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ASCOLI-TYPE THEOREMS FOR LOCALLY BOUNDED QUASICONTINUOUS FUNCTIONS, MINIMAL USCO AND MINIMAL CUSCO MAPS

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ABSTRACT. Let X be a locally compact space and let (Y, d) be a nontrivial metric space such that d has the Heine–Borel property. We prove Ascoli-type theorem for locally bounded quasicontinuous functions from X to Y. Using the above result we also prove an Ascoli-type theorems for the spaces of minimal usco and minimal cusco maps from X to $\mathbb{R}$.

1. Introduction

In our paper we prove Ascoli-type theorems for locally bounded quasicontinuous functions, minimal usco and minimal cusco maps.

The notion of quasicontinuity of real-valued functions of real variable was introduced by Kempisty in [16]. The property of quasicontinuity was pehaps the first time used by Baire in [1] in the study of points of continuity of separately continuous functions. For general topological spaces X and Y a function $f : X \to Y$ is quasicontinuous [19] at $x \in X$ if for every open set $V \subset Y, f(x) \in V$ and open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

Quasicontinuous functions are also very important in the study of minimal usco and minimal cusco maps. In fact, every selection of of a minimal usco map is quasicontinuous.

There is a rich literature concerning the study of quasicontinuity (see, for instance [19], [17], [4]).

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The acronym usco (cusco) stands for a (convex) upper-semicontinuous non-empty compact-valued set-valued map. Such set-valued maps are interesting because they describe common features of maximal monotone operators, of the convex subdifferential and of Clarke generalized gradient. (see [5])

Minimal usco and minimal cusco maps are used in many papers (see [5], [7], [13], [10], [11], [12]). Minimal usco maps are a very convenient tool in the theory of games (see [6]) or in functional analysis. (see [3]).

2. Compact subsets of $(Q^*(X,Y), \tau_{UC})$

Let X, Y be Hausdorff topological spaces. By $\mathbb{N}$ we denote the set of positive integers, $\mathbb{R}$ be the space of real numbers with the usual Euclidean metric and $\mathbb{R}^n$ be the finite dimensional Euclidean space. The symbol $\overline{A}$ and $IntA$ will stand for the closure and interior of the set A in a topological space.

We say that a subset of X is quasi-open (or semi-open) if it is contained in the closure of its interior. Then a function $f : X \to Y$ is quasicontinuous if and only if $f^{-1}(V)$ is quasi-open for every open set $V \subset Y$.

Denote by $F(X,Y)$ the set of all functions from X to Y, by $C(X,Y)$ the set of all continuous functions in $F(X,Y)$, and by $Q(X,Y)$ the set of all quasicontinuous functions from X to Y. Notice that if $f : X \to Y$ is a function, we will use the symbol f also for the graph of f.

By τ_p and τ_{UC} we mean the topology of pointwise convergence on $F(X,Y)$ and the topology of uniform convergence on compact sets on $F(X,Y)$, respectively.

Let $H \subset F(X,Y)$ and let $x \in X$, denote by $H[x]$ the set $\{f(x) \in Y;\ f \in H\}$.

If X is a locally compact space and (Y,d) is a metric space, the Ascoli theorem [15] says that a subset $\mathcal{E}$ of $(C(X,Y), \tau_{UC})$ is compact if and only if it is closed in $(C(X,Y), \tau_{UC})$, $\mathcal{E}[x]$ has a compact closure for each $x \in X$ and is equicontinuous, where a subset $\mathcal{E}$ of $C(X,Y)$ is equicontinuous provided that for each $x \in X$ and $\epsilon > 0$ there is a neighbourhood U of x with $d(f(x), f(z)) < \epsilon$ for all $z \in U$ and $f \in \mathcal{E}$.

If f is a function from X to a metric space (Y,d), we say that f is locally bounded, if for every $x \in X$ there is an open set $U \subset X$, $x \in U$ such that $f(U) = \{f(u) : u \in U\}$ is a bounded subset of (Y,d). Denote by $F^*(X,Y)$ the space of all locally bounded functions from $F(X,Y)$. If G is a subset of $F^*(X,Y)$ then we denote this set also by G^*.

Inspired by papers [9], [18] we give the following definition.

Definition 2.1. Let X be a topological space and (Y,d) be a metric space. We say that a subset $\mathcal{E}$ of $F^*(X,Y)$ is densely equiquasicontinuous at a point x of X provided that for every $\epsilon > 0$, there exists a finite family $\mathcal{B}$ of subsets of X which are either quasi-open or nowhere dense such that $\cup\mathcal{B}$ is a neighbourhood of x and such that for every $f \in \mathcal{E}$, for every $B \in \mathcal{B}$ and for every $p, q \in B$, $d(f(p), f(q)) < \epsilon$. Then $\mathcal{E}$ is densely equiquasicontinuous provided that it is densely equiquasicontinuous at every point of X.

Lemma 2.2. *Let X be a topological space and (Y, d) be a metric space. If $\mathcal{E}$ is densely equiquasicontinuous subset of $C(X, Y)$, then $\mathcal{E}$ is equicontinuous.*

Proof. Let $x \in X$ and $\epsilon > 0$. Because $\mathcal{E}$ is densely equiquasicontinuous, there exist a finite family $\mathcal{B} = \{B_1, B_2, \cdots, B_n\}$ of quasi-open or nowhere dense subsets of X such that $\cup\mathcal{B}$ is a neighborhood of x and such that for every $f \in \mathcal{E}$, for every $B \in \mathcal{B}$ and for every $p, q \in B$, $d(f(p), f(q)) < \frac{\epsilon}{2}$. Since $\mathcal{E} \subset C(X, Y)$, for every $f \in \mathcal{E}$, for every $B \in \mathcal{B}$ and for every $p, q \in \overline{B}$, $d(f(p), f(q)) < \epsilon$. Without loss of generality we can suppose that $x \in \overline{B_i}$ for every $i \in \{1,\ 2,\ \cdots,\ n\}$. Now let $z \in \cup\mathcal{B}$ and $f \in \mathcal{E}$. Then there is $i \in \{1,\ 2,\ \cdots,\ n\}$ such that $z \in B_i$ and so $d(f(x), f(z)) < \epsilon$. □

Remark 2.3. Note that if $\{A_1, A_2, \cdots, A_n\}$ is a family of quasi-open sets of X, then the set $A = A_1 \cap \cdots \cap A_n$ is nowhere dense or it is a union of nowhere dense set and quasi-open set. In fact, consider first the case $IntA = \emptyset$. If $x \in A$, then there is $i \in \{1,\ 2,\ \cdots,\ n\}$ such that $x \in A_i \setminus IntA_i$, hence $A \subset A_1 \setminus IntA_1 \cup \cdots \cup A_n \setminus IntA_n$. Since A_i is quasi-open for every $i \in \{1,\ 2,\ \cdots,\ n\}$, A is nowhere dense. Now let $IntA \neq \emptyset$. Then we can show as above that the set $A \setminus IntA$ is nowhere dense and so $A = (A \setminus IntA) \cup IntA$ is a union of a nowhere dense set $A \setminus IntA$ and a quasi-open set $IntA$.

