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# SHANNON TYPE INEQUALITIES OF A RELATIVE OPERATOR ENTROPY INCLUDING TSALLIS AND RÉNYI ONES

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Abstract. Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions, that is, $A_i, B_i > 0$ $(1 \le i \le n)$ and $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = I$. We give a new relative operator entropy of two operator distributions as follows: For $t, s \in \mathbb{R} \setminus \{0\}$,

$$K_{t,s}(\mathbb{A}|\mathbb{B}) \equiv \frac{(\sum_{i=1}^n A_i \natural_t B_i)^s - I}{ts},$$

where $A \natural_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}}$. This includes relative operator entropy $S(\mathbb{A}|\mathbb{B})$, Rényi relative operator entropy $I_t(\mathbb{A}|\mathbb{B})$ and Tsallis relative operator entropy $T_t(\mathbb{A}|\mathbb{B})$.

In this paper, firstly, we discuss fundamental properties of $K_{t,s}(\mathbb{A}|\mathbb{B})$. Secondly, we obtain Shannon type operator inequalities by using $K_{t,s}(\mathbb{A}|\mathbb{B})$, which include previous results by Furuta, Yanagi–Kuriyama–Furuichi and ourselves.

## 1. Introduction

In this paper, an operator means a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator $T$ is said to be positive (denoted by $T \ge 0$) if $(Tx, x) \ge 0$ for all $x \in \mathcal{H}$, and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

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For $A, B > 0$, we consider a path passing through $A$ and $B$ as follows: $A \natural_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}}$ for $t \in \mathbb{R}$. We remark that for $0 \leq t \leq 1$, $A \natural_t B$ is known as the weighted geometric mean, which is denoted by $A \sharp_t B$.

For discrete probability distributions $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$, that is, $p_i, q_i > 0$ $(1 \leq i \leq n)$ and $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$, the relative entropy (Kullback–Leibler divergence, Kullback–Leibler distance) is defined by $s(p|q) \equiv \sum_{i=1}^n p_i \log \frac{p_i}{q_i}$ ([12, 14]). We remark that if $q = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, then $s(p|q) = \log n - h(p)$, where $h(p) \equiv -\sum_{i=1}^n p_i \log p_i$ is the famous Shannon entropy. The relative entropy plays an important role in the classical information theory as a notion to measure the difference between two probability distributions. We also remark that $s(p|q) \geq 0$ holds for any probability distributions $p$ and $q$, which is equivalent to Shannon inequality (Shannon lemma, Gibbs' inequality) $-\sum_{i=1}^n p_i \log p_i \leq -\sum_{i=1}^n p_i \log q_i$.

As an operator version of $s(p|q)$, for $A, B > 0$, relative operator entropy

$$S(A|B) \equiv A^{\frac{1}{2}} \log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$$

is introduced in [4], and Furuta [2] introduced its generalization

$$S_t(A|B) \equiv A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t \log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$$

for $t \in \mathbb{R}$. We remark that $S(A|I) = -A \log A$ is operator entropy given by Nakamura–Umegaki [13], and we denote $S(A|I)$ by $H(A)$. $S_t(A|B)$ can be considered as a tangent at $t$ of $A \natural_t B$. If $t = 0$, then $S(A|B) = S_0(A|B)$. Yanagi–Kuriyama–Furuichi introduced Tsallis relative operator entropy in [16]. For $A, B > 0$ and $t \in \mathbb{R}$, Tsallis relative operator entropy is defined by

$$T_t(A|B) \equiv \frac{A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^t A^{\frac{1}{2}} - A}{t} = \frac{A \natural_t B - A}{t} \quad (t \neq 0),$$
$$T_0(A|B) \equiv \lim_{t \to 0} T_t(A|B) = S(A|B).$$

We call an operator sequence $\mathbb{A} = (A_1, \cdots, A_n)$ operator distribution if $A_i > 0$ $(1 \leq i \leq n)$ and $\sum_{i=1}^n A_i = I$, since it can be regarded as an operator version of discrete probability distribution. We remark that operator distribution is called POVM (positive operator valued measure) in quantum information theory. In [8, 10], we define relative operator entropy $S(\mathbb{A}|\mathbb{B})$, generalized relative operator entropy $S_t(\mathbb{A}|\mathbb{B})$, Tsallis relative operator entropy $T_t(\mathbb{A}|\mathbb{B})$ and Rényi relative operator entropy $I_t(\mathbb{A}|\mathbb{B})$ of two operator distributions $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ as follows: For $t \in \mathbb{R}$,

$$S(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n S(A_i|B_i), \quad S_t(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n S_t(A_i|B_i),$$

$$T_t(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^n T_t(A_i|B_i),$$

$$I_t(\mathbb{A}|\mathbb{B}) \equiv \frac{1}{t} \log \sum_{i=1}^n A_i \natural_t B_i \quad (\text{if } t \neq 0) \quad \text{and } I_0(\mathbb{A}|\mathbb{B}) \equiv \lim_{t \to 0} I_t(\mathbb{A}|\mathbb{B}).$$

