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WEYL TYPE THEOREMS FOR ALGEBRAICALLY QUASI-$\mathcal{HNP}$ OPERATORS

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Abstract. In this paper, by introducing the class of quasi hereditarily normaloid polaroid operators, we obtain a theoretical and general framework from which Weyl type theorems may be promptly established for many of these classes of operators. This framework also entails Weyl type theorems for perturbations $f(T+A)$, where A is algebraic and commutes with T, and f is an analytic function, defined on an open neighborhood of the spectrum of $T+A$, such that f is non constant on each of the components of its domain.

1. Introduction

Let $\mathscr{X}$ (or $\mathscr{H}$) be a complex Banach (Hilbert, respectively) space and $\mathscr{B}(\mathscr{X})$ (or $\mathscr{B}(\mathscr{H})$) be the set of all bounded linear operators on $\mathscr{X}$ ($\mathscr{H}$, respectively). A part of an operator is its restriction to an invariant subspace. An operator $T \in \mathscr{B}(\mathscr{H})$ is hereditarily normaloid, denoted $T \in \mathcal{HN}$, if every part of T is normaloid. An operator $T \in \mathscr{B}(\mathscr{X})$ is polaroid if the isolated points of the spectrum of T are poles of the resolvent of T. An operator $T \in \mathscr{B}(\mathscr{H})$ is hereditarily polaroid, denoted $T \in \mathcal{HP}$, if every part of T is normaloid. Class $\mathcal{HNP}$ denote the class of operators $T \in \mathscr{B}(\mathscr{H})$ such that $T \in \mathcal{HP} \cap \mathcal{HN}$. An operator $T \in \mathscr{B}(\mathscr{H})$ is totally hereditarily normaloid, denoted $T \in \mathcal{THN}$, if every part of T, and (also) invertible part of T, is normaloid. The class $\mathcal{THN}$ is large: it contains a number of the often considered classes of Hilbert space operators. The class $\mathcal{THN}$ is introduced in [12] by Duggal and Djordjevic. Evidently, $(\mathcal{THN}) \subset (\mathcal{HNP})$. In [2], Aiena et.,al investigated $Quasi-\mathcal{THN}$ operators. An operator $T \in \mathscr{B}(\mathscr{X})$,

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$\mathscr{X}$ a Banach Space, is said to be k-Quasi totally hereditarily normaloid, k is a nonnegative integer, if the restriction $T \mid \overline{T^k(\mathscr{X})}$ is $\mathcal{THN}$. Class $\mathcal{THN} \Longrightarrow Quasi - \mathcal{THN}$ [2].

If the range $T(\mathscr{X})$ of $T \in \mathscr{B}(\mathscr{X})$ is closed and $\alpha(T)$=dim$(T^{-1}(0)) < \infty$ (resp., $\beta(T)$=dim$(\mathscr{X} \setminus T(\mathscr{X}) < \infty)$ then T is upper semi-Fredholm (resp., lower semi- Fredholm) operator. Let $SF_+(\mathscr{X})$ (resp.,$SF_-(\mathscr{X})$) denote the semigroup of upper semi Fredholm (resp., lower semi Fredholm) operator on $\mathscr{X}$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be semi-Fredholm, $T \in SF$, if $T \in SF_+(\mathscr{X}) \cup SF_-(\mathscr{X})$ and Fredholm if $T \in SF_+(\mathscr{X}) \cap SF_-(\mathscr{X})$. If T is semi-Fredholm then the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. The classes of upper semi-Weyl operators $W_+(\mathscr{X})$ and lower semi-Weyl operators $W_-(\mathscr{X})$ are defined by

$$W_+(\mathscr{X}) = \{T \in B(\mathscr{X}) : T \,\text{is upper semi Fredholm and}\, \text{ind}(T) \leq 0\},$$

$$W_-(\mathscr{X}) = \{T \in B(\mathscr{X}) : T \,\text{is lower semi Fredholm and}\, \text{ind}(T) \geq 0\}.$$

Recall that the ascent of an operator $T \in \mathscr{B}(\mathscr{X})$ is the smallest non negative integer $p := p(T)$ such that $T^{-p}(0) = T^{-(p+1)}(0)$. If such p does not exist, then $p(T) = \infty$. The descent of T is defined as the smallest non negative integer $q := q(T)$ such that $T^q(\mathscr{X}) = T^{q+1}(\mathscr{X})$. If no such q exist, then $q(T) = \infty$. It is well known that if p and q are both finite then they are equal [1, Theorem 3.3].

A bounded linear operator T acting on a Banach space $\mathscr{X}$ is Weyl, $T \in W$, if $T \in W_+(\mathscr{X}) \cap W_-(\mathscr{X})$ and Browder, $T \in \mathscr{B}(\mathscr{X})$, if T is Fredholm of finite ascent and descent. Let $\mathbb{C}$ denote the set of complex numbers and let $\sigma(T)$ denote the spectrum of T. The Wolf spectrum $\sigma_{SF}(T)$, Weyl spectrum $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF\},$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin W\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin B\}.$$

Let $E^0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < (T - \lambda) < \infty\}$ and $\sigma_0(T)$ denote the set of all normal eigenvalues (Riesz points) of T. According to Coburn [9], Weyl's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$ and Browder's theorem holds for T if $\sigma(T) \setminus \sigma_w(T) = \sigma_0(T)$.