Lemma 2.4. *Let X be a topological space and (Y, d) be a metric space such that every bounded set is totally bounded. If $\mathcal{E}$ is a densely equiquasicontinuous subset of $F^*(X, Y)$ and $f \in Q^*(X, Y)$, then $\mathcal{E} \cup \{f\}$ is densely equiquasicontinuous.*

Proof. Let $x \in X$ and let $\epsilon > 0$. Since $\mathcal{E}$ is densely equiquasicontinuous at x there exists a finite family $\mathcal{B} = \{B_1, B_2, \cdots, B_n\}$ of subsets of X which are either quasi-open or nowhere dense such that $\cup\mathcal{B}$ is a neighbourhood of x and such that for every $f \in \mathcal{E}$, for every $B \in \mathcal{B}$ and for every $p, q \in B$, $d(f(p), f(q)) < \epsilon$. Let O be a neighbourhood of x such that $O \subset \cup\mathcal{B}$ and $f(O)$ is bounded. Choose $y_0 \in Y$ and let $r > 0$ be such that the set $f(O)$ is contained in the closed ball $B_r(y_0)$. Let $V_1,\ V_2,\ \cdots,\ V_m$ be a finite open cover of $B_r(y_0)$ where radius of V_j is less than ϵ for every $j \in \{1,\ \cdots,\ m\}$. For every $j \in \{1,\ 2,\ \cdots,\ m\}$ let $H_j = f^{-1}(V_j)$. Then $\mathcal{H} = \{H_j :\ j \in \{1,\ 2,\ \cdots,\ m\}\}$ is a finite family of quasi-open subsets of X such that $O \subset \cup\mathcal{H}$. For every $i \in \{1,\ 2,\ \cdots,\ n\}$, $j \in \{1,\ 2,\ \cdots,\ m\}$ put $P_{i,j} = (B_i \cap H_j) \setminus Int(B_i \cap H_j)$ and $R_{i,j} = Int(B_i \cap H_j)$, where $P_{i,j}$ is a nowhere dense set (if B_i and H_j are both quasi-open see Remark 2.3).

Denote by $\mathcal{D}$ the finite family containing all nonempty sets $P_{i,j}$ and $R_{i,j}$ where $i \in \{1,\ 2,\ \cdots,\ n\}$, $j \in \{1,\ 2,\ \cdots,\ m\}$. Evidently $O \subset \cup\mathcal{D}$ and so $\cup\mathcal{D}$ is a neighbourhood of x. It is easy to see that for every $g \in \mathcal{E} \cup \{f\}$, for every $D \in \mathcal{D}$ and for every $p, q \in D$, $d(g(p), g(q)) < \epsilon$. □

It should be noted that bounded sets are totally bounded for a metric d if and only if the metric of its completion has the Heine–Borel property.

Lemma 2.5. *Let X be a topological space and (Y, d) be a metric space. Let $\{f_\lambda :\ \lambda \in \Lambda\}$ be a net in $Q^*(X, Y)$ pointwise convergent to $f \in F(X, Y)$ and let the set $\{f_\lambda :\ \lambda \in \Lambda\}$ be densely equiquasicontinuous. Then $f \in Q^*(X, Y)$.*

Proof. Let $x \in X$. Suppose, by the way of contradiction, that f is not quasicontinuous at x. Then there exist $\epsilon > 0$ and an open set U, $x \in U$ such that for every nonempty open set $W \subset U$ there is $z \in W$ such that $d(f(x), f(z)) > \epsilon$.

Since $\{f_\lambda : \lambda \in \Lambda\}$ is densely equiquasicontinuous at x, there exists a finite family $\mathcal{B} = \{B_1, B_2, \cdots, B_n\}$ of subsets of X which are either quasi-open or nowhere dense such that $\cup\mathcal{B}$ is a neighbourhood of x and such that for every $\lambda \in \Lambda$, for every $B \in \mathcal{B}$ and for every $p, q \in B$, $d(f_\lambda(p), f_\lambda(q)) < \frac{\epsilon}{4}$. Without loss of generality we can suppose that $x \in \overline{B_i}$ for every $i \in \{1,\ 2,\ \cdots,\ n\}$.

Denote by C the set of all numbers from $j \in \{1,\ 2,\ \cdots,\ n\}$ where $Int B_j \neq \emptyset$. We choose a $z_j \in Int B_j \cap U$ such that $d(f(x), f(z_j)) > \epsilon$ for every $j \in C$. Since the net $\{f_\lambda : \lambda \in \Lambda\}$ is pointwise convergent to f, there is $\lambda \in \Lambda$ such that $d(f(x), f_\lambda(x)) < \frac{\epsilon}{4}$ and $d(f(z_j), f_\lambda(z_j)) < \frac{\epsilon}{4}$ for every $j \in C$.

We show that for $\frac{\epsilon}{4}$, the open set $U \cap (Int \cup \mathcal{B})$ and for every nonempty open set $W \subset (U \cap (Int \cup \mathcal{B}))$ there exists $w \in W$ such that $d(f_\lambda(x), f_\lambda(w)) > \frac{\epsilon}{4}$. So let $W \subset (U \cap (Int \cup \mathcal{B}))$. Let $j \in C$ be such that $Int B_j \cap W \neq \emptyset$ and let $w \in Int B_j \cap W$. Then $d(f_\lambda(w), f_\lambda(z_j)) < \frac{\epsilon}{4}$. We have

$$d(f_\lambda(w), f(z_j)) \leq d(f_\lambda(w), f_\lambda(z_j)) + d(f_\lambda(z_j), f(z_j)) < \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \tfrac{2\epsilon}{4}$$

and $d(f_\lambda(x), f(x)) < \frac{\epsilon}{4}$. So from inequality $d(f(x), f(z_j)) > \epsilon$ follows that $d(f_\lambda(x), f_\lambda(w)) > \frac{\epsilon}{4}$. Hence f_λ is not quasicontinuous at x, a contradiction.