On the other hand, for two operator distributions $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$, Furuta [2] had shown an operator version of Shannon inequality (briefly, operator Shannon inequality) and its reverse one.

$$-\log\left[\sum_{i=1}^{n} A_i B_i^{-1} A_i\right] \leq S(\mathbb{A}|\mathbb{B}) \leq 0. \tag{1.1}$$

Yanagi–Kuriyama–Furuichi [16] showed a substitution of (1.1) by using Tsallis relative operator entropy.

$$\frac{(\sum_{i=1}^{n} A_i B_i^{-1} A_i)^{-t} - I}{t} \leq T_t(\mathbb{A}|\mathbb{B}) \leq 0 \quad \text{for } 0 < t \leq 1. \tag{1.2}$$

Recently, as an extension of operator Shannon inequality, we obtained

$$S(\mathbb{A}|\mathbb{B}) \leq I_t(\mathbb{A}|\mathbb{B}) \leq T_t(\mathbb{A}|\mathbb{B}) \leq 0, \tag{1.3}$$

$$0 \leq -T_{1-t}(\mathbb{B}|\mathbb{A}) \leq -I_{1-t}(\mathbb{B}|\mathbb{A}) \leq S_1(\mathbb{A}|\mathbb{B}), \tag{1.4}$$

$$T_t(\mathbb{A}|\mathbb{B}) \leq S_t(\mathbb{A}|\mathbb{B}) \leq -T_{1-t}(\mathbb{B}|\mathbb{A}) \tag{1.5}$$

for $0 < t < 1$ in [8].

In [10] (see also [9]), we introduced generalizations of $S(\mathbb{A}|\mathbb{B})$, $I_t(\mathbb{A}|\mathbb{B})$ and $T_t(\mathbb{A}|\mathbb{B})$ by using power mean, and we obtain generalizations of (1.3), (1.4) and (1.5).

Here, we introduce a new operator entropy including $T_t(\mathbb{A}|\mathbb{B})$ and $I_t(\mathbb{A}|\mathbb{B})$ as a different generalization from [10].

**Definition 1.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions. For $t, s \in \mathbb{R} \setminus \{0\}$,*

$$K_{t,s}(\mathbb{A}|\mathbb{B}) \equiv \frac{(\sum_{i=1}^{n} A_i \natural_t B_i)^s - I}{ts}.$$

*We define $K_{t,0}(\mathbb{A}|\mathbb{B})$ and $K_{0,s}(\mathbb{A}|\mathbb{B})$ by the limit of $K_{t,s}(\mathbb{A}|\mathbb{B})$ as a parameter tends to 0, respectively.*

In this paper, firstly, we discuss fundamental properties of $K_{t,s}(\mathbb{A}|\mathbb{B})$. Secondly, we obtain Shannon type operator inequalities by using $K_{t,s}(\mathbb{A}|\mathbb{B})$, which include (1.1), (1.2) and (1.3). Lastly, we introduce Shannon–Nakamura–Umegaki operator entropy and give fundamental properties.

## 2. Properties of $K_{t,s}(\mathbb{A}|\mathbb{B})$

First, we state relations among $T_t(\mathbb{A}|\mathbb{B})$, $I_t(\mathbb{A}|\mathbb{B})$ and $K_{t,s}(\mathbb{A}|\mathbb{B})$.

**Proposition 2.1.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions.*

(i) $K_{t,0}(\mathbb{A}|\mathbb{B}) = I_t(\mathbb{A}|\mathbb{B})$ *for* $t \in \mathbb{R} \setminus \{0\}$.

(ii) $K_{t,1}(\mathbb{A}|\mathbb{B}) = T_t(\mathbb{A}|\mathbb{B})$ *for* $t \in \mathbb{R} \setminus \{0\}$.

(iii) *For* $t > 0$ *and* $0 < s < 1$, $S(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \le T_t(\mathbb{A}|\mathbb{B})$.