Let $SF_+^-(\mathscr{X}) = \{T \in SF_+ : \text{ind}\,(\text{T}) \leq 0\}$. The *upper semi Weyl spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathscr{X})\}$. According to Rakočević [17], an operator $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \text{iso}\,\sigma_\text{a}(\text{T}) : 0 < \alpha(\text{T} - \lambda\text{I}) < \infty\}.$$

It is known [17] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

An operator $T \in \mathscr{B}(\mathscr{X})$ is called B-Fredholm, $T \in \mathcal{BF}$, if there exist a natural number n, for which the induced operator $T_n = T \mid T^n(\mathscr{X}), T_0 = T$ is Fredholm in the usual sense [7]. The class of B-Weyl operator $T \in \mathscr{B}(\mathscr{X})$ is defined by $\mathcal{BW} = \{T \in \mathcal{BF} : ind(T_n) = 0\}$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{BW}\}$ [7]. As a stronger version of Weyl's theorem, generalized Weyl's theorem was introduced by Berkani [8]. Let $E(T)$ is the set of all eigenvalues of T which are isolated in $\sigma(T)$. We say that T satisfies generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$.

Following [7], we say that T satisfies *generalized Browders's theorem*, if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, where $\pi(T)$ is the set of poles of T.

Let $SBF_+^-(\mathscr{X})$ denote the class of all is *upper B-Fredholm* operators such that ind (T) ≤ 0. The *upper B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathscr{X})\}.$$

Following [5], we say that *generalized a-Weyl's theorem* holds for $T \in \mathscr{B}(\mathscr{X})$ if $\Delta_a^g(S) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in iso\sigma_a(T) : \alpha(T - \lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathscr{B}(\mathscr{X})$ obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [4, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [5, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [6, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Weyl type theorems have been studied in the last two decades by several authors and most of them have essentially proved that such theorems hold for special classes of operators. Many times the arguments used, to prove Weyl type theorems for each one of these classes of operators, are rather similar. In this paper we show that it is possible to bring back up these theorems from some general common ideas. Actually, we determine a very useful and unique theoretical framework, from which we can deduce that Weyl type theorems hold for all these classes of operators. This framework is created by introducing the class of *quasi hereditarily normaloid polaroid* operators and by proving that these operators are *hereditarily polaroid*. Many classes of operators T on Hilbert spaces are quasi hereditarily normaloid polaroid, and this fact, together with SVEP, permits to us to extend all Weyl type theorems to the perturbations $f(T + A)$, where A is algebraic and commutes with T, f is an analytic function, defined on an open neighborhood of the spectrum of $T + A$, such that f is nonconstant on each of the components of its domain. Consequently, our results subsume and extend many results existing in literature.

2. Hereditarily Normaloid Polaroid Operators

A bounded operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *polaroid* if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent. An operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *hereditarily polaroid*, i.e. any restriction to an invariant closed subspace is polaroid. An example of polaroid operator which is not hereditarily polaroid may be found in [11, Example 2.6]. An operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *normaloid* if $\|T\| = r(T)$, $r(T)$ the spectral radius of T. An operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *hereditarily normaloid*, $T \in \mathcal{HN}$, if the restriction $T|M$ of T, to any closed T-invariant subspace M, is normaloid. An operator $T \in \mathscr{B}(\mathscr{X})$ is said to be *totally hereditarily normaloid*, $T \in \mathcal{THN}$, if $T \in \mathcal{HN}$ and every invertible restriction $T|M$ has a normaloid inverse. Totally hereditarily operators were introduced in [12]. Finally, $T \in \mathscr{B}(\mathscr{X})$ is said to be *hereditarily normaloid polaroid*, $T \in \mathcal{HNP}$, if $T \in \mathcal{HP} \cap \mathcal{HN}$.

Two important subspaces in local spectral theory and Fredholm theory are defined in the sequel. The quasi-nilpotent part of an operator $T \in \mathscr{B}(\mathscr{X})$ is the set

$$H_0(T) = \{x \in X : \lim_{n \longrightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$. If $T \in B(X)$, the analytic core $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$. It is known that $T \in \mathscr{B}(\mathscr{X})$ is polaroid if and only if there exists $p := p(\lambda - T) \in \mathbb{N}$ such that

$$H_0(\lambda - T) = \ker(\lambda - T)^p \quad \text{for all} \quad \lambda \in \mathrm{iso}\sigma(T), \tag{2.1}$$

where $\mathrm{iso}\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$. A very important class of hereditarily polaroid operators is the class of $H(p)$ operators, where $T \in \mathscr{B}(\mathscr{X})$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all} \quad \lambda \in \mathbb{C}$$

The class $H(p)$ has been introduced by Oudghiri in [16]. Property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [16]. Therefore, algebraically p-hyponormal or algebraically w-hyponormal operators are $H(p)$, see [18]. we know that every operator T which belongs to the class $H(p)$ has SVEP. Moreover, from (2.1) it follows that every $H(p)$ operator T is polaroid. The restriction to closed invariant subspaces of any $H(p)$ operator is also $H(p)$, see [16], so every $H(p)$ is hereditarily polaroid.

We start below by listing some examples of $\mathcal{HNP}$ operators.