Suppose, by the way of contradiction, that f is not locally bounded at x. Choose any $y_0 \in Y$. Let U be a neighbourhood of x. For every $n \in \mathbb{N}$ there is $x_n \in U \cap Int \cup \mathcal{B}$ such that $d(f(x_n), y_0) > n$. By passing to a subsequence there is $B_i \in \mathcal{B}$ such that $\{x_n : n \in \mathbb{N}\} \subset B_i$. The net $\{f_\lambda : \lambda \in \Lambda\}$ pointwise converges to f, so for every $n \in \mathbb{N}$ there is $\lambda_n \in \Lambda$ such that $d(f_\lambda(x_n), y_0) > n$ for every $\lambda > \lambda_n$. Then, since the set $\{f_\lambda : \lambda \in \Lambda\}$ is densely equiquasicontinuous at x, $d(f_\lambda(z), y_0) > n - \frac{\epsilon}{4}$ for every $z \in B_i$ and for every $\lambda > \lambda_n$. From this it follows that $d(f(z), y_0) \geq n - \frac{\epsilon}{4}$ for every $n \in \mathbb{N}$ and for every $z \in B_i$, a contradiction. □

Corollary 2.6. *Let X be a topological space and (Y, d) be a metric space. Let $\mathcal{E}$ be a densely equiquasicontinuous and closed subset of $(Q^*(X, Y), \tau_p)$. Then $\mathcal{E}$ is closed also in $(F(X, Y), \tau_p)$.*

Proof. The proof follows from Lemma 2.5 □

Theorem 2.7. *Let X be a topological space and (Y, d) be a metric space such that every bounded set is totally bounded. Let $\mathcal{E}$ be a densely equiquasicontinuous subset of $Q^*(X, Y)$. Then the topologies τ_p and τ_{UC} on $\mathcal{E}$ coincide.*

Proof. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net in $\mathcal{E}$ which τ_p-converges to $f \in Q^*(X, Y)$. We show that $\{f_\lambda : \lambda \in \Lambda\}$ also τ_{UC}-converges to f.

By Lemma 2.4 the set $E \cup \{f\}$ is densely equiquasicontinuous, so for every $x \in X$ and every $m \in \mathbb{N}$ there exists a finite family $\mathcal{B}^{m,x}$ of quasi-open or nowhere dense subsets of X such that $\cup\mathcal{B}^{m,x}$ is a neighbourhood of x and such that for every $g \in \mathcal{E} \cup \{f\}$, for every $B \in \mathcal{B}^{m,x}$ and for every $p, q \in B$, $d(g(p), g(q)) < \frac{1}{m}$.

Let $x \in X$, $m \in \mathbb{N}$. We claim that there is $\lambda_{m,x} \in \Lambda$ such that $d(f(u)), f_\lambda(u)) \leq \frac{3}{m}$ for every $\lambda \geq \lambda_{m,x}$ and every $u \in \cup\mathcal{B}^{m,x}$. For every $B \in \mathcal{B}^{m,x}$ we choose a $z \in B$. Since $\{f_\lambda : \lambda \in \Lambda\}$ τ_p-converges to f, there is $\lambda_{z,B}$ such that $d(f(z)), f_\lambda(z)) < \frac{1}{m}$ for every $\lambda \geq \lambda_{z,B}$. Let $u \in B$, $\lambda \geq \lambda_{z,B}$.

$$
\begin{aligned}
&d(f(u), f_\lambda(u)) \leq d(f(u), f(z)) + d(f(z)), f_\lambda(u)) \leq \\
\leq &d(f(u), f(z)) + d(f(z)), f_\lambda(z)) + d(f_\lambda(z)), f_\lambda(u)) \leq \\
\leq &\frac{1}{m} + \frac{1}{m} + \frac{1}{m} < \frac{3}{m},
\end{aligned}
$$

Then let $\lambda_{m,x}$ be such that $\lambda_{m,x} \geq \lambda_{z,B}$ for every $B \in \mathcal{B}^{m,x}$. Then we have that $d(f(u), f_\lambda(u)) \leq \frac{3}{m}$ for every $u \in \cup\mathcal{B}^{m,x}$ and every $\lambda \geq \lambda_{m,x}$.

Let K be a compact subset of X. For every $m \in \mathbb{N}$ there is finite set of points $x_j \in K$, where $j \in \{1,\ 2,\ \cdots,\ n_m\}$ such that the family $\cup\mathcal{B}^{m,x_1}$, $\cup\mathcal{B}^{m,x_1}, \cdots, \cup\mathcal{B}^{m,x_{n_m}}$ is a finite cover of K. Let λ_m be such that $\lambda_m \geq \lambda_{m,x_j}$ for every $j \in \{1,\ 2,\ \cdots,\ n_m\}$. Then $d(f(u)), f_\lambda(u)) \leq \frac{3}{m}$ for every $u \in K$ and every $\lambda \geq \lambda_m$. Thus the net $\{f_\lambda :\ \lambda \in \Lambda\}$ uniformly converges to f on K. $\square$

Let Y be a metric space. We say that a subset H of $F(X, Y)$ is pointwise bounded provided that for each $x \in X$ the set $H[x]$ is bounded in Y.

Let (Y, d) be a metric space. We say that d has the Heine–Borel property if each closed bounded subset is compact. This notion is also known in the literature as the boundedly compact space [2]. Therefore (Y, d) is a locally compact, separable metric space and d is complete. In fact, any locally compact, separable metric space has a compatible metric with the Heine–Borel property ([20], [9]).