(iv) *For* $t < 0$ *and* $0 < s < 1$, $T_t(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B}) \le S(\mathbb{A}|\mathbb{B})$.

If $0 < t \le 1$, then (iii) in Proposition 2.1 ensures

$$S(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \le T_t(\mathbb{A}|\mathbb{B}) \le 0 \tag{2.1}$$

by (1.2), so that (2.1) is an improvement of (1.3) by using $K_{t,s}(\mathbb{A}|\mathbb{B})$.

Jensen's operator inequality [7] plays an important role to prove our results.

**Theorem 2.A** (Jensen's operator inequality [7])**.** *Let $f(x)$ be an operator concave function on an interval $J$. Let $\{C_i\}_{i=1}^n$ be operators with $\sum_{i=1}^n C_i^* C_i = I$. Then*

$$f\left(\sum_{i=1}^n C_i^* A_i C_i\right) \ge \sum_{i=1}^n C_i^* f(A_i) C_i$$

*holds for every selfadjoint operators $\{A_i\}_{i=1}^n$ whose spectra are contained in $J$.*

*Proof of Proposition 2.1.* (i) Since $\lim_{s\to 0} \frac{a^s - 1}{s} = \log a$ for $a > 0$, we have

$$K_{t,0}(\mathbb{A}|\mathbb{B}) = \frac{1}{t} \lim_{s\to 0} \frac{(\sum_{i=1}^n A_i \natural_t B_i)^s - I}{s} = \frac{1}{t} \log \sum_{i=1}^n A_i \natural_t B_i = I_t(\mathbb{A}|\mathbb{B}).$$

(ii) We obtain (ii) as follows:

$$K_{t,1}(\mathbb{A}|\mathbb{B}) = \frac{\sum_{i=1}^n A_i \natural_t B_i - I}{t} = \sum_{i=1}^n \frac{A_i \natural_t B_i - A_i}{t} = T_t(\mathbb{A}|\mathbb{B}).$$

(iii) Since $\log a \le \frac{a^s - 1}{s} \le a - 1$ holds for $a > 0$ and $0 < s < 1$,

$$\frac{1}{t} \log \sum_{i=1}^n A_i \natural_t B_i \le \frac{(\sum_{i=1}^n A_i \natural_t B_i)^s - I}{st} \le \frac{\sum_{i=1}^n A_i \natural_t B_i - I}{t},$$

that is, $I_t(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \le T_t(\mathbb{A}|\mathbb{B})$ holds for $t > 0$ and $0 < s < 1$. The relation $S(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B})$ has already shown in (1.3).

(iv) By the similar way to the proof of (iii), we have $T_t(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B})$ for $t < 0$ and $0 < s < 1$.

Since $f(x) = \log x$ is operator concave for $x > 0$, we have

$$\begin{aligned} I_t(\mathbb{A}|\mathbb{B}) &= \frac{1}{t}\log \sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^t A_i^{\frac{1}{2}} \\ &\leq \frac{1}{t}\sum_{i=1}^{n} A_i^{\frac{1}{2}}\log(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})^t A_i^{\frac{1}{2}} \\ &= \sum_{i=1}^{n} A_i^{\frac{1}{2}}\log(A_i^{\frac{-1}{2}}B_iA_i^{\frac{-1}{2}})A_i^{\frac{1}{2}} = S(\mathbb{A}|\mathbb{B}). \end{aligned}$$

for $t < 0$ by using Theorem 2.A. Hence the proof is complete. □

**Remark.** We remark that $I_0(\mathbb{A}|\mathbb{B}) \equiv \lim_{t\to 0} I_t(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B})$ holds by (iii) and (iv) in Proposition 2.1.

Next, we discuss monotonicity of $K_{t,s}(\mathbb{A}|\mathbb{B})$ on $s$.

**Proposition 2.2.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions.*

(i) *For $t > 0$, $K_{t,s}(\mathbb{A}|\mathbb{B})$ is increasing for $s \in \mathbb{R}$.*
(ii) *For $t < 0$, $K_{t,s}(\mathbb{A}|\mathbb{B})$ is decreasing for $s \in \mathbb{R}$.*
(iii) *$K_{0,s}(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B})$ for $s \in \mathbb{R} \setminus \{0\}$.*

*Proof.* We have (i) and (ii) immediately since for $a > 0$,

$$f(s) = \begin{cases} \dfrac{a^s - 1}{s} & (s \neq 0) \\ \log a & (s = 0) \end{cases}$$

is increasing for $s \in \mathbb{R}$.