Example 2.1. $T \in \mathscr{B}(\mathscr{X})$ is completely hereditarily normaloid, $T \in \mathcal{CHN}$, if either $T \in \mathcal{HNP}$ or $T - \lambda$ is normaloid for every $\lambda \in \mathbb{C}$. $\mathcal{CHN}$ operators are simply hereditarily polaroid, i.e., the poles of every part of the operator are simple (or order one) [11, Proposition 2.1]. In particular, paranormal operators (i.e., operators $T \in \mathscr{B}(\mathscr{X})$ such that $\|T^2 x\| \geq \|Tx\|^2$ for every unit vector $x \in \mathscr{X}$ are $\mathcal{HNP}$ operators.

Example 2.2. $T \in \mathscr{B}(\mathscr{H})$ is a 2-isometry (or, a 2-isometric operator) if $T^{*2}T^2 - 2T^*T + I = 0$. Every 2-isometric operator is left invertible; if T is not invertible, then $\sigma(T)$ is the closed unit disc ($\mathrm{iso}\sigma(T) = \emptyset$), and if T is invertible, then it is a unitary [1]. Evidently, the restriction of a 2-isometry to an invariant subspace is a 2-isometry. Hence, 2-isometric operators are $\mathcal{HNP}$ operators.

Example 2.3. An operator $T \in \mathscr{B}(\mathscr{X})$ is polynomially $\mathcal{HP}$ if there exists a non-trivial polynomial g such that $g(T) \in \mathcal{HN}$. Polynomially $\mathcal{HP}$ operators are $\mathcal{HP}$, as the following argument shows. Let $A = T|M$, where M is an invariant subspace of T ; let $A_0 = A|H_0(A - \lambda)$ and $A_1 = A|K(A - \lambda)$. If $\lambda \in \mathrm{iso}\sigma(A)$, then $M = H_0(A-\lambda) \oplus K(A-\lambda)$, $\sigma(A_0) = \{\lambda\}$ and A_1 is invertible. Evidently, $\sigma(g(A_0))$ and (since $g(A)$ is polaroid) there exists a positive integer n such that $H_0(g(A) - g(\lambda)) = \ker(g(A) - g(\lambda))^n$ if and only if $g(A) - g(\lambda)^n = 0$. Letting $g(A) - g(\lambda)^n = c_0(A_0 - \lambda)^t \prod_{i=1}^{s}(A_0 - \lambda_i)$ for some scalars c_0 and λ_i $(1 \le i \le s)$, and positive integers s and t , it follows that $(A_0 - \lambda)^t = 0$ then $H_0(A_0 - \lambda) = \ker(A - \lambda)^t$. Hence $M = \ker(A-\lambda)^t \oplus K(A-\lambda)$ and so, $M = \ker(A-\lambda)^t \oplus (A-\lambda)^t M$ i.e., λ is a pole of the resolvent of A.

We define k-Quasi $\mathcal{HNP}$ Operators as follows.

Definition 2.4. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be k-Quasi $\mathcal{HNP}$, k a nonnegative integer, if the restriction $T \mid \overline{T^k(\mathscr{H})}$ is $\mathcal{HNP}$.

Clearly,

$$\mathcal{HNP} \subset k-\text{Quasi}-\mathcal{HNP} \qquad \text{and} \qquad k-\text{Quasi}-\mathcal{THN} \subset k-\text{Quasi}-\mathcal{HNP}.$$

Remark 2.5. It is rather simple to see that if $T \in \mathscr{B}(\mathscr{X})$ is $\mathcal{HNP}$ and M is a T-invariant closed subspace of $\mathscr{X}$ then the restriction $T|M$ is also $\mathcal{HNP}$.

Theorem 2.6. *Let $T \in \mathscr{B}(\mathscr{X})$ is Quasi $\mathcal{HNP}$ operator and $\mathcal{M}$ be an invariant subspace of T. Then the restriction $T \mid \mathcal{M}$ is Quasi $\mathcal{HNP}$*

Proof. Let k a nonnegative integer such that $T_k := T|T_k(\mathscr{X})$ is $\mathcal{HNP}$. Let T_M denote the restriction $T|M$. Clearly, $\overline{T^k_M(M)} \subset \overline{T^k(\mathscr{X})}$, so $\overline{T^k_M(M)}$ is T_k-invariant subspace of $\overline{T^k(\mathscr{X})}$. By Remark 2.5 it then follows that $T_M|\overline{T^k_M(M)} = T_k|\overline{T^k_M(M)}$ is $\mathcal{HNP}$. □

We recall now some elementary algebraic facts. Suppose that $T \in \mathscr{B}(\mathscr{X})$ and $\mathscr{X} = M \oplus N$, with M and N closed subspace of $\mathscr{X}$, M invariant under T. With respect to this decomposition of $\mathscr{X}$ it is known that T may be represented by a upper triangular operator matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A \in \mathscr{B}(M)$, $C \in \mathscr{B}(N)$ and $B \in \mathscr{B}(M,N)$. It is easily seen that for every $x = \begin{pmatrix} x \\ 0 \end{pmatrix} \in M$ we have $Tx = Ax$, so $A = T|M$. Let us consider now the case of operators T acting on a Hilbert space $\mathscr{H}$, and suppose that $T^k(\mathscr{H})$ is not dense in $\mathscr{H}$. In this case we can consider the nontrivial orthogonal decomposition

$$\mathscr{H} = \overline{T^k(\mathscr{H})} \oplus \overline{T^k(\mathscr{H})}^{\perp}, \tag{2.2}$$

where $\overline{T^k(\mathscr{H})}^{\perp} = \ker(T^*)^k$, T^* the adjoint of T. Note that the subspace $\overline{T^k(\mathscr{H})}$ is T-invariant, since

$$T(\overline{T^k(\mathscr{H})}) \subseteq \overline{T(T^k(\mathscr{H})} = \overline{T^{k+1}(\mathscr{H})} \subseteq \overline{T^k(\mathscr{H})}.$$