Theorem 2.8. *Let X be a locally compact space and (Y, d) be a nontrivial metric space such that d has the Heine–Borel property. A subset $\mathcal{E}$ of $(Q^*(X, Y), \tau_{UC})$ is compact if and only if it is closed, pointwise bounded and densely equiquasicontinuous.*

Proof. Let $\mathcal{E}$ be closed in $(Q^*(X, Y), \tau_{UC})$, pointwise bounded and densely equiquasicontinuous subset of $Q^*(X, Y)$. By Theorem 2.7 $\mathcal{E}$ is also τ_p-closed. Since $\mathcal{E}$ is pointwise bounded for every $x \in X$, from the Heine–Borel property of d, there exist a compact set B_x such that $\mathcal{E}[x] \subset B_x$. The product $\prod_{x\in X}\{B_x :\ x \in X\}$ is a compact subset of $Y^X = \prod_{x\in X}\{Y_x :\ x \in X\}$ with the relative product topology. Since $\mathcal{E}$ is a τ_p-closed subset of the τ_p-compact set $\prod_{x\in X}\{B_x :\ x \in X\}$ it follows that $\mathcal{E}$ is τ_p-compact and hence by Theorem 2.7 $\mathcal{E}$ is τ_{UC}-compact.

For the converse, suppose that $\mathcal{E}$ is a compact subset of $(Q^*(X, Y), \tau_{UC})$. The set $\mathcal{E}$ is closed because $(Q^*(X, Y), \tau_{UC})$ is a Hausdorff space.

The evaluation at x defined by $e_x(f) = f(x)$ for all $f \in Q^*(X, Y)$ is continuous with respect to τ_p topology on $Q^*(X, Y)$ [15], hence it is continuous also with respect to τ_{UC} topology on $Q^*(X, Y)$ (Theorem 2.7) and so the image $\mathcal{E}[x]$ of $\mathcal{E}$ is compact and therefore is bounded.

For the proof of equiquasicontinuity of $\mathcal{E}$ we use an idea of the proof of Theorem 5.7 in [9]. Let $x \in X$ and let O be an open neighbourhood such that $\overline{O} = A$ is compact. Let $\epsilon > 0$, we define a finite family $\mathcal{B}$ of quasi-open subsets of X at follows. Since $\mathcal{E}$ is compact in $(Q^*(X, Y), \tau_{UC})$, there is $f_1,\ \cdots,\ f_n \in \mathcal{E}$ such that

$$\mathcal{E} \subset W(f_1, A, \tfrac{\epsilon}{3}) \cup \cdots \cup W(f_n, A, \tfrac{\epsilon}{3}).$$

Since every function from $\mathcal{E}$ is locally bounded and A is compact, for every $i \in \{1,\ 2,\ \cdots,\ n\}$ the set $f_i(A)$ is bounded. Choose $y_0 \in Y$ and let $r > 0$ be such that the set $f_1(A) \cup \cdots \cup f_n(A)$ is contained in the closed ball $B_r(y_0)$. Let $V_1,\ V_2,\ \cdots,\ V_m$ be a finite open cover of $B_r(y_0)$ where radius of V_j is less than $\frac{\epsilon}{3}$ for every $j \in \{1,\ \cdots,\ m\}$. For every $i \in \{1,\ 2,\ \cdots,\ n\}$, $j \in \{1,\ 2,\ \cdots,\ m\}$ put $B^i_j = f_i^{-1}(V_j)$, which is a quasi-open set since f_i is quasicontinuous.

Denote by $\mathcal{F}$ the set of all functions from $\{1,\ 2,\ \cdots,\ n\}$ to $\{1,\ 2,\ \cdots,\ m\}$. For every $h \in \mathcal{F}$ put

$$P_h = (O \cap B^1_{h(1)} \cap \cdots \cap B^n_{h(n)}) \setminus Int(O \cap B^1_{h(1)} \cap \cdots \cap B^n_{h(n)})$$

and

$$R_h = Int(O \cap B^1_{h(1)} \cap \cdots \cap B^n_{h(n)}).$$

By Remark 2.3 for every $h \in \mathcal{F}$ the set P_h is a nowhere dense set.

Denote by $\mathcal{B}$ the family containing all nonempty sets P_h and R_h where $h \in \mathcal{F}$. We show that $\cup\mathcal{B}$ is a neighbourhood of x. Let $z \in O$, then there is $h \in \mathcal{F}$ such that $f_i(z) \in V_{h(i)}$ and so $z \in B^i_{h(i)}$ for every $i \in \{1,\ 2,\ \cdots,\ n\}$. Then $z \in P_h$ or $z \in R_h$ and thus $z \in \cup\mathcal{B}$.

Now let $f \in \mathcal{E}$, let $B \in \mathcal{B}$ and let $p, q \in B$. Of course there is a $h \in \mathcal{F}$ such that $B \subset B^1_{h(1)} \cap \cdots \cap B^n_{h(n)}$. Because $\mathcal{E} \subset W(f_1, A, \frac{\epsilon}{3}) \cup \cdots \cup W(f_n, A, \frac{\epsilon}{3})$ there exist a $i \in \{1,\ 2,\ \cdots,\ n\}$ such that $f \in W(f_i, A, \frac{\epsilon}{3})$. So $d(f(p), f_i(p)) < \frac{\epsilon}{3}$ and $d(f(q), f_i(q)) < \frac{\epsilon}{3}$. Because $p, q \in B^i_{h(i)}$ we have that $f_i(p) \subset V_{h(i)}$ and $f_i(q) \subset V_{h(i)}$ Then $d(f(p), f(q)) \le d(f(p), f_i(p)) + d(f_i(p), f_i(q)) + d(f_i(q), f(q)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Hence $\mathcal{E}$ is densely equiquasicontinuous. □

3. Compact subsets of $(MU(X,Y), \tau_{UC})$ and $(MC(X,Y), \tau_{UC})$

Let X, Y be Hausdorff topological spaces. By 2^Y we denote the space of all closed subsets of Y and by $CL(Y)$ the space of all nonempty closed subsets of Y. The space of all functions from X to 2^Y we denote by $F(X, 2^Y)$ and the space of all functions from X to $CL(Y)$ we denote by $F(X, CL(Y))$. We also call the functions from $F(X, 2^Y)$ set-valued maps, or multifunctions, from X to Y. If F is a set-valued map from X to Y, then its graph is the set $\{(x, y) \in X \times Y :\ y \in F(x)\}$. Conversely, if F is a subset of $X \times Y$ and $x \in X$, define $F(x) = \{y \in Y :\ (x, y) \in F\}$. Then we can assign to each subset F of $X \times Y$ a set-valued map which takes the value $F(x)$ at each point $x \in X$ and which graph is F. In this way, we identify set-valued maps with their graphs. Following [7] the term map is reserved for a set-valued map.

