b)$ the direct product of groups $S(b)$. It is clear that if $a_1(b_1)a_2(b_2)\cdots a_k(b_k)$ is an element of H, then for any $b \in B$, the mapping

$$b^* : a_1(b_1)a_2(b_2)\cdots a_k(b_k) \to a_1(b_1b)a_2(b_2b)\cdots a_k(b_kb)$$

is an automorphism of D and $b \to b^*$ is an embedding of B into $Aut\,H$. Thus, we can form a semidirect product $G = B \rtimes H$. This semigroup is called *the wreath* *product* of the semigroup S and the group B, and will be denoted by $G = S \wr B$. We will identify the group S with subgroup $S(1)$ of H, where $1 \in B$. Hence, we can assume that S is a subgroup of H.

Let us denote, by C, the group of order four having generators b, c and defining relations: $b^2 = c^2 = 1$, $bc = cb$. Consider the semigroup group $S \wr C$.

Lemma 4.7. *Suppose that $f \in PK_4(S \wr C)$. If for some $x, y, z \in S$ we have*

$$|f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)| = \delta > 0$$

then for some $x_1, y_1, z_1 \in H$ we have

$$|f(x_1y_1z_1) + f(x_1) + f(y_1) + f(z_1) - f(x_1y_1) - f(x_1z_1) - f(y_1z_1)| = 9\delta.$$

Proof. Let $x_1 = xyz$, $y_1 = x_1^b$, $z_1 = x_1^c$. We have $x_1 \in S(1), x_1^b \in S(b), x_1^c \in S(c)$, therefore subsemigroup generated by x_1, x_1^b, x_1^c is an abelian semigroup. Applying Lemma 4.5 we get

$$f(xyz(xyz)^b(xyz)^c) = f(xx^bx^cyy^by^czz^bz^c),$$

$$\begin{aligned} |\, & f(xx^bx^cyy^by^czz^bz^c) + f(xx^bx^c) + f(yy^by^c) + f(zz^bz^c) \\ & - f(xx^bx^cyy^by^c) - f(xx^bx^czz^bz^c) - f(yy^by^czz^bz^c)\,| \\ &= 9|f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)| = 9\delta \end{aligned}$$

and the proof is now complete. □

Lemma 4.8. *Suppose that $f \in PK_2(S \wr C)$. If for some $x, y, z \in S$ we have*

$$|f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)| = \delta > 0$$

then for some $x_1, y_1, z_1 \in H$ we have

$$|f(x_1y_1z_1) + f(x_1) + f(y_1) + f(z_1) - f(x_1y_1) - f(x_1z_1) - f(y_1z_1)| = 3\delta.$$

Proof. Let $x_1 = xyz$, $y_1 = x_1^b$, $z_1 = x_1^c$. We have $x_1 \in S(1)$, $x_1^b \in S(b)$, $x_1^c \in S(c)$, therefore subsemigroup generated by x_1, x_1^b, x_1^c is an abelian semigroup. Applying Lemma 4.6 we get

$$f(xyz(xyz)^b(xyz)^c) = f(xx^bx^cyy^by^czz^bz^c),$$

$$\begin{aligned} |\, & f(xx^bx^cyy^by^czz^bz^c) + f(xx^bx^c) + f(yy^by^c) + f(zz^bz^c) \\ & - f(xx^bx^cyy^by^c) - f(xx^bx^czz^bz^c) - f(yy^by^czz^bz^c)\,| \\ &= 3\,|f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz)| = 3\delta \end{aligned}$$

and the proof is now complete. □

Theorem 4.9. *Let S be a semigroup with left (or right) law of reduction. Then S can be embedded into a semigroup G with the left (or right respectively) law of reduction and the equation (2.1) is stable on G. Moreover, if S is a group then G is a group too.*

Proof. Let C_i, for $i \in \mathbb{N}$, be a group of order 4 with two generators b_i, c_i and defining relations $b_i^2 = 1, c_i^2 = 1, b_i c_i = c_i b_i$. Consider the chain of groups defined as follows:

$$S_1 = S,\; S_2 = S_1 \wr C_1,\; S_3 = S_2 \wr C_2,\; \dots,\; S_{k+1} = S_k \wr C_k, \dots$$

Define a chain of embeddings

$$S_1 = S \;\to\; S_2 = S_1 \wr C_1 \;\to\; S_3 = S_2 \wr C_2 \;\to\; \cdots \to\; S_{k+1} = S_k \wr C_k \to\; \dots \qquad (4.4)$$

by identifying S_k with $S_k(1)$ a subgroup of S_{k+1}. Let G be the direct limit of the chain (4.4). Then we have $G = \cup_{k\in\mathbb{N}} S_k$ and

$$S_1 \subset S_2 \subset \cdots \subset S_k \subset S_{k+1} \subset \dots\dots \;\subset G.$$

Let $f \in PK_4(G)$, and let for $k \in \mathbb{N}$

$$\delta_k = \sup \big\{ |\, f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) \,| ;\; x, y, z \in S_k \big\}.$$

Let us verify that $\delta_k = 0$ for any k. Suppose that $\delta_1 > 0$. Then for some x_1, y_1, z_1 in S_1, we have

$$|\, f(x_1y_1z_1) + f(x_1) + f(y_1) + f(z_1) - f(x_1y_1) - f(x_1z_1) - f(y_1z_1) \,| = \delta > 0.$$

By Lemma 4.7 there are $x_2, y_2, z_2 \in S_2$ such that

$$|\, f(x_2y_2z_2) + f(x_2) + f(y_2) + f(z_2) - f(x_2y_2) - f(x_2z_2) - f(y_2z_2) \,| = 9\delta > 0.$$

By repeated applications of Lemma 4.7 we obtain, for any $k \in \mathbb{N}$, there are $x_k, y_k, z_k \in S_k$ such that

$$|\, f(x_ky_kz_k) + f(x_k) + f(y_k) + f(z_k) - f(x_ky_k) - f(x_kz_k) - f(y_kz_k). \,| = 9^{k-1}\delta > 0$$

This gives a contradiction to the assumption that $f \in PK_4(G)$. Therefore $\delta_1 = 0$. Similarly, using Lemma 4.8, we verify that $\delta_n = 0$ for any $n \in \mathbb{N}$. So, $PK_4(G) = K_4(G)$. Similarly we verify that $PK_2(G) = K_2(G)$. Thus by Proposition 3.3 we get $PK(G) = K(G)$ and the equation (2.1) is stable on G. This finishes the proof of the theorem. □

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