Thus we can represent, with respect the decomposition (2.2), T as an upper triangular operator matrix

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where $T_1 = T|\overline{T^k(\mathscr{H})}$. Moreover, T_3 is nilpotent. Indeed, if $x \in \overline{T^k(\mathscr{H})}^{\perp}$, an easy computation yields $T^k x = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^k \begin{pmatrix} 0 \\ x \end{pmatrix} = T_3^k x$. Hence $T_3^k x = 0$, since $T^k x \in \overline{T^k(\mathscr{H})} \cup \overline{T^k(\mathscr{H})}^{\perp} = \{0\}$. Therefore we have:

Theorem 2.7. *Suppose that $T \in \mathscr{B}(\mathscr{H})$ and $T^k(\mathscr{H})$ non dense in $\mathscr{H}$. Then, according the decomposition (2.2),$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is quasi-$\mathcal{HNP}$ if and only if T_1 is $\mathcal{HNP}$. Furthermore,*

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

Proof. The first assertion is clear, since $T_1 = T \mid \overline{T^k(\mathscr{H})}$. The second assertion follows from the following general result: if $T := \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is an upper triangular operator matrix acting on some direct sum of Banach spaces and $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C)$; see [14]. □

Let $H_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in H_{nc}(\sigma(T))$.
The following result has been proved in [3, Theorem 2.4].

Theorem 2.8. *For an operator $T \in \mathscr{B}(\mathscr{X})$ the following statements are equivalent:*

(i) *T is polaroid;*
(ii) *there exists $f \in H_{nc}(\sigma(T))$ such that $f(T)$ is polaroid;*
(iii) *$f(T)$ is polaroid for every $f \in H_{nc}(\sigma(T))$.*

An operator $T \in \mathscr{B}(\mathscr{X})$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \longrightarrow \mathscr{X}$ which satisfies the equation $(\lambda - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. The operator T is said to have SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. It follows from the identity theorem for analytic functions that T has SVEP at every point of the boundary of the spectrum. In particular, T and its dual T^* have SVEP at every isolated point of $\sigma(T)$.

Theorem 2.9. *Let $T \in \mathscr{B}(\mathscr{X})$ is k-Quasi $\mathcal{HNP}$ operator. Then T satisfies SVEP.*

Proof. Suppose $T \in \mathscr{B}(\mathscr{X})$ be a k-Quasi-$\mathcal{HNP}$ operator. If $T^k(\mathscr{X})$ is dense, T is $\mathcal{HNP}$, so T has Bishop's property (β). Suppose that $T^k(\mathscr{X})$ is not dense and write T as matrix representation on $\mathscr{X} = \overline{T^k(\mathscr{X})} \oplus \overline{T^k(\mathscr{X})}^{\perp}$,

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

. where $T_1 = T$
$overlinemidT^k(\mathscr{X})$ is $\mathcal{HNP}$ and T_3 is nilpotent. Since $\mathcal{HNP}$ satisfies SVEP, from [2, theorem 4.3], T has SVEP. $\square$

An operator $T \in \mathscr{B}(\mathscr{H})$ is algebraically Quasi-$\mathcal{HNP}$ operator if there exist a non-constant complex polynomial p such that $p(T) \in Quasi - \mathcal{HNP}$

Lemma 2.10. *Let $T \in \mathscr{B}(\mathscr{H})$ be an algebraically Quasi-$\mathcal{HNP}$ operator and $\sigma(T) = \{\mu\}$. Then $T - \mu I$ is nilpotent.*

Proof. Suppose that $T \in \mathscr{B}(\mathscr{H})$ be an algebraically Quasi-$\mathcal{HNP}$ operator. Then $p(T)$ is of Quasi-$\mathcal{HNP}$ for some non constant polynomial p . If $\sigma(T) = \{\lambda\}$, then $\sigma\{p(T)\} = p(\lambda)$ and so $p(T) = p(\lambda)I$. Let

$$p(T) - p(\lambda_0) = c(T - \lambda_0)^{k_0}(T - \lambda_1)^{k_1}(T - \lambda_2)^{k_1}.....(T - \lambda_n)^{k_n}$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$. Then

$$0 = p(T) - p(\lambda_0)^m = c^m(T - \lambda_0)^{mk_0}(T - \lambda_1)^{mk_1}(T - \lambda_2)^{mk_2}.....(T - \lambda_n)^{mk_n}$$

we must have $(T - \lambda_0)^{mk_0} = 0$. $\square$

Theorem 2.11. *If $T \in \mathscr{B}(\mathscr{X})$ is an analytically quasi-$\mathcal{HNP}$ operator, then T is polaroid.*