Given two maps $F, G : X \to Y$, we write $G \subset F$ and say that G is contained in F if $G(x) \subset F(x)$ for every $x \in X$.

A map $F : X \to Y$ is upper-semicontinuous at a point $x \in X$ if for every open set V containing $F(x)$, there exists an open set $U \subset X$, $x \in U$ such that $F(U) = \bigcup\{F(u) : u \in U\} \subset V$. F is upper-semicontinuous if it is upper-semicontinuous at each point of X.
Following Christensen [6] we say that a map F is usco if it is upper-semicontinuous and takes nonempty compact values. A map F from a topological space X to a linear topological space Y is cusco if it is usco and $F(x)$ is convex for every $x \in X$.

Finally, a map F from a topological space X to a topological (linear topological space) Y is said to be minimal usco (minimal cusco) if it is a minimal element in the family of all usco (cusco) maps (with domain X and range Y); that is, if it is usco (cusco) and does not contain properly any other usco (cusco) map from X into Y. By an easy application of the Kuratowski-Zorn principle we can guarantee that every usco (cusco) map from X to Y contains a minimal usco (cusco) map from X to Y (see [5], [7]).

We say that a (single-valued) function $f : X \to Y$ is subcontinuous (see [8]) at $x \in X$ if for every net $\{x_\sigma : \sigma \in \Sigma\}$ in X converging to x, there is a convergent subnet of $\{f(x_\sigma) : \sigma \in \Sigma\}$. A function f is subcontinuous if it is subcontinuous at every point of X.

Let $F : X \to Y$ be a map with nonempty values. Then a function $f : X \to Y$ is called a selection of F if $f(x) \in F(x)$ for every $x \in X$.

Let Y be a Hausdorff locally convex (linear topological) space. Let $F : X \to Y$ be a map with nonempty compact values. Then a selection f of F such that $f(x)$ is a extreme point of $F(x)$ for every $x \in X$ is called an extreme function of F [11].

In our papers [10], [11] we gave characterizations of minimal usco and minimal cusco maps via their selections.

Theorem 3.1. *(see [10]) Let X, Y be topological spaces and Y be a T_1 regular space. Let F be a map from X to Y. The following are equivalent:*

(1) *F is a minimal usco map;*
(2) *There exist a quasicontinuous and subcontinuous selection f of F such that $\overline{f} = F$;*
(3) *Every selection f of F is quasicontinuous, subcontinuous and $\overline{f} = F$.*

Theorem 3.2. *(see [11]) Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space. Let F be a map from X to Y. Then the following are equivalent:*

(1) *F is a minimal cusco map;*
(2) *F has nonempty compact values and there is a quasicontinuous, subcontinuous selection f of F such that $\overline{co}\overline{f}(x) = F(x)$ for every $x \in X$;*
(3) *F has nonempty compact, convex values, F has a closed graph, every extreme function of F is quasicontinuous, subcontinuous and any two extreme functions of F have the same closures of their graphs;*
(4) *F has nonempty compact values, every extreme function f of F is quasicontinuous, subcontinuous and $F(x) = \overline{co}\overline{f}(x)$ for every $x \in X$.*

To define a densely continuous form from X to Y [9], denote by $DC(X, Y)$ the set of all functions $f \in F(X, Y)$ such that the set $C(f)$ of points of continuity is dense in X. We call such functions densely continuous.

Of course $DC(X, Y)$ contains the set $C(X, Y)$ of all continuous functions from X to Y. If $Y = \mathbb{R}$ and X is a Baire space, then all upper and lower semicontinuous functions on X belongs to $DC(X, Y)$ and if X is a Baire space and Y is a metric space then every quasicontinuous function $f : X \to Y$ has a G_δ dense set $C(f)$ of the points of continuity [19]; i.e. $Q(X,Y) \subset DC(X,Y)$.

For every $f \in DC(X,Y)$ we denote by $\overline{f \restriction C(f)}$ the closure of the graph of $f \restriction C(f)$. If D is any dense subset of $C(f)$ then the closure of $f \restriction D$ in $X \times Y$ is equal to $\overline{f \restriction C(f)}$. We define the set $D(X,Y)$ of densely continuous forms by

$$D(X,Y) = \{\overline{f \restriction C(f)} : \ f \in DC(X,Y)\}.$$

Then the densely continuous forms from X to Y may be considered as maps, where for each $x \in X$ and $F \in D(X,Y)$ $F(x) = \{y \in Y : \ (x,y) \in F\}$.

Densely continuous forms from X to Y have a kind of minimality property found in the theory of minimal USCO maps. In particular, if X is a Baire space X and (Y,d) is locally compact metric space then $D^*(X,Y) = MU(X,Y)$, where $D^*(X,Y)$ is the set of all members of $D(X,Y)$ that are locally bounded [14].

We have the following characterizations of elements of $D(X,Y)$.

Theorem 3.3. *(see [10]) Let X be a Baire space and Y be a metric space. Let $F : X \to Y$ be such that $F(x) \neq \emptyset$ for every $x \in X$. The following are equivalent:*

(1) $F \in D(X,Y)$
(2) *There is a quasicontinuous function $f : X \to Y$ such that $\overline{f} = F$;*
(3) *Every selection f of F is quasicontinuous and $\overline{f} = F$.*

It is easy to see that if $f \in Q(X,Y)$ and D is a dense subset of X then $\overline{f} = \overline{f \restriction D}$.

Let (Y,d) be a metric space. The open d-ball with center $z_0 \in Y$ and radius $\epsilon > 0$ will be denoted by $S_\epsilon(z_0)$ and the ϵ-enlargement $\bigcup_{a \in A} S_\epsilon(a)$ for a subset A of Y will be denoted by $S_\epsilon(A)$.
If $A \in CL(Y)$, the distance functional $d(.,A) \ : Y \mapsto [0,\infty)$ is described by the familiar formula

$$d(z,A) = \inf\{d(z,a) : \ a \in A\}.$$

Let A and B be nonempty subsets of (Y,d). The excess of A over B with respect to d is defined by the formula

$$e_d(A,B) = \sup\{d(a,B) : \ a \in A\}.$$

The Hausdorff (extended-valued) metric H_d on $CL(Y)$ [2] is defined by

$$H_d(A,B) = \max\{e_d(A,B), e_d(B,A)\}.$$

We can also use the following equality on $CL(Y)$:

$$H_d(A,B) = \inf\{\epsilon > 0 : \ A \subset S_\epsilon(B) \text{ and } B \subset S_\epsilon(A)\}.$$

The topology generated by H_d is called the Hausdorff metric topology.