We prove (iii) as follows: For each $s \in \mathbb{R} \setminus \{0\}$, there exists a natural number $m$ such that $|s| \leq m$. Therefore, if $t > 0$, then

$$K_{t,-m}(\mathbb{A}|\mathbb{B}) \leq K_{t,s}(\mathbb{A}|\mathbb{B}) \leq K_{t,m}(\mathbb{A}|\mathbb{B}) \tag{2.2}$$

holds by (i). Similarly, if $t < 0$, then

$$K_{t,m}(\mathbb{A}|\mathbb{B}) \leq K_{t,s}(\mathbb{A}|\mathbb{B}) \leq K_{t,-m}(\mathbb{A}|\mathbb{B}) \tag{2.3}$$

holds by (ii).

Put $X_t = \sum_{i=1}^{n} A_i \natural_t B_i$. Then, since $X_0 = I$, we have

$$\begin{aligned} K_{t,m}(\mathbb{A}|\mathbb{B}) &= \frac{X_t^m - I}{tm} = \frac{(X_t - I)(X_t^{m-1} + X_t^{m-2} + \cdots + I)}{tm} \\ &= T_t(\mathbb{A}|\mathbb{B}) \cdot \frac{X_t^{m-1} + X_t^{m-2} + \cdots + I}{m} \to S(\mathbb{A}|\mathbb{B}) \quad \text{as } t \to 0 \end{aligned}$$

and

$$K_{t,-m}(\mathbb{A}|\mathbb{B}) = \frac{X_t^{-m} - I}{-tm} = X_t^{-m} \cdot \frac{X_t^m - I}{tm} \to S(\mathbb{A}|\mathbb{B}) \quad \text{as } t \to 0.$$

Therefore we obtain $\lim_{t\to 0} K_{t,s}(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B})$ by (2.2) and (2.3). □

**Remark.** The same argument to the proof of (iii) in Proposition 2.2 yields

$$K_{0,0}(\mathbb{A}|\mathbb{B}) \equiv \lim_{(t,s)\to(0,0)} K_{t,s}(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B}),$$

so we can allow the case $t = 0$ in (i) and (ii) in Proposition 2.1, and the case $s = 0$ in (iii) in Proposition 2.2.

When we consider monotonicity of $K_{t,s}(\mathbb{A}|\mathbb{B})$ on $t$, we might want to put $s = \frac{\beta}{t}$. Then we have the following result on monotonicity of $K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B})$ on $t$.

**Theorem 2.3.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions.*

(i) *For $\beta > 0$, $K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B})$ is increasing for $t$ such that $t \le -\beta$ or $\beta \le t$.*

(ii) *For $\beta > 0$, $K_{t,\frac{-\beta}{t}}(\mathbb{A}|\mathbb{B})$ is increasing for $t$ such that $t \le -\beta$ or $\beta \le t$.*

(iii) *$I_t(\mathbb{A}|\mathbb{B})$ is increasing for $t \in \mathbb{R}$.*

In order to prove Theorem 2.3, we use the following lemma.

**Lemma 2.4.** *Let $\{C_i\}_{i=1}^n$ be invertible operators with $\sum_{i=1}^n C_i^* C_i = I$ and $\{X_i\}_{i=1}^n$ be strictly positive operators.*

(i) *For a fixed $\beta > 0$, the function $g_1(\lambda) = \left(\sum_{i=1}^n C_i^* X_i^\lambda C_i\right)^{\frac{\beta}{\lambda}}$ defined on $J \equiv (-\infty, -\beta] \cup [\beta, \infty)$ is increasing on the interval $J$.*

(ii) *The function $g_2(\lambda) = \frac{1}{\lambda} \log \left(\sum_{i=1}^n C_i^* X_i^\lambda C_i\right)$ is increasing for $\lambda \in \mathbb{R}$, where $g_2(0) \equiv \sum_{i=1}^n C_i^* (\log X_i) C_i$.*