Proof. We show that for every isolated point λ of $\sigma(T)$ we have $p(\lambda - T) = q(\lambda - T) < \infty$. Let λ be an isolated point of $\sigma(T)$, and denote by P_λ denote the spectral projection associated with $\{\lambda\}$. Then $M := K(\lambda - T) = \ker P_\lambda$ and $N := H_0(\lambda - T) = P_\lambda(\mathscr{X})$, see [1, Theorem 3.74]. Therefore, $H = H_0(\lambda - T) \oplus K(\lambda - T)$. Furthermore, since $\sigma(T|N) = \{\lambda\}$, while $\sigma(T|M) = \sigma(T) \setminus \{\lambda\}$, so the restriction $\lambda - T|N$ is quasi-nilpotent and $\lambda - T|N$ is invertible. Since $\lambda - T|N$ is analytically quasi $\mathcal{THN}$, then Lemma 2.10 implies that $\lambda - T|N$ is nilpotent. In other words, $\lambda - T|N$ is an operator of Kato Type.
Now, both T and the dual T^* have SVEP at λ, since λ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.16 and Theorem 3.17 of [1], that both $p(\lambda - T)$ and $q(\lambda - T)$ are finite. Therefore, λ is a pole of the resolvent. $\square$

Let $\mathcal{C}$ be any class of operators. We say that T is an analytically $\mathcal{C}$-operator if there exists some analytic function $f \in H_{nc}(\sigma(T))$ such that $f(T) \in \mathcal{C}$.

Recall that an invertible operator $T \in \mathscr{B}(\mathscr{X})$ is said to be doubly power-bounded if $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$.

Theorem 2.12. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ is quasi-nilpotent. If T is an analytically $\mathcal{HNP}$ operator, then T is nilpotent.*

Proof. Let $T \in \mathscr{B}(\mathscr{X})$ and suppose that $f(T)$ is a $\mathcal{HNP}$-operator for some $f \in H_{nc}(\sigma(T))$. From the spectral mapping theorem we have

$$\sigma(f(T)) = f(\sigma(T)) = \{f(0)\}.$$

We claim that $f(T) = f(0)I$. To see this, let us consider the two possibilities: $f(0) = 0$ or $f(0) \neq 0$.
If $f(0) = 0$ then $f(T)$ is quasi-nilpotent and $f(T)$ is normaloid, and hence $f(T) = 0$. The equality $f(T) = f(0)I$ then trivially holds.

Suppose the other case $f(0) \neq 0$, and set $f_1(T) := \frac{1}{f(0)} f(T)$. Clearly,$\sigma(f_1(T)) = \{1\}$ and $\|f_1(T)\| = 1$. Further, $f_1(T)$ is invertible and is $\mathcal{HNP}$. This easily implies that its inverse $f_1(T)^{-1}$ has norm 1. The operator $f_1(T)$ is then doubly power-bounded and, by a classical theorem due to Gelfand, it then follows that $f_1(T) = I$, and consequently $f(T) = f(0)I$, as claimed.

Now, let $g(\lambda) := f(0) - f(\lambda)$. Clearly, $g(0) = 0$, and g may have only a finite number of zeros in $\sigma(T)$. Let $\{0, \lambda_1, \cdots, \lambda_n\}$ be the set of all zeros of g, where $\lambda_i \neq \lambda_j$, for all $i \neq j$, and λ_i has multiplicity $n_i \in \mathbb{N}$. We have

$$g(\lambda) = \mu\lambda^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} h(\lambda),$$

where $h(\lambda)$ has no zeros in $\sigma(T)$. From the equality $g(T) = f(0)I - f(T) = 0$ it then follows that

$$0 = g(T) = \mu T^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} h(\lambda) \qquad \text{with} \quad \lambda_i \neq 0,$$

where all the operators $\lambda_i I - T$ and $h(T)$ are invertible. This, obviously, implies that $T^m = 0$, i.e. T is nilpotent. □

Theorem 2.13. *Suppose that $T \in \mathscr{B}(\mathscr{H})$, is analytically k-quasi-$\mathcal{HNP}$ and quasi-nilpotent. Then T is nilpotent.*

Proof. Suppose first that T is quasi-nilpotent and k-quasi-$\mathcal{HNP}$. If $T^k(\mathscr{H})$ is dense then T is $\mathcal{HNP}$, so T is nilpotent by Theorem 2.12. Suppose that $T^k(H)$ is not dense and write $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is $\mathcal{HNP}$, $T_3^k = 0$, and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since $\sigma(T) = \{0\}$ and $\sigma(T_1)$ is not empty, we then have $\sigma(T_1) = \{0\}$, thus T_1 is a quasi-nilpotent $\mathcal{HNP}$ operator and hence $T_1 = 0$. Therefore $T = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}$. An easy computation yields that

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0,$$

so that T is nilpotent.