Following [9] we define the topology τ_p of pointwise convergence on $F(X, 2^Y)$. The topology τ_p of pointwise convergence on $F(X,Y)$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

$$W(A,\epsilon) = \{(\Phi,\Psi) : \ \forall \ x \in A \ \ H_d(\Phi(x),\Psi(x)) < \epsilon\},$$

where A is from the family of all nonempty finite subsets of X and $\epsilon > 0$.

We will define the topology τ_{UC} of uniform convergence on compact sets on $F(X, 2^Y)$ [9]. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K, \epsilon) = \{(\Phi, \Psi) : \ \forall \ x \in K \quad H_d(\Phi(x), \Psi(x)) < \epsilon\},$$

where K is from the family of all nonempty compact subsets of X and $\epsilon > 0$.

Theorem 3.4. *Let X be a locally compact space and (Y, d) be a nontrivial metric space such that d has the Heine–Borel property. If $\mathcal{E}$ is a compact subset of $(Q(X, Y), \tau_{UC})$ then $\{\overline{f} : \ f \in \mathcal{E}\}$ is a compact subset of $(D(X, Y), \tau_{UC})$.*

Proof. Let $\mathcal{E}$ be a compact subset of $(Q(X, Y), \tau_{UC})$ and let $\{\overline{f}_\lambda : \ \lambda \in \Lambda\}$ be a net in $\{\overline{f} : \ f \in \mathcal{E}\}$. By Theorem 3.3 if $f \in \mathcal{E}$, then $\overline{f} \in D(X, Y)$. By passing to a subnet, there is a function $f \in \mathcal{E}$ such that the net $\{f_\lambda : \ \lambda \in \Lambda\}$ τ_{UC}-converges to f. We prove that $\{\overline{f}_\lambda : \ \lambda \in \Lambda\}$ τ_{UC}-converges to $\overline{f}$. For every $x \in X$ choose a compact set $K_x \subset X$, such that $x \in IntK_x$. Let $x_0 \in X$. First we show that $\{\overline{f}_\lambda : \ \lambda \in \Lambda\}$ uniformly converges on $IntK_{x_0}$ to $\overline{f}$.

Let $\epsilon > 0$. The net $\{f_\lambda : \ \lambda \in \Lambda\}$ uniformly converges to f on K_{x_0}, so there is $\lambda_0 \in \Lambda$ such that $d(f(x), f_\lambda(x)) < \frac{\epsilon}{2}$ for every $\lambda \geq \lambda_0$ and every $x \in K_{x_0}$.

Choose any $x \in IntK_{x_0}$ and $z \in \overline{f}(x)$. There is a net $\{x_\omega : \ \omega \in \Omega\}$ in $IntK_{x_0}$ which converges to x such that the net $\{f(x_\omega) : \ \omega \in \Omega\}$ converges in Y to z. Without loss of generality we may assume that $\{f(x_\omega) : \ \omega \in \Omega\}$ is bounded. Let λ be arbitrary such that $\lambda \geq \lambda_0$. Then $d(f(x_\omega), f_\lambda(x_\omega)) < \frac{\epsilon}{2}$ for every $\omega \in \Omega$. The net $\{f_\lambda(x_\omega) : \ \omega \in \Omega\}$ has a cluster point $y \in \overline{f}_\lambda(x)$ since is bounded and d has the Heine–Borel property. Then $d(z.y) \leq \frac{\epsilon}{2}$ and since z is arbitrary element of $\overline{f}(x)$ we have that $\overline{f}(x) \subset S_\epsilon(\overline{f}_\lambda(x))$. The inclusion $\overline{f}_\lambda(x) \subset S_\epsilon(\overline{f}(x))$ can be proved similarly, hence $H_d(\overline{f}(x), \overline{f}_\lambda(x)) < \epsilon$. This shows that for every $x \in IntK_{x_0}$ and every $\lambda \geq \lambda_0$ $H_d(\overline{f}(x), \overline{f}_\lambda(x)) < \epsilon$. So the net $\{\overline{f}_\lambda : \ \lambda \in \Lambda\}$ uniformly converges on $IntK_{x_0}$ to $\overline{f}$.

Let K be a compact subset of X. There are finitely many points $x_1, x_2, \cdots, x_n$ from X such that $K \subset \cup\{IntK_{x_i} : \ i \in \{1, \ 2, \ \cdots, \ n\}\}$. The uniform convergence of $\{\overline{f}_\lambda : \ \lambda \in \Lambda\}$ to $\overline{f}$ on K follows from uniform convergence of this net to $\overline{f}$ on $IntK_{x_i}$ for $i \in \{1, \ 2, \ \cdots, \ n\}$. □

Denote by $MU(X, Y)$ the set of all minimal usco maps from X to Y and if Y is a Hausdorff locally convex linear topological space, by $MC(X, Y)$ the set of all minimal cusco maps from X to Y.

If F is a map from X to a metric space (Y, d), we say that F is locally bounded, if for every $x \in X$ there is an open set $U \subset X$, $x \in U$ such that $F(U) = \bigcup\{F(u) : u \in U\}$ is a bounded subset of (Y, d).

Note that if f is a selection of $F \in MU(X, Y)$ then $f \in Q(X, Y)$ (Theorem 3.1) and since F is upper-semicontinuous, $f \in Q^*(X, Y)$.

Theorem 3.5. *Let X be a locally compact space and (Y, d) be a nontrivial metric space such that d has the Heine–Borel property. Let $\mathcal{E}$ be a subset of $MU(X, Y)$, where for every $F \in \mathcal{E}$ there is a selection f_F of F such that $\{f_F : \ F \in \mathcal{E}\}$ is closed in $(Q^*(X, Y), \tau_{UC})$, densely equiquasicontinuous and pointwise bounded. Then $\mathcal{E}$ is compact subset of $(MU(X, Y), \tau_{UC})$.*

Proof. Let the set $\{f_F : F \in \mathcal{E}\}$ be closed in $(Q^*(X,Y), \tau_{UC})$, densely equiquasicontinuous and pointwise bounded. Then by Theorem 2.8 the set $\{f_F : F \in \mathcal{E}\}$ is a compact subset of $(Q^*(X,Y), \tau_{UC})$ and by Theorem 3.4 $\{\overline{f_F} : F \in \mathcal{E}\}$ is a compact subset of $(D(X,Y), \tau_{UC})$. By Theorem 3.1 $\mathcal{E} = \{\overline{f_F} : F \in \mathcal{E}\}$, hence $\mathcal{E}$ is a compact subset of $(MU(X,Y), \tau_{UC})$. □

Let Y be a Hausdorff locally convex (linear topological) space. Define the function $\varphi : MU(X,Y) \to F(X, CL(Y))$ as follows: $\varphi(F)(x) = \overline{co}F(x)$.