*Proof.* (i) By Theorem 2.A,

$$\left(\sum_{i=1}^n C_i^* X_i^\lambda C_i\right)^{\frac{\mu}{\lambda}} \ge \sum_{i=1}^n C_i^* X_i^\mu C_i \quad \text{for } 0 < \mu \le \lambda \tag{2.4}$$

since $f(x) = x^{\frac{\mu}{\lambda}}$ is operator concave. Applying Loewner–Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*" to (2.4), we have

$$g_1(\mu) = \left(\sum_{i=1}^{n} C_i^* X_i^{\mu} C_i\right)^{\frac{\beta}{\mu}} \leq \left(\sum_{i=1}^{n} C_i^* X_i^{\lambda} C_i\right)^{\frac{\beta}{\lambda}} = g_1(\lambda) \tag{2.5}$$

for $0 < \beta \leq \mu \leq \lambda$. Replacing $X_i$ by $X_i^{-1}$ and taking inverses to (2.5), we have

$$g_1(-\lambda) = \left(\sum_{i=1}^{n} C_i^* X_i^{-\lambda} C_i\right)^{\frac{\beta}{-\lambda}} \leq \left(\sum_{i=1}^{n} C_i^* X_i^{-\mu} C_i\right)^{\frac{\beta}{-\mu}} = g_1(-\mu)$$

for $-\lambda \leq -\mu \leq -\beta < 0$. Therefore $g_1(\lambda)$ is increasing for $\lambda$ such that $\lambda \leq -\beta$ or $\beta \leq \lambda$.

(ii) Taking logarithm to (2.4), we have

$$g_2(\mu) = \frac{1}{\mu}\log\left(\sum_{i=1}^{n} C_i^* X_i^{\mu} C_i\right) \leq \frac{1}{\lambda}\log\left(\sum_{i=1}^{n} C_i^* X_i^{\lambda} C_i\right) = g_2(\lambda)$$

for $0 < \mu \leq \lambda$. By Theorem 2.A,

$$g_2(\mu) = \frac{1}{\mu}\log\left(\sum_{i=1}^{n} C_i^* X_i^{\mu} C_i\right) \geq \frac{1}{\mu}\sum_{i=1}^{n} C_i^* (\log X_i^{\mu}) C_i = \sum_{i=1}^{n} C_i^* (\log X_i) C_i = g_2(0)$$

since $f(x) = \log x$ is operator concave. Therefore $g_2(0) \leq g_2(\mu) \leq g_2(\lambda)$ for $0 < \mu \leq \lambda$. We also get $g_2(-\lambda) \leq g_2(-\mu) \leq g_2(0)$ for $-\lambda \leq -\mu < 0$ by replacing $X_i$ by $X_i^{-1}$. Hence the proof is complete. □

*Proof of Theorem 2.3.* (i) By putting $X_i = A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}}$ and $C_i = A_i^{\frac{1}{2}}$ in (i) in Lemma 2.4, we have that

$$\left\{\sum_{i=1}^{n} A_i^{\frac{1}{2}} (A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}})^t A_i^{\frac{1}{2}}\right\}^{\frac{\beta}{t}} = \left(\sum_{i=1}^{n} A_i \natural_t B_i\right)^{\frac{\beta}{t}} \tag{2.6}$$

is increasing for $t$ such that $t \leq -\beta$ or $\beta \leq t$.

Therefore we obtain that $K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) = \dfrac{(\sum_{i=1}^{n} A_i \natural_t B_i)^{\frac{\beta}{t}} - I}{\beta}$ is increasing for $t$ such that $t \leq -\beta$ or $\beta \leq t$.

(ii) Since $K_{t,\frac{-\beta}{t}}(\mathbb{A}|\mathbb{B}) = \dfrac{\{(\sum_{i=1}^{n} A_i \natural_t B_i)^{\frac{\beta}{t}}\}^{-1} - I}{-\beta}$, we have (ii) by (2.6).

(iii) Similarly to (i), we obtain (iii) by using (ii) in Lemma 2.4. □

By Proposition 2.2 and Theorem 2.3, we can obtain monotonicity of $K_{t,s}(\mathbb{A}|\mathbb{B})$ on $t$.

**Theorem 2.5.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions.*

(i) *For* $0 \le s \le 1$, $K_{t,s}(\mathbb{A}|\mathbb{B})$ *is increasing for* $t \in \mathbb{R}$.

(ii) $T_t(\mathbb{A}|\mathbb{B})$ *is increasing for* $t \in \mathbb{R}$.

*Proof.* (i) We may assume $0 < s \le 1$ since $K_{t,0}(\mathbb{A}|\mathbb{B}) = I_t(\mathbb{A}|\mathbb{B})$ by (i) in Proposition 2.1 and $I_t(\mathbb{A}|\mathbb{B})$ is increasing for $t \in \mathbb{R}$ by (iii) in Theorem 2.3.

If $0 < t \le u$, then

$$K_{t,s}(\mathbb{A}|\mathbb{B}) \le K_{u,\frac{ts}{u}}(\mathbb{A}|\mathbb{B}) \le K_{u,s}(\mathbb{A}|\mathbb{B}) \tag{2.7}$$

holds by (i) in Theorem 2.3 and (i) in Proposition 2.2 since $0 < ts \le t \le u$.