Finally, suppose that T is quasi-nilpotent and analytically k-quasi $\mathcal{HNP}$. Let $h \in H_{nc}(\sigma(T))$ be such that $h(T)$ is quasi-$\mathcal{HNP}$. We claim that $h(T)$ is nilpotent. If $h(T)^k$ has dense range then $h(T)$ is $\mathcal{HNP}$ and hence, by Theorem 2.12, $h(T)$ is nilpotent. Suppose that $h(T)^k$ has not dense range. Then with respect the decomposition $X = \overline{h(T)^k(\mathscr{H})} \oplus \overline{h(T)^k(\mathscr{H})}^{\perp}$, the operator $h(T)$ has a triangulation $h(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, such that $A = h(T)|\overline{h(T)^k(\mathscr{H})}$ is $\mathcal{HNP}$ and

$$\sigma(h(T)) = \sigma(A) \cup \{0\},$$

By the spectral mapping theorem we have $\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}$. Consequently, $0 \in \{h(0)\}$, i.e. $h(0) = 0$, and therefore $h(T)$ is quasi-nilpotent. Since

$h(T)$ is quasi-$\mathcal{HNP}$, by the first part of proof it then follows that $h(T)$ is nilpotent. Now, $h(0) = 0$ so we can write

$$h(\lambda) = \mu\lambda^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} g(\lambda),$$

where $g(\lambda)$ has no zeros in $\sigma(T)$ and $\lambda_i \neq 0$ are the other zeros of g with multiplicity n_i. Hence

$$h(T) = \mu T^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} g(T),$$

where all $\lambda_i I - T$ and $g(T)$ are invertible. Since $h(T)$ is nilpotent then also T is nilpotent. $\square$

Theorem 2.14. *If $T \in \mathscr{B}(\mathscr{H})$ is analytically quasi $\mathcal{HNP}$, then T is hereditarily polaroid.*

Proof. Let $f \in H_{nc}(\sigma(T))$ such that $f(T)$ is quasi-$\mathcal{HNP}$. If M is a closed T-invariant subspace of X, we know that $f(T)|M$ is quasi-$\mathcal{HNP}$, by Theorem 2.6, and $f(T)|M = f(T|M)$, so $f(T|M)$ is polaroid, by Theorem 2.11, and consequently, $T|M$ is polaroid, by Theorem 2.8. $\square$

Upper triangular operator matrices have been studied by many authors, see for instance [14]. In the sequel we give some examples of operators which are quasi-hereditarily normaloid polaroid.

Example 2.15. The class of quasi-paranormal operators may be extended as follows: $T \in \mathscr{B}(\mathscr{H})$ is said to be (n,k)-quasiparanormal if

$$\left\|T^{k+1}x\right\| \leq \left\|T^{n+1}(T^k x)\right\|^{\frac{1}{n+1}} \left\|T^k x\right\|^{\frac{n}{n+1}} \text{ for all } x \in \mathscr{H}.$$

The class of $(1,k)$-quasiparanormal operators has been studied in [15]. If $T^k(\mathscr{H})$ is not dense then, in the triangulation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, $T_1 = T|\overline{T^k(\mathscr{H})}$ is n-quasiparanormal, and hence $\mathcal{HNP}$, see [21].

Example 2.16. An extension of class A operators is given by the class of all k-quasiclass A operators, where $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{H}$ a separable infinite dimensional Hilbert space, is said to be a k-quasiclass A operator if

$$T^{*k}(|T^2| - |T|^2)T^k \geq 0.$$

Every k-quasiclass A operator is quasi-$\mathcal{HNP}$. Indeed, if T has dense range then T is a class A operator and hence paranormal. If T does not have dense range then T with respect the decomposition $\mathscr{H} = \ker(T^{*k}) \oplus \overline{T^k(\mathscr{H})}$ may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, $T_1 = T|\overline{T^k(\mathscr{H})}$ is a class A operator, and hence $\mathcal{HNP}$, see [19]. As it has been observed in [10, Example 0.2], a quasi-class A operator (i.e. $k = 1$), need not to be normaloid. This shows that, in general, a quasi-$\mathcal{HNP}$ operator is not normaloid, so the class of quasi-$\mathcal{HNP}$ operators properly contains the class of $\mathcal{HNP}$ operators.

Example 2.17. An operator $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{H}$ a separable infinite dimensional Hilbert space, is said to be k-quasi $*$-paranormal, $k \in \mathbb{N}$, if

$$\left\|T^*T^kx\right\|^2 \leq \left\|T^{k+2}x\right\| \left\|T^kx\right\| \text{ for all unit vectors } x \in \mathscr{H}.$$

This class of operators contains the class of all quasi-$*$-paranormal operators (which corresponds to the value $k = 1$). Every k-quasi-$*$-paranormal operator is quasi-$\mathcal{HNP}$. Indeed, if T^k has dense range then T is $*$-paranormal and hence $\mathcal{HNP}$. If T^k does not have dense range then T, may be decomposed, according the decomposition $\mathscr{H} = \ker(T^{*k}) \oplus \overline{T^k(\mathscr{H})}$ may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, $T_1 = T|\overline{T^k(\mathscr{H})}$ is $*$-paranormal, hence $\mathcal{HNP}$, see [15, Proposition 2.3].

Example 2.18. An extension of p-quasihyponormal operators is defined as follows: an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be (p,k)-quasihyponormal for some $0 < p \leq 1$ and $k \in \mathbb{N}$, if

$$T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \geq 0.$$

Every (p,k)-quasihyponormal operator T with respect to the decomposition $\mathscr{H} = \ker(T^{*k}) \oplus \overline{T^k(\mathscr{H})}$ may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, $T_1 = T|\overline{T^k(\mathscr{H})}$ is is k-hyponormal (hence paranormal) and consequently $\mathcal{HNP}$, see [20].