Theorem 3.6. *(see [12]) Let X be a locally compact space and Y be a Banach space. The map φ from $(MU(X,Y), \tau_{UC})$ onto $(MC(X,Y), \tau_{UC})$ is homeomorphism.*

Remark 3.7. Let X be a topological space and Y be a Hausdorff locally convex (linear topological) space in which the closed convex hull of a compact set is compact. Note that if $G \in MC(X,Y)$ and $F \in MU(X,Y)$, where F is contained in G then $\varphi(F)(x) = G(x)$ (see [12]). In fact, By Theorem 2.5. in [12] for G there is a unique minimal usco map F_G contained in G, so $F_G = F$. By Lemma 2.1 (see [12]) the map $x \to \overline{co}F(x)$ is a cusco map such that $\overline{co}F(x) \subset G(x)$ for every $x \in X$. Since G is minimal cusco, $G(x) = \overline{co}F(x)$ for every $x \in X$.

Theorem 3.8. *Let X be a locally compact space. Let $\mathcal{E}$ be a subset of $MC(X, \mathbb{R}^n)$ where for every $F \in \mathcal{E}$ there is a selection f_F of F such that $\{f_F : F \in \mathcal{E}\}$ is closed in $(Q^*(X, \mathbb{R}^n), \tau_{UC})$, densely equiquasicontinuous and pointwise bounded. Then $\mathcal{E}$ is a compact subset of $(MC(X, \mathbb{R}^n), \tau_{UC})$.*

Proof. Let the set $\{f_F : F \in \mathcal{E}\}$ be closed in $(Q^*(X, \mathbb{R}^n), \tau_{UC})$, densely equiquasicontinuous and pointwise bounded. Then by Theorem 2.8 the set $\{f_F : F \in \mathcal{E}\}$ is a compact subset of $(Q^*(X, \mathbb{R}^n), \tau_{UC})$ and by Theorem 3.4 $\{\overline{f_F} : F \in \mathcal{E}\}$ is compact subset of $(D(X, \mathbb{R}^n), \tau_{UC})$. By Theorem 3.1 $\{\overline{f_F} : F \in \mathcal{E}\} \subset MU(X, \mathbb{R}^n)$ and hence $\{\overline{f_F} : F \in \mathcal{E}\}$ is a compact subset of $(MU(X,Y), \tau_{UC})$. By Remark 3.7 $\mathcal{E} = \varphi(\{\overline{f_F} : F \in \mathcal{E}\})$ and by Theorem 3.6 $\mathcal{E}$ is a compact subset of $(MC(X, \mathbb{R}^n), \tau_{UC})$. □

Let $F : X \to \mathbb{R}$ be a map with bounded values. Define the functions s^F and l_F as follows:

$$s^F(x) = \sup\{y : y \in F(x)\}, \qquad l_F(x) = \inf\{y : y \in F(x)\}.$$

If $F \in MU(X, \mathbb{R}) \cup MC(X, \mathbb{R})$, then of course s^F and l_F are selections of F, where s^F is upper-semicontinuous, l_F is lower-semicontinuous and both of them are locally bounded. If $F \in MU(X, \mathbb{R})$, by Theorem 3.1, s^F and l_F are quasicontinuous and if $F \in MC(X, \mathbb{R})$ then s^F and l_F are extreme functions and so by Theorem 3.2 are also quasicontinuous.

Denote by $UC(X, \mathbb{R})$ the set of all upper-semicontinuous functions from X to $\mathbb{R}$.

Define the mapping $\mathcal{S} : MU(X, \mathbb{R}) \cup MC(X, \mathbb{R}) \to Q^\star(X, \mathbb{R}) \cap UC(X, \mathbb{R})$ by $\mathcal{S}(F) = s^F$.

Theorem 3.9. *(see [10]) Let X be a locally compact topological space. Then the mapping $\mathcal{S}$ from $(MU(X,\mathbb{R}),\ \mathfrak{U}_{UC})$ onto $(Q^\star(X,\mathbb{R})\cap UC(X,\mathbb{R}),\mathfrak{U}_{UC})$ is a uniform isomorphism.*

Proposition 3.10. *Let X be a Hausdorff topological space. The mapping $\mathcal{S}: MC(X,\mathbb{R}) \to Q^\star(X,\mathbb{R})\cap UC(X,\mathbb{R})$ is a bijection.*

Proof. The map φ^{-1} from $MC(X,\mathbb{R})$ to $MU(X,\mathbb{R})$ and map $\mathcal{S}$ from $MU(X,\mathbb{R})$ to $Q^\star(X,\mathbb{R})\cap UC(X,\mathbb{R})$ are bijections. Then also the composition $\mathcal{S}\varphi^{-1}$ from $MC(X,\mathbb{R})$ to $Q^\star(X,\mathbb{R})\cap UC(X,\mathbb{R})$ is a bijection. Let $F\in MU(X,\mathbb{R})$ and let $G(x)=\overline{co}F(x)$ for every $x\in X$. Clearly $s^F=s^G$. Then for every $G\in MC(X,\mathbb{R})$ we have that $\mathcal{S}(\varphi^{-1}(G))=\mathcal{S}(F)=s^F=s^G=\mathcal{S}(G)$. $\square$

Remark 3.11. It is easy to see that if A and B are nonempty compact subsets of $\mathbb{R}$, then $d(\sup A,\sup B)\leqq H_d(A,B)$.