If $u \le t < 0$, then

$$K_{u,s}(\mathbb{A}|\mathbb{B}) \le K_{u,\frac{ts}{u}}(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \tag{2.8}$$

holds by (ii) in Proposition 2.2 and (ii) in Theorem 2.3 since $u \le t \le ts < 0$.

By letting $t \to +0$ in (2.7) and $t \to -0$ in (2.8), we have

$$K_{v,s}(\mathbb{A}|\mathbb{B}) \le K_{0,s}(\mathbb{A}|\mathbb{B}) = S(\mathbb{A}|\mathbb{B}) \le K_{u,s}(\mathbb{A}|\mathbb{B})$$

for $v < 0 < u$ by (iii) in Proposition 2.2. Therefore $K_{t,s}(\mathbb{A}|\mathbb{B})$ is increasing for $t \in \mathbb{R}$.

(ii) We have (ii) by putting $s = 1$ in (i), or immediately obtain since for $a > 0$,

$$f(t) = \begin{cases} \dfrac{a^t - 1}{t} & (t \ne 0) \\ \log a & (t = 0) \end{cases}$$

is increasing for $t \in \mathbb{R}$. □

## 3. Shannon type inequalities

In this section, we get several Shannon type inequalities. First, we obtain an extension of (1.1) and (1.2).

**Theorem 3.1.** *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *and* $\mathbb{B} = (B_1, \cdots, B_n)$ *be operator distributions.*

(i) *For* $0 < \beta \le t \le 1$,

$$K_{-1,-\beta}(\mathbb{A}|\mathbb{B}) \le K_{-t,\frac{\beta}{-t}}(\mathbb{A}|\mathbb{B}) \le K_{-\beta,-1}(\mathbb{A}|\mathbb{B}) \\ \le T_\beta(\mathbb{A}|\mathbb{B}) \le K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le 0.$$

(ii) *For* $0 < \beta \le t \le 1$,

$$K_{-1,\beta}(\mathbb{A}|\mathbb{B}) \le K_{-t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le T_{-\beta}(\mathbb{A}|\mathbb{B}) \\ \le K_{\beta,-1}(\mathbb{A}|\mathbb{B}) \le K_{t,\frac{-\beta}{t}}(\mathbb{A}|\mathbb{B}) \le 0.$$

(iii) *For* $0 < t \le 1$,

$$I_{-1}(\mathbb{A}|\mathbb{B}) \le I_{-t}(\mathbb{A}|\mathbb{B}) \le S(\mathbb{A}|\mathbb{B}) \le I_t(\mathbb{A}|\mathbb{B}) \le 0.$$

We remark that (iii) in Theorem 3.1 implies (1.1) since

$$I_{-1}(\mathbb{A}|\mathbb{B}) = -\log\left[\sum_{i=1}^{n} A_i B_i^{-1} A_i\right],$$

and also (i) in Theorem 3.1 implies (1.2) since

$$K_{-1,-\beta}(\mathbb{A}|\mathbb{B}) = \frac{(\sum_{i=1}^{n} A_i B_i^{-1} A_i)^{-\beta} - I}{\beta}.$$

*Proof of Theorem 3.1.* (i) By (i) in Theorem 2.3, we have

$$K_{-1,-\beta}(\mathbb{A}|\mathbb{B}) \le K_{-t,\frac{\beta}{-t}}(\mathbb{A}|\mathbb{B}) \le K_{-\beta,-1}(\mathbb{A}|\mathbb{B})$$

and

$$T_\beta(\mathbb{A}|\mathbb{B}) = K_{\beta,1}(\mathbb{A}|\mathbb{B}) \le K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le K_{1,\beta}(\mathbb{A}|\mathbb{B}) = 0.$$

Since $f(x) = -x^{-1}$ is operator concave, by Theorem 2.A, we have

$$\begin{aligned} K_{-\beta,-1}(\mathbb{A}|\mathbb{B}) = \frac{(\sum_{i=1}^{n} A_i \natural_{-\beta} B_i)^{-1} - I}{\beta} &= \frac{\left\{\sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}})^{-\beta} A_i^{\frac{1}{2}}\right\}^{-1} - I}{\beta} \\ &\le \frac{\sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{\frac{-1}{2}} B_i A_i^{\frac{-1}{2}})^{\beta} A_i^{\frac{1}{2}} - I}{\beta} = T_\beta(\mathbb{A}|\mathbb{B}). \end{aligned}$$

Therefore the desire inequality is obtained.