Theorem 2.19. *Let $T \in \mathscr{B}(\mathscr{H})$ is algebraically Quasi-$\mathcal{HNP}$ operator. Then T is polaroid.*

Proof. Suppose that *p(T)* is of Quasi $\mathcal{HNP}$ for some non constant polynomial p and λ is an isolated point of $\sigma(T)$. To prove λ is simple pole of resolvent of T it is enough to show that $T - \lambda I$ has finite ascent and descent. Let P_λ denote the spectral projection associated with λ. We can represent T as

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \mid \{\lambda\}$ From Lemma 2.10 it follows that $T_1 - \lambda I$ is nilpotent and so $T_1 - \lambda I$ has finite ascent and descent. The invertibility of $T_2 - \lambda I$ implies that $T_2 - \lambda I$ has finite ascent and descent and so $T - \lambda I$ has finite ascent and descent. This completes the proof. $\square$

Theorem 2.20. *Let T be algebraically Quasi-$\mathcal{HNP}$ operator. Then the generalized Weyl's theorem holds for both T and T^**

Proof. Since T is algebraically Quasi-$\mathcal{HNP}$, T is polaroid and T has SVEP. Then result follows from [2, theorem 4.1]. $\square$

Theorem 2.21. *Let T is algebraically Quasi-$\mathcal{HNP}$ operator. Then the equality $\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$ holds for every $f \in Hol(\sigma(T))$*

Proof. Suppose T is algebraically Quasi-$\mathcal{HNP}$. Then T has SVEP. Since T has SVEP, $f(T)$ has SVEP. Thus, $f(T)$ satisfies the generalized Browder's theorem and hence $\sigma_{BW}(f(T)) = \sigma_D(f(T))$. Since $\sigma_D(f(T)) = f(\sigma_D(T))$, $\sigma_{BW}(f(T)) = f(\sigma_D(T))$. From Theorem 2.20 T satisfies the generalized Weyl's theorem and so T satisfies the generalized Browder's theorem. Thus we have $f(\sigma_D(T)) = f(\sigma_{BW}(T))$. Hence,$\sigma_{BW}(f(T)) = f(\sigma_{BW}(T))$. $\square$

The following theorem is immediate consequence of the above theorem.

Theorem 2.22. *Let T is Algebraically Quasi-$\mathcal{HNP}$ operator. Then $f(T)$ satisfies generalized Weyl's theorem for every $f \in Hol(\sigma(T))$*

3. Perturbations

An operator $R \in \mathscr{B}(\mathscr{X})$ is a Riesz operator if $R - \lambda$ is Fredholm for every non-zero $\lambda \in \mathbb{C}$. Since operators of Algebraically k-Quasi-$\mathcal{HNP}$ have SVEP, we have immediately

Proposition 3.1. *Let $T \in \mathscr{B}(\mathscr{H})$ be algebraically k-quasi-$\mathcal{HNP}$ operator and $R \in \mathscr{B}(\mathscr{H})$ be a Riesz operator which commutes with T, then $f(T + R)$ satisfies generalized a-Browder's theorem for every $f \in H_{nc}(\sigma(T + R))$.*

Recall that $T \in \mathscr{B}(\mathscr{H})$ is finitely isoloid if isolated points of $\sigma(T)$ are eigenvalues of finite multiplicity.

Definition 3.2. Let $T \in \mathscr{B}(\mathscr{H})$. We say that T satisfies

(i) property (t) if $\sigma(T) \setminus \sigma_{SF_-^+} = E^0(T)$.

(ii) property (gt) if $\sigma(T) \setminus \sigma_{SBF_-^+} = E(T)$.

Proposition 3.3. *If $T \in \mathscr{B}(\mathscr{H})$ is an algebraically k-quasi-$\mathcal{HNP}$ operator and commutes with an injective quasi-nilpotent $Q \in \mathscr{B}(\mathscr{H})$, then $T + Q$ satisfies property (t). Furthermore, if T is finitely isoloid, then $T^* + Q^*$ satisfies property (gt).*

Proof. T and $T + Q$ being quasi-nilpotent equivalent, $T + Q$ has SVEP and this implies that $\sigma(T+Q) = \sigma(T^*+Q^*) = \sigma_a(T^*+Q^*)$ and $\sigma_w(T+Q) = \sigma_{SF_-^+}(T+Q)$. The commutativity of T and Q implies that if $(0 \neq)x \in \ker(T + Q - \lambda)$ for some $\lambda \in \sigma(T + Q)$, then $Q^m x \in \ker(T + Q - \lambda)$ for all non-negative integers m. Let $p(t) = \sum_{i=1}^{n} c_i t^i = c_i \prod_{i=1}^{n} (t - \lambda_i)$ be a polynomial such that $p(Q) = 0$. Then the injectivity of Q implies that $c_n = 0$; hence, by a finite induction argument, $c_i = 0$ for all $0 \leq i \leq n$. Since this implies that $\{Q^n x\}$ is a linearly independent set of vectors in $\ker(T + Q - \lambda)$, eigenvalues of $T + Q$, hence also of T since $T = (T+Q)-Q$, have infinite multiplicity. In particular, $E^0(T+Q) = \emptyset$. Clearly, $T + Q$ satisfies generalized a-Browder's theorem; hence $T + Q$ satisfies Browder's theorem, i.e., $\sigma(T+Q)\setminus\sigma_w(T+Q) = \sigma_0(T+Q)$. Since $\sigma_0(T+Q) \subseteq E^0(T+Q) = \emptyset$ and $\sigma(T + Q) \setminus \sigma_{SF_-^+}(T + Q) = E^0(T + Q)$ and so $T + Q$ satisfies property (t). Assume now that T is finitely isoloid. Then it follows from the above that $\text{iso}\sigma(T) = \text{iso}(T + Q) = \emptyset$. By [11, Proposition 3.2], $T^* + Q^*$ satisfies generalized