Theorem 3.12. *Let X be a locally compact topological space. Then the mapping $\mathcal{S}$ from $(MC(X,\mathbb{R}),\ \mathfrak{U}_{UC})$ onto $(Q^\star(X,\mathbb{R})\cap UC(X,\mathbb{R}),\mathfrak{U}_{UC})$ is a uniform isomorphism.*

Proof. As we proved above the mapping $\mathcal{S}$ from $MC(X,\mathbb{R})$ to $Q^\star(X,\mathbb{R})\cap UC(X,\mathbb{R})$ is a bijection. By Remark 3.11 we have that $\mathcal{S}:(MC(X,\mathbb{R}),\mathfrak{U}_{UC})\to(O^\star(X,\mathbb{R})\cap UC(X,\mathbb{R}),\mathfrak{U}_{UC})$ is uniformly continuous.

To prove that also $\mathcal{S}^{-1}$ is uniformly continuous, let K be a nonempty compact subset of X and $\varepsilon>0$. The local compactness of X implies that there is an open set O in X such that $K\subset O$ and $\overline{O}$ is compact. Let $f,g\in Q^\star(X)\cap UC(X)$ be such that $d(f(x),g(x))<\varepsilon$ for every $x\in\overline{O}$. We prove that $H_d(co\overline{f}(x),co\overline{g}(x))\leq\varepsilon$ for every $x\in K$. Suppose, by the way of contradiction that this is not true. Then there exist $x_0\in K$ such that $H_d(co\overline{f}(x_0),co\overline{g}(x_0))>\epsilon$.

Since $f(x)=\sup co\overline{f}(x))$, $g(x)=\sup co\overline{g}(x))$ and $d(f(x),g(x))<\varepsilon$ for every $x\in X$ we have that $d(\inf co\overline{f}(x_0),\inf co\overline{g}(x_0))>\epsilon$, where one of following two cases can occurs: $\inf co\overline{f}(x_0)<\inf co\overline{g}(x_0)$ or $\inf co\overline{g}(x_0)<\inf co\overline{f}(x_0)$. Suppose the first case occurs; the proof for the other one is analogous.

Put $\beta=d(\inf co\overline{f}(x_0),\inf co\overline{g}(x_0))-\epsilon$. Let $\{x_\lambda:\lambda\in\Lambda\}$ be a net in X converging to x_0, such that the net $\{f(x_\lambda):\lambda\in\Lambda\}$ converges to $\inf co\overline{f}(x_0)$. Then for $\frac{\beta}{4}$ there is $\lambda_0\in\Lambda$ such that $f(x_\lambda)\in S_{\frac{\beta}{4}}(\inf co\overline{f}(x_0))$ for all $\lambda>\lambda_0$. The upper semicontinuity of $co\overline{g}(x)$ at x_0 implies that there is an open neighbourhood U of x_0 such that $co\overline{g}(x)\subset S_{\frac{\beta}{4}}(co\overline{g}(x_0))$ for all $x\in U$. Let $\lambda\in\Lambda$ be such that $\lambda>\lambda_0$ and $x_\lambda\in U\cap O$. Then of course $d(f(x_\lambda),g(x_\lambda))>\epsilon$, a contradiction. $\square$

We can give a similar result for functions l_F, where $F\in MU(X,\mathbb{R})$ or $F\in MC(X,\mathbb{R})$.

Theorem 3.13. *Let X be a locally compact space. Let $\mathcal{E}$ be a subset of $(MU(X,\mathbb{R}),\tau_{UC})$. Then $\mathcal{E}$ is compact if and only if the set $\{s^F:\ F\in\mathcal{E}\}$ is closed in $(Q^*(X,\mathbb{R}),\tau_{UC})$, densely equiquasicontinuous and pointwise bounded.*

Proof. Let $\mathcal{E}$ be a compact subset of $(MU(X,\mathbb{R}),\tau_{UC})$. By Theorem 3.9 the set $\{s^F:\ F\in\mathcal{E}\}$ is compact subset of $(Q^*(X,Y),\tau_{UC})$ and by Theorem 2.8 it is closed in $(Q^*(X,\mathbb{R}),\tau_{UC})$, densely equiquasicontinuous and pointwise bounded.

The converse follows from Theorem 3.5. □

Theorem 3.14. *Let X be a locally compact space. Let $\mathcal{E}$ be a subset of $(MC(X,\mathbb{R}),\tau_{UC})$. Then $\mathcal{E}$ is compact if and only if the set $\{s^F: F\in\mathcal{E}\}$ is closed in $(Q^*(X,\mathbb{R}),\tau_{UC})$, densely equiquasicontinuous and pointwise bounded.*

Proof. Let $\mathcal{E}$ be a compact subset of $(MC(X,\mathbb{R}),\tau_{UC})$. By Theorem 3.12 the set $\{s^F: F\in\mathcal{E}\}$ is compact subset of $(Q^*(X,Y),\tau_{UC})$ and by Theorem 2.8 it is closed in $(Q^*(X,\mathbb{R}),\tau_{UC})$, densely equiquasicontinuous and pointwise bounded.

The converse follows from Theorem 3.8. □

Theorem 3.15. *Let X be a locally compact space. Let $\mathcal{E}$ be a subset of $(MU(X,\mathbb{R}),\tau_{UC})$ such that the set $\{s^F: F\in\mathcal{E}\}$ is densely equiquasicontinuous subset of $Q^*(X,\mathbb{R})$. Then the topologies τ_p and τ_{UC} on $\mathcal{E}$ coincide.*

Proof. Let $\{F_\lambda: \lambda\in\Lambda\}$ be a net in $\mathcal{E}$ which τ_p-converges to a $F\in\mathcal{E}$. We show that $\{F_\lambda: \lambda\in\Lambda\}$ also τ_{UC}-converges to F. It is easy to see that $\{s^{F_\lambda}: \lambda\in\Lambda\}$ τ_p-converges to s^F. By Theorem 2.7 $\{s^{F_\lambda}: \lambda\in\Lambda\}$ τ_{UC}-converges to s^F and by Theorem 3.9 $\{F_\lambda: \lambda\in\Lambda\}$ τ_{UC}-converges to F. □

Theorem 3.16. *Let X be a locally compact space. Let $\mathcal{E}$ be a subset of $(MC(X,\mathbb{R}),\tau_{UC})$ such that the set $\{s^F: F\in\mathcal{E}\}$ is densely equiquasicontinuous subset of $Q^*(X,\mathbb{R})$. Then the topologies τ_p and τ_{UC} on $\mathcal{E}$ coincide.*

Proof. The proof is similar to the proof of the above Theorem but we must use Theorem 3.12 instead of Theorem 3.9. □

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