(ii) is shown by the similar way to the proof of (i).

(iii) holds by (iii) in Theorem 2.3, or we can get (iii) by letting $\beta \to +0$ in (i) or (ii). □

Combining Theorem 3.1 and the results shown in section 2, we have the following Theorem 3.2. We remark that (i) in Theorem 3.2 is a generalization of (2.1).

**Theorem 3.2.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ be operator distributions.*

(i) *For $0 < \beta \le t \le 1$ and $0 < s < 1$,*

$$\begin{aligned} &K_{-1,\beta}(\mathbb{A}|\mathbb{B}) \le K_{-t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le T_{-\beta}(\mathbb{A}|\mathbb{B}) \le K_{-\beta,s}(\mathbb{A}|\mathbb{B}) \le I_{-\beta}(\mathbb{A}|\mathbb{B}) \\ &\le S(\mathbb{A}|\mathbb{B}) \le I_\beta(\mathbb{A}|\mathbb{B}) \le K_{\beta,s}(\mathbb{A}|\mathbb{B}) \le T_\beta(\mathbb{A}|\mathbb{B}) \le K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le 0. \end{aligned}$$

(ii) *For $0 < t \le 1$ and $0 < s \le 1$,*

$$K_{-t,-s}(\mathbb{A}|\mathbb{B}) \le K_{t,s}(\mathbb{A}|\mathbb{B}) \le 0 \quad \text{and} \quad K_{-t,s}(\mathbb{A}|\mathbb{B}) \le K_{t,-s}(\mathbb{A}|\mathbb{B}) \le 0.$$

(iii) *For $0 < \beta \le 1$,*

$$I - \sum_{i=1}^{n} A_i B_i^{-1} A_i \le K_{-1,\beta}(\mathbb{A}|\mathbb{B}) \le I_{-1}(\mathbb{A}|\mathbb{B}) \le K_{-1,-\beta}(\mathbb{A}|\mathbb{B}) \le T_\beta(\mathbb{A}|\mathbb{B}) \le 0.$$

*Proof.* (i) We get

$$T_\beta(\mathbb{A}|\mathbb{B}) \le K_{t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le 0$$

and

$$K_{-1,\beta}(\mathbb{A}|\mathbb{B}) \le K_{-t,\frac{\beta}{t}}(\mathbb{A}|\mathbb{B}) \le T_{-\beta}(\mathbb{A}|\mathbb{B})$$

by (i) and (ii) in Theorem 3.1. The rest holds by (iii) and (iv) in Proposition 2.1.

(ii) By putting $\beta = ts$ in (i) and (ii) in Theorem 3.1, we have (ii).

(iii) By (ii) in Proposition 2.2, we have

$$K_{-1,1}(\mathbb{A}|\mathbb{B}) \le K_{-1,\beta}(\mathbb{A}|\mathbb{B}) \le K_{-1,0}(\mathbb{A}|\mathbb{B}) \le K_{-1,-\beta}(\mathbb{A}|\mathbb{B}),$$

that is,

$$I - \sum_{i=1}^{n} A_i B_i^{-1} A_i \le K_{-1,\beta}(\mathbb{A}|\mathbb{B}) \le I_{-1}(\mathbb{A}|\mathbb{B}) \le K_{-1,-\beta}(\mathbb{A}|\mathbb{B})$$

for $0 < \beta \le 1$. The rest holds by (i) in Theorem 3.1. □

## 4. Operator valued $\alpha$-divergence and Shannon–Nakamura–Umegaki operator entropy

In this section, we state two topics related to relative operator entropies.

Based on $\alpha$-divergence of two probability distributions by Amari [1], Fujii [3] defined an operator valued $\alpha$-divergence as follows: For strictly positive operators $A$ and $B$, and for $\alpha \in (0,1)$,

$$D_\alpha(A|B) \equiv \frac{1}{\alpha(1-\alpha)}(A \nabla_\alpha B - A \sharp_\alpha B) = \frac{1}{1-\alpha}\{B - A - T_\alpha(A|B)\},$$

where $A \nabla_\alpha B \equiv (1-\alpha)A + \alpha B$ is the weighted arithmetic mean. Fujii-Mićić-Pečarić-Seo [5, 6] showed

$$D_0(A|B) \equiv \lim_{\alpha\to 0} D_\alpha(A|B) = B - A - S(A|B) \quad \text{and}$$
$$D_1(A|B) \equiv \lim_{\alpha\to 1} D_\alpha(A|B) = A - B + S_1(A|B).$$

We remark that $D_0(A|B)$ is a divergence introduced by Petz [15]. It is easily seen that $D_1(A|B) = D_0(B|A)$ since we can show $S_1(A|B) = -S(B|A)$ by using an equation $f(XX^*)X = Xf(X^*X)$ for a continuous function $f$ on an interval $J$ and an operator $X$ such that spectra of $XX^*$ and $X^*X$ are contained in $J$.