a-Browder's theorem and this implies that $\sigma_a(T^*+Q^*)\setminus\sigma_{SBF_-^+}(T^*+Q^*)=\pi_a(T^*+Q^*)\subseteq E^a(T^*+Q^*)=E(T^*+Q^*)$. Since $\lambda\in E(T^*+Q^*)$ implies $\lambda\in \mathrm{iso}\sigma(T+Q)=\emptyset$, it follows that $\sigma(T^*+Q^*)\setminus\sigma_{SBF_-^+}(T^*+Q^*)=E(T^*+Q^*)$, so that T^*+Q^* satisfies property (gt). □

An operator $K\in\mathscr{B}(\mathscr{X})$ is an algebraic operator if there exists a non-trivial polynomial $q(.)$ such that $q(A)=0$. Operators $F\in\mathscr{B}(\mathscr{X})$ such that F^n is finite dimensional for some $n\in\mathbb{N}$ are algebraic.

Theorem 3.4. *Let $T\in\mathscr{B}(\mathscr{X})$ be an algebraically k-quasi-$\mathcal{HNP}$ operator and let $A\in\mathscr{B}(\mathscr{X})$ be an algebraic operator which commutes with T. Then $f(T^*+A^*)$ satisfies property (gt) for every $f\in H_{nc}(\sigma(T+A))$.*

Proof. The operator A being algebraic, $\sigma(A)=\{\mu_1,\cdots,\mu_n\}$ for some scalars $\mu_i, 1\leq i\leq n$. Let $A_i=A|H_0(A-\mu_i)$ and $T_i=T|H_0(A-\mu_i)$, $1\leq i\leq n$. The commutativity of A with T then implies that A_i commutes with T_i for all $1\leq i\leq n$ (for the reason that the projection $H_0(A-\mu_i)$ corresponding to μ_i commutes with T for all $1\leq i\leq n$), $T=\bigoplus_{i=1}^n T_i$ and $T+A=\bigoplus_{i=1}^n T_i+A_i$. Since $A_i-\mu_i$ is nilpotent for all $1\leq i\leq n$ by [13, Lemma 3.5], the upper triangular operator $\bigoplus_{i=1}^n (T_i+A_i-\mu_i)=\bigoplus_{i=1}^n T_i$, with entries $A_i-\mu_i$ along the main diagonal, is nilpotent. Hence $T+A-\mu$ and T are quasi-nilpotent equivalent. Since T has SVEP; hence $T+A-\mu$, equivalently $T+A$, has SVEP (so that both $T+A$ and T^*+A^* satisfy generalized a-Browder's theorem). Arguing as in the proof of [13, Lemma 6] it is now seen that $H_0(T+A-\lambda)=\ker(T+A-\lambda)^m$, for some $m\in\mathbb{N}$, at every $\lambda\in\mathrm{iso}\sigma(T+A)$. Hence $T+A$ is polaroid and so it satisfies the generalized Weyl's theorem, i.e., $\sigma(T+A)\setminus E(T+A)=\sigma_{BW}(T+W)(=\sigma(T+A)\setminus\pi(T+A))$. The operator $T+A$ being isoloid, a familiar argument [1, Lemma 3.89] shows that $f(\sigma(T+A)\setminus E(T+A))=\sigma(f(T+A))\setminus E(f(T+A))$ for every $f\in H_{nc}(\sigma(T+A))$. Since $f(\sigma_{BW}(T+A))=\sigma_{BW}(f(T+A))$ for every $f\in H_{nc}(\sigma(T+A))$, $\sigma(f(T+A))\setminus E(f(T+A))=f(\sigma_{BW}(T+A))=\sigma_{BW}(f(T+A))$, i.e., $f(T+A)$ satisfies the generalized Weyl's theorem. Observe that SVEP implies $\sigma(T+A)=\sigma(T^*+A^*)=\sigma_a(T^*+A^*), E_a(T^*+A^*)=E(T^*+A^*)$ and the polaroid property of $T+A$, and therefore of T^*+A^*, implies that $E(T^*+A^*)=\pi(T^*+A^*)=\pi(T+A)=E(T+A)$. Recall from the proof of Proposition 3.2 of [11]that $\sigma_{SBF_-^+}(T^*+A^*)=\sigma_{BW}(T^*+A^*)=\sigma_{BW}(T+A)$. Hence

$$\sigma(T+A)\setminus E(T+A)=\sigma_{BW}(T+A)\Longrightarrow\sigma(T^*+A^*)\setminus E(T^*+A^*)=\sigma_{SBF_-^+}(T^*+A^*),$$

i.e., T^*+A^* satisfies property (gt). Since T^*+A^* is (evidently) a-isoloid, $f(T^*+A^*)$ satisfies property (gt) for every $f\in H_{nc}(\sigma(T+A))$. □

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