In [11], we define operator valued $\alpha$-divergence $D_\alpha(\mathbb{A}|\mathbb{B})$ of two operator distributions $\mathbb{A} = (A_1, \cdots, A_n)$ and $\mathbb{B} = (B_1, \cdots, B_n)$ as

$$D_\alpha(\mathbb{A}|\mathbb{B}) \equiv \sum_{i=1}^{n} D_\alpha(A_i|B_i).$$

We note that it was shown in [11] that

$$D_\alpha(\mathbb{A}|\mathbb{B}) = \frac{-1}{1-\alpha} T_\alpha(\mathbb{A}|\mathbb{B}) \text{ for } \alpha \in (0,1),$$
$$D_0(\mathbb{A}|\mathbb{B}) = -S(\mathbb{A}|\mathbb{B}) \quad \text{and} \quad D_1(\mathbb{A}|\mathbb{B}) = D_0(\mathbb{B}|\mathbb{A}) = S_1(\mathbb{A}|\mathbb{B}),$$

so that (2.1) is equivalent to

$$0 \le (1-\alpha)D_\alpha(\mathbb{A}|\mathbb{B}) \le -K_{t,s}(\mathbb{A}|\mathbb{B}) \le -I_t(\mathbb{A}|\mathbb{B}) \le D_0(\mathbb{A}|\mathbb{B})$$

for $0 < t \le 1$ and $0 < s < 1$.

Next, as an operator version of Shannon entropy and as an expression of Nakamura–Umegaki operator entropy $H(A)$ by using an operator distribution, we introduce Shannon–Nakamura–Umegaki operator entropy of an operator distribution $\mathbb{A} = (A_1, \cdots, A_n)$ as

$$H(\mathbb{A}) \equiv \sum_{i=1}^{n} H(A_i) = \sum_{i=1}^{n} A_i \log A_i^{-1}.$$

Then the following fundamental properties hold.

**Proposition 4.1.** *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be an operator distribution and $\frac{1}{n}\mathbb{I} = (\frac{1}{n}I, \cdots, \frac{1}{n}I)$.*

(i) $S(\mathbb{A}|\frac{1}{n}\mathbb{I}) = H(\mathbb{A}) - (\log n)I$.

(ii) $0 \le H(\mathbb{A}) \le (\log n)I$.

*Proof.* (i) Since $S(A_i|\frac{1}{n}I) = A_i \log(A_i^{-1} \cdot \frac{1}{n}I) = A_i \log A_i^{-1} - (\log n)A_i$ holds for $i = 1, \cdots, n$,

$$S(\mathbb{A}|\tfrac{1}{n}\mathbb{I}) = \sum_{i=1}^{n} \{A_i \log A_i^{-1} - (\log n)A_i\} = H(\mathbb{A}) - (\log n)I.$$

(ii) We have $H(\mathbb{A}) \ge 0$ since $-x\log x \ge 0$ for $0 < x \le 1$ and the spectrum of $A_i$ is included in $(0, 1]$ for $i = 1, \cdots, n$. We have $H(\mathbb{A}) \le (\log n)I$ by (1.1) and (i). □

**Remark.** In the proof of (ii) in Proposition 4.1, by using (2.1) instead of (1.1), we have

$$\begin{aligned} 0 \le H(\mathbb{A}) &\le \frac{1}{t}\log \sum_{i=1}^{n} A_i^{1-t} \le \frac{1}{ts}\left\{\frac{1}{n^{ts}}\left(\sum_{i=1}^{n} A_i^{1-t}\right)^s + (\log n^{ts} - 1)I\right\} \\ &\le \frac{1}{t}\left\{\frac{1}{n^t}\sum_{i=1}^{n} A_i^{1-t} + (\log n^t - 1)I\right\} \le (\log n)I \end{aligned}$$

for $0 < t \le 1$ since $A_i \natural_t (\frac{1}{n}I) = \frac{1}{n^t}A_i^{1-t}$ holds for $i = 1, \cdots, n$